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Unraveling the Mystery of the Origin of Mathematical Problems: Using a Problem-Posing Framework With Prospective Mathematics Teachers

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In this article, I model how a problem-posing framework can be used to enhance our abilities to systematically generate mathematical problems by modifying the attributes of a given problem. The problem-posing model calls for the application of the following fundamental mathematical processes: proving, reversing, specializing, generalizing, and extending. The given problem turned out to be a rich source of interesting, worthwhile mathematical problems appropriate for secondary mathematics teachers and high school students.

As a student and teacher of mathematics, I was intrigued about the origin of mathematical problems, especially nontrivial problems whose solutions are not obtained by a formula or algorithm, problems that somehow extend the frontiers of our personal mathematical knowledge. When, as a student, I solved nonroutine textbook problems, I thought, not only of devising a plan or a solution, but also of how the textbook authors and mathematicians generated mathematical problems. Their origin remained an enigma to me until later when, as a teacher, my curiosity led me to read *The Art of Problem Posing* (Brown & Walter, 1990). This book provided insights into the origin of mathematical problems and motivated me to examine more closely the relationships among related problems. In their book, Brown and Walter propose the “What-if” strategy as a generic means to modify a given problem to create additional related problems. Because I am a geometry lover, I first applied Brown and Walter’s “What-if” problem-posing strategy to geometric problems. As a result, I developed a problem-posing framework that has guided my students and me to pose mathematical problems *systematically*. This problem-posing framework calls for the application of the following prototypical problem-posing strategies: proof problems, converse problems, special problems, general problems, and extended problems. I often use the Geometer’s Sketchpad (GSP) (Jackiw, 2001) to verify the reasonability of the resulting conjectures.

The main purpose of this article is twofold. First, I model how the framework can be used to generate nonroutine mathematical problems from a given problem and, as a consequence, to discover mathematical patterns and relationships. Second, I discuss some of the difficulties that my students, prospective secondary mathematics teachers, experience when generating mathematical problems. The approach described below reflects the approach that I have followed in class. Because the focus of this article is on problem posing, proofs for most of the resulting theorems are not provided.

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The Problem-Posing Framework in Action: Generating Problems From a Problem Involving Isosceles Triangles and Medians

Many mathematical problems are rich sources of additional related problems. I illustrate the use of the problem-posing framework with the following problem:

What special property do the medians corresponding to the congruent sides of an isosceles triangle have?

Because problem posing and problem solving go hand in hand, it is not only important to pose problems but also to solve them. Before engaging ourselves in generating problems from this given problem, let us solve it. As Figure 1 suggests, the medians corresponding to the congruent sides of an isosceles triangle seem to be congruent. A straightforward proof shows that, indeed, the conjecture is true, and, therefore, it is a mathematical theorem. Of course, we can derive other conclusions: The medians of the congruent sides of an isosceles triangle divide each other in the ratio 2:1, they create congruent triangles, etc.

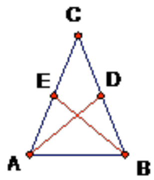


Figure 1. Medians of an isosceles triangle

How can we generate additional mathematical problems from a given problem? Some problems can be modified to generate new problems by applying the following fundamental mathematical processes: proving, reversing, specializing, generalizing, and extending. Such processes are fundamental to mathematics because they are common means of generating or establishing mathematical knowledge. By applying these processes, we generate the following types of problems: proof problems, converse problems, special problems, general problems, and extended problems (Figure 2). Every problem that can be modified to generate related problems is called a base problem.

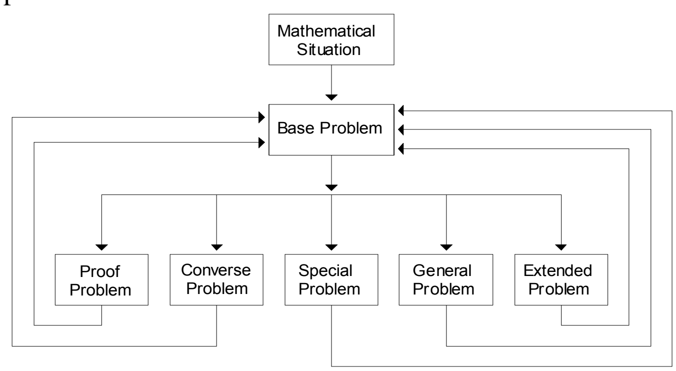


Figure 2. A problem-posing framework

As indicated in Figure 2, in some cases we may face a mathematical situation that does not contain a mathematical problem, e.g., we could have been asked to generate problems based only on the geometric diagram displayed in Figure 1. In these cases, our first task is to formulate a mathematical problem using information contained in the situation.

Posing Proof and Converse Problems

As indicated by the framework, an important type of problem to generate is a proof problem (i.e., a problem asking for a proof). In some cases, a problem that is not written in proof form can be reformulated as a proof problem. Proving is a critical activity of doing mathematics. First, proof is the fundamental mathematical process by which mathematicians establish the validity of their claims, and, as a consequence, the means by which mathematical knowledge is enlarged. Second, a proof allows us to gain insights into why a mathematical theorem holds. Finally, a proof guaranties that our proof problem is well posed in the sense that is solvable. That is, reformulating a problem as a proof problem involves more than changing the syntactic structure of the problem: it implies that we either know that a proof exists or we can develop a proof. Of course, if this is not the case, we can always formulate a problem beginning with the phrase “If possible, prove that….” Our original problem above can be reformulated as a proof problem as follows: Prove that the medians corresponding to the congruent sides of an isosceles triangle are congruent.

We can also reverse a known and an unknown of a problem to generate a converse problem. In other words, we formulate a converse problem when we substitute a known attribute of a base problem by an unknown attribute and vice versa. Why should we consider formulating a converse problem? Generating and investigating the converse of a problem is also a valuable mathematical activity. The converse of a mathematical problem is often a potential avenue for discovering new mathematical relationships, thus expanding mathematical knowledge. While a direct problem allows us to investigate the necessary conditions or properties of a mathematical object, a converse problem allows us to investigate its sufficient conditions or properties. In this way, we gain a more complete characterization of the properties of a mathematical object. Every problem has the potential for generating one or more converse problems. Of course, in many situations the resulting converse conjecture does not hold. In some of these situations we may need to impose additional conditions or restrictions for the converse theorem to hold. In any event, we often broaden our mathematical knowledge by investigating the converse of a problem.

The converse of our medians problem follows: Prove that a triangle with two congruent medians is isosceles or, more specifically, prove that, in a triangle with two congruent medians, the sides corresponding to the medians are congruent. As it is often the case, the proof of the corresponding converse theorem is more challenging than the proof of the original theorem. This case is no exception. A proof follows:

Let $\overline{AD}$ and $\overline{BE}$ be two congruent medians of triangle ABC (Figure 3). Since $\overline{AD}$ and $\overline{BE}$ are medians we have that AE = EC and BD = DC. Euclid’s fifth postulate allows constructing through B the parallel line to $\overline{AD}$. Let F be the point of intersection of this parallel line to $\overline{AD}$ and line $\overleftrightarrow{ED}$. Since D and E are the midpoints of two sides of a triangle, we know that $\overline{EF}$ is parallel to $\overline{AB}$. Therefore, quadrilateral ABFD is a parallelogram. As a consequence, BF = AD. Since AD = BE, we know that BF = BE. Since the opposite angles of a parallelogram are congruent we have that $\angle BAD \cong \angle BFD$. On the other hand, $\angle BFD \cong \angle BEF$ by the isosceles triangle theorem. By the alternate interior angle theorem, $\angle BEF \cong \angle ABE$. Thus, $\angle BAD \cong ABE$. Therefore, $\Delta ABD \cong \Delta BAE$ by the SAS congruence criterion. As a consequence, $AE = BD$, and, hence, $AC = BC$. That is, ΔABC is isosceles.

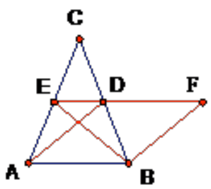


Figure 3. A triangle with two congruent medians

Posing Special Problems

Another potential way to generate mathematical problems is through specialization. We accomplish this by substituting for a mathematical object or attribute of the base problem with a particular example or case of the original mathematical object or attribute. In a special problem we impose additional restrictions on one or more of the attributes. A compelling reason for generating special problems is that, in some situations, specializing a problem allows us to detect implicit relationships among mathematical concepts that are not apparent at first sight. In other cases, specializing a problem permits us to find stronger relationships among the involved problem attributes. By detecting implicit relationships or finding stronger relationships between or among problem attributes, our mathematical knowledge becomes deeper.

Prove that the medians corresponding to the congruent sides of an isosceles triangle are congruent.		
Angle bisectors	Equilateral	Parallelogram
Altitudes	Right	Rhombus
Perpendicular bisectors	Scalene	

Figure 4. Potential changes to the original base problem

How can we generate a special problem? A useful idea is to underline the attributes of the base problem that can be changed, list some possible changes, and examine which attributes have special cases (Figure 4). For this problem, the attribute that has a special case is *isosceles*. Therefore, we can pose a problem involving equilateral triangles because an equilateral triangle is a special case of an isosceles triangle. Our special problem can be formulated as follows: Prove that the medians of an equilateral triangle are congruent (Figure 5).

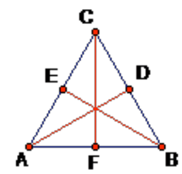


Figure 5. The medians of an equilateral triangle

We can also formulate and prove the converse of the previous problem but I will leave this task to the reader. Of course, other special problems can be generated (e.g., prove that the medians corresponding to the congruent sides of a right isosceles triangle are congruent). The above special problem is exemplary because an isosceles triangle is defined as a triangle with *at least two* congruent sides while an equilateral triangle is defined as a triangle in which *all sides* are congruent. In addition, the corresponding theorem for equilateral triangles reveals additional properties about the medians (i.e., the three medians are congruent) while the corresponding theorem for right isosceles triangles does not reveal any additional or implicit properties about the medians of the congruent sides.

Posing General Problems

Another potential source of mathematical problems is generalization. We create a general problem by substituting a mathematical object or attribute of the base problem with another for which the original is an example. A compelling reason for formulating a general problem is that in some general cases the same relationship holds while in others a weaker or more subtle relationship exists. Of course, in other general cases, there is not a relationship at all. In any case, mathematicians, as searchers of mathematical patterns, want to discover all possible relationships and the conditions under which a relationship exists or does not exist. As we relax our original conditions, we gain a more complete understanding of the properties of mathematical objects.

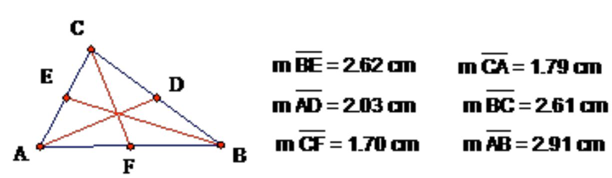


Figure 6. The medians of a triangle

Because I proved that a triangle is isosceles if and only if two medians are congruent, I will pose the general problem in an open-ended format: Is there a relationship among the medians of a triangle? As Figure 6 shows, there is not an apparent relationship among the medians of a scalene triangle. However, by measuring the lengths of the sides of the triangle a more elusive, subtle relationship may be noticed: In any triangle, the longest median corresponds to the shortest side and vice versa. Sure, this relationship involves more than medians, but it is still a valid generalization from the original statement. By specializing this new general relationship for isosceles triangles, we obtain our former relationship involving the congruence of two of the medians of an isosceles triangle as the following argument shows.

Let ABC be an isosceles triangle with $AC = BC$ and medians $\overline{AD}$ and $\overline{BE}$ (Figure 1). If $AD > BE$ then $AC > BC$, which contradicts our hypothesis. A similar contradiction exists if $AD < BE$. Therefore, $AD = BE$.

I should confess that I did not discover this general relationship by myself. I discovered it when I was looking for a proof of a related challenging problem (presented below) that I generated by modifying other attributes of the original problem.

Posing Extended Problems

Another potential source of mathematical problems is extension. We pose an extended problem when we substitute a mathematical object or attribute of a base problem by another similar or analogous mathematical object or attribute. In this case, none of the mathematical objects is a special case of the other. Why should we consider generating extended problems? After all, at this point we might have already formulated a general relationship as well a special one. A compelling reason is that, in some situations, the same relationship exists for an extended case while in others a similar or analogous relationship exists. If the extended situation is a special case of the general case, we may discover a stronger relationship than the general relationship or we may find an implicit relationship. If the extended situation is not a special case of the general case, we may find that the same or a similar relationship exists. In any event, mathematicians, as searchers of mathematical patterns and relationships, want to discover, examine, or characterize all possible relationships that exist between specific mathematical objects. Extension is a common means of enlarging mathematical knowledge.

Since a right triangle is not a special case or a general case of an isosceles triangle, we may say that a right triangle is an extended case of an isosceles triangle. As I reflected on how to pose a problem involving a right triangle, I remembered that the median corresponding to the hypotenuse is half as long as the hypotenuse and that the hypotenuse is the longest side of a right triangle. As a result of this thinking, I generated the following extended problem: Prove that the medians of a right triangle are greater than or equal to half the length of the hypotenuse. As we can notice, the problem takes advantage of properties of both scalene and right triangles, resulting in a stronger relationship for right triangles than for generic scalene triangles.

Posing Further Extended Problems

So far we have posed problems involving special, general, and extended cases of an isosceles triangle. We can continue generating problems by modifying other problem attributes as indicated in Figure 4. To distinguish these new problems from the extended problem generated previously, I call the new generated problems *further extended problems*. Notice, however, that we can distinguish between extended and further extended problems only when (a) one of the problem attributes has special, general, and extended cases, and (b) there is another attribute that can be changed. In the present situation, (a) an isosceles triangle has special cases (e.g., an equilateral triangle), general cases (i.e., a scalene triangle), and extended cases (e.g., a right scalene triangle) and (b) other attributes can be changed (e.g., medians.)

Instead of medians, we can consider related attributes such as altitudes, angle bisectors, and perpendicular bisectors. However, since these geometric figures are lines or rays, they do not have a finite length. To circumvent this obstacle, and at the same time salvage our potential problem, we can consider the length of an altitude as the distance from the corresponding vertex to the opposite (extended) side (e.g., BE in Figure 7) of a triangle. The length of an angle bisector can be defined in a similar way (e.g., AD in Figure 7). The length of a perpendicular bisector, however, is more troublesome because a perpendicular bisector intersects both of the other two (extended) sides of a non-right triangle. To elude this problem, I defined such a length as the distance between the midpoint of the segment and the point of intersection of the perpendicular bisector with the adjacent side or its extension following a clockwise direction (e.g., FG in Figure 7). Notice that for right triangles, the length of the perpendicular bisector of one leg is still infinite.

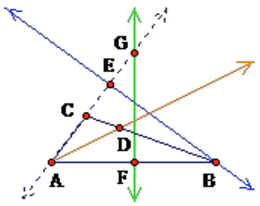


Figure 7. Examples of lengths of altitudes (BE), angle bisectors (AD), and perpendicular bisectors (FG) associated to a triangle.

Using this definition, I was still not able to find any significant relationship between the lengths of the perpendicular bisectors of the congruent sides of an isosceles triangle. However, as suggested by Figure 8, there is, indeed, a relationship involving the perpendicular bisectors of the sides of an isosceles triangle: EG = DF and EI = DH. This discovery led me to redefine the length of the perpendicular bisectors of the congruent sides of an isosceles triangle in two ways as suggested by Figure 8. One way involves defining the length of the perpendicular bisector of a congruent side as the distance between the corresponding midpoint and the point of intersection with the other congruent side or its extension (DF and EG in Figure 8). Another way involves defining such as length as the distance between the corresponding midpoint and the point of intersection with the (extended) base of the triangle (DH and EI in Figure 8). Notice that in each case the length of the perpendicular bisector of one of the congruent sides is measured to the adjacent side clockwise while the length of the perpendicular bisector of the other congruent side in measured to the adjacent side counterclockwise.

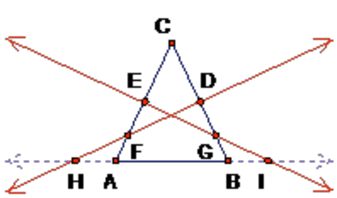


Figure 8. Lengths of the perpendicular bisectors of the congruent sides of an isosceles triangle

As a result of defining the lengths of altitudes, angle bisectors, and perpendicular bisectors as described above, we can pose some further extended problems of our original base problem (Figure 9).

Prove that the altitudes corresponding to the congruent sides of an isosceles triangle are congruent.

If a triangle is isosceles, prove that the lengths of the angles bisectors of the congruent angles are equal.

Prove that the perpendicular bisectors of the congruent sides of an isosceles triangle are congruent (Figure 8).

Figure 9. Some further extended problems of the original base problem

As suggested by the framework, each of the further extended problems displayed in Figure 9 can be taken as a base problem to generate additional converse, special, general, and extended problems. Figure 10 displays some resulting theorems related to triangles and altitudes. Similar problems can also be posed for angle bisectors and perpendicular bisectors. Figure 11 displays a further extended problem and its solution that, with guidance, my students have solved.

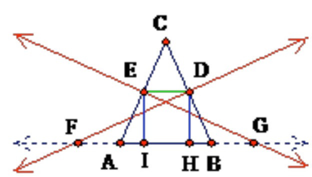
A triangle is isosceles if and only if the altitudes corresponding to the congruent sides are congruent.

A triangle is equilateral if and only the three altitudes are congruent.

In any triangle, the longest altitude corresponds to the shortest side and vice versa.

Figure 10. Theorems related to triangles and altitudes

Problem: Let $\overleftrightarrow{DF}$ and $\overleftrightarrow{EG}$ be the perpendicular bisectors of two sides of a triangle as indicated on the figure. If $DF = EG$, prove that ΔABC is isosceles.



Since $\overline{ED}$ is a mid-segment of ΔABC we know that $\overline{ED}$ is parallel to $\overline{AB}$. Construct $\overline{EI}$ and $\overline{DH}$ perpendicular to $\overline{AB}$ as indicated on the figure. $\Delta EGI \cong \Delta DFH$ by the HL congruence criterion. This implies that $\Delta AEG \cong \Delta BDF$ by the ASA congruence criterion. Therefore, $\angle EAG \cong \angle DBF$, which implies that ΔABC is isosceles.

Figure 11. A problem involving a triangle with congruent perpendicular bisectors and its solution

A challenging problem that I myself was not able to solve is as follows: If a triangle has two congruent angle bisectors, prove that it is isosceles. Because all of my attempts were futile, I proposed this problem to some of my colleagues, one of which is a mathematician, but none could solve it. An internet search revealed that this problem is a classic and challenging problem in Euclidean geometry with an interesting history. This problem is known as the Steiner-Lehmus problem. A short historical sketch can be found in Lewin (1974), and a simple and elegant proof is provided by Coxeter (1969). Coxeter's proof uses the theorem stating that if a triangle has two non-congruent angles, then the greater angle has the shorter angle bisector. This theorem inspired me to formulate the subtle generalization among the medians and sides of a generic triangle mentioned above.

We can continue generating problems by considering the exterior angles of a triangle and defining the length of an angle bisector as the distance from the vertex of the angle to the point of intersection of the exterior angle bisector with the extension of the side opposite that vertex (e.g., AD in Figure 12). One such problem can be formulated as follows: Prove that the angle bisectors of the exterior angles of an isosceles triangle are congruent (Figure 12). Again, we can take this problem as a base problem to generate additional problems. For example, we can formulate its converse as follows: Two exterior angle bisectors of triangle are congruent. Is the triangle necessarily isosceles? Justify your response. I formulated this problem in an open-ended form because, even though I was "sure" that a triangle with two congruent exterior angles is isosceles, I was not able to develop a proof in spite of strenuous efforts. Again, I proposed this problem to some of my colleagues but the solution remained elusive. (I challenge the reader to solve this problem before continuing reading. Hint: Use interactive geometry software!) I was so sure that the triangle is isosceles that I did not attempt immediately to find a counterexample with GSP. After working frantically on this problem for a couple of weeks, I did use GSP and I was amazed for what I discovered: The mystery of the equal exterior angle bisectors problem was revealed before my eyes. As Figure 13 suggests, there are some non-isosceles triangles with congruent exterior angle bisectors. This discovery inspired me to pose the following question that I am still trying to investigate: Is there a necessary and sufficient condition for a triangle to have two congruent exterior angle bisectors? If so, what is it?

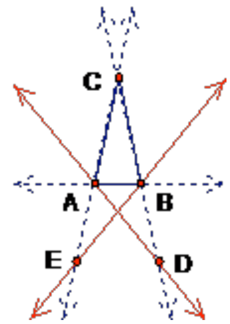


Figure 12. Diagram for the equal exterior angle bisector problem

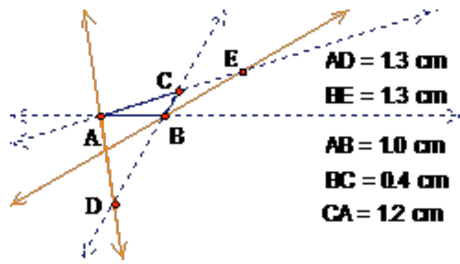


Figure 13. A triangle with two congruent exterior angle bisectors

To continue generating additional related problems we can extend some of the previous ideas to geometric figures other than triangles (e.g., parallelograms, trapezoids, etc.). We can also challenge our definitions for the lengths of medians, altitudes, angle bisectors, etc. For medians, we can consider the distance from the vertex of a triangle to the centroid or to the circumscribed circle along the median (Figure 14). As I did this, I was able to generate additional problems and, as a consequence, I was able to discover and prove more mathematical relationships. Needless to say, I am still trying to solve some of these problems. Without a doubt, generating problems may become interminable when each new problem becomes the source of additional problems.

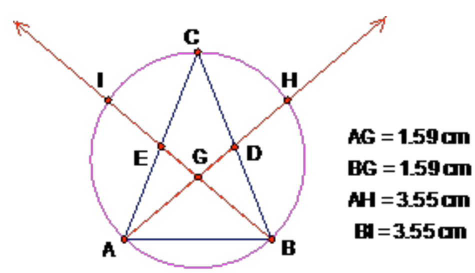


Figure 14. Congruent segments related to medians of an isosceles triangle

Some Prospective Secondary Mathematics Teachers' Thinking and Difficulties When Working on Problem-Posing Tasks

The problem-posing tasks described in this article have been implemented with prospective secondary mathematics teachers enrolled in a college geometry course designed for them. I have implemented the problem-posing tasks using two formats: a student-centered approach and an instructor-centered approach. I use the student-centered approach after the students have had some experience with at least one problem-posing task. In this approach students pose most of the problems using the problem-posing framework as a guide. Performing the problem-posing tasks described in this article usually lasts more than two class periods (about 3 hours). I use the instructor-centered approach when students have not had any experience posing problems. In this case I model how to pose problems using the problem-posing framework as a guide.

Prospective secondary mathematics teachers' abilities to generate problems are underdeveloped (Contreras & Martínez-Cruz, 1999). Their approaches to generate problems tend to be unsystematic, ad-hoc, and nongeneralizable. In addition, their generated problems tend to be trivial and unproductive to pursue (Knuth, 2002). For example, I asked my students to modify the following problem to pose different but related mathematical problems or questions: Prove that the medians corresponding to the congruent sides of an isosceles triangle are congruent. Among the problems posed were: What is a median? How many congruent sides does an isosceles triangle have? How difficult is it to prove?

The problem-posing strategies described in this article are prototypical strategies that can be used *systematically* in a variety of problem situations to generate *worthwhile* mathematical problems. Yet, as I have noticed with my students, without adequate experiences, students rarely use these prototypical strategies to generate problems. Therefore, there seems to be a need to provide students with experiences in generating proof problems, converse problems, special problems, general problems, and extended problems. Prospective secondary mathematics teachers' thinking and difficulties with each of these problem types are elaborated below.

Proof Problems

As stated above, proving is a fundamental mathematical process that permeates mathematical thinking and research. In fact, a proposition for which a proof has not been developed is called a conjecture and not a theorem. Even though proof is a vital part of mathematics, my students are often reluctant to pose proof problems. For example, Contreras and Martínez-Cruz (1999) asked 17 prospective secondary mathematics teachers to pose problems related to each of four given geometric situations. The researchers found that only one student out of 17 generated a proof problem for one geometric situation. Even after instruction, students avoid developing a proof corresponding to a proof problem. In addition, many of them do not use the full power of a proof when adapted to a special case. In other words, they do not establish the truth of special theorems as corollaries of more general theorems.

To illustrate, after my students proved that the medians corresponding to the congruent sides of an isosceles triangle are congruent, most of them did not use this theorem to prove that the three medians of an equilateral triangle are congruent. Instead, they provided a proof from scratch. Only a couple of students used an argument along the following lines:

Since $AC = BC$ (Figure 5) we know that $AD = BE$ because the medians corresponding to the congruent sides of an isosceles triangle are congruent. Applying the same theorem again, we conclude that $BE = CF$ because $AB = AC$. In conclusion, $AD = BE = CF$.

This research and personal experience suggest that students should have extensive experiences posing and solving proof problems.

Converse Problems

From a problem-posing perspective, the critical mathematical process of reversing involves generating a converse problem. Whereas converse problems permeate mathematical thinking and research, it is not natural for students to generate them. For example, Contreras and Martínez-Cruz (1999) found that students generated only one converse problem out of more than 68 potential converse problems. Ideally, each of the 17 students could have generated a converse problem for each of the four geometric situations.

In addition, formulating the converse of a problem is challenging for some students. Some of my students have formulated the converse of the problem "the medians corresponding to the congruent sides of an isosceles triangle are congruent" as follows "If the medians corresponding to the *congruent sides* of a triangle are congruent, prove that the triangle is isosceles." Notice that these students are assuming that the triangle is already isosceles. Examples like this emphasize how critical it is that students have a wide range of experiences posing converse problems of varying difficulty.

Special Problems

Specializing a mathematical problem allows us to investigate implicit or stronger relationships between or among the corresponding problem attributes. Yet, my experience has shown that students rarely consider specializing as a problem-posing strategy. In addition, the way a problem is formulated affects the quality of the generated special problem.

The following problem was posed to a group of secondary mathematics majors enrolled in a geometry class: "If $\overline{AD}$ and $\overline{BE}$ are the medians corresponding to the congruent sides of an isosceles triangle ABC, prove that $\overline{AD} \cong \overline{BE}$." When asked to generate a special problem, *all* students formulated a problem like this: "If $\overline{AD}$ and $\overline{BE}$ are the medians corresponding to the congruent sides of an equilateral triangle ABC, prove that $\overline{AD} \cong \overline{BE}$." While this may be a well-posed problem, it does not prompt us to investigate whether a stronger relationship holds for the special case, which is one of the assets of generating a special problem. Such evidence suggests that students should be given a broad variety of experiences in generating special problems.

General Problems

Despite the importance of generalizing, Contreras and Martínez-Cruz (1999) found that the 17 students only generated 38 general problems out of more than 100 possible general problems. That is, each student had the opportunity to generate at least 6 general problems for the four geometric situations.

In addition, generating well-posed general problems is challenging for some students. While students generating ill-posed problems provides pedagogical opportunities, these problems often reveal that students do not fully understand the connections among the discovered relationships or the structural aspects of the problem. For example, some of my students formulated the following general problem related to the previous problem: Prove that the medians of a triangle are congruent. This ill-posed problem was generated after my students and I established that a triangle with two congruent medians is isosceles. Of course, we can reformulate the ill-posed problem as a well-posed problem as follows: "Does a (generic) triangle have congruent medians? Prove your answer" or "Is there a relationship among the medians of a (generic) triangle? Justify your response." If our students have adequate experiences in posing general problems, they may gain the expertise necessary to overcome their difficulties and, as a result, more frequently pose general problems.

Extended Problems

My experience suggests that students do not often extend mathematical problems. When they do, they sometimes generate ill-posed extended problems. For example, some students have generated the following extended problem: prove that the medians of a right triangle are congruent. Thus, providing students with opportunities to pose extended problems is essential.

Reflection and Conclusion

Mathematicians (e.g., Halmos, 1980; Polya, 1954), mathematics educators (e.g., Brown & Walter, 1990; Freudenthal, 1973), the National Council of Teachers of Mathematics (NCTM, 1989, 2000) and the National Research Council (Kilpatrick, Swafford, & Findell, 2001) consider problem posing as a core element of mathematical proficiency. The *Principles and Standards for School Mathematics* (NCTM, 2000), for example, calls for teachers to regularly ask students to pose interesting problems based on a wide variety of situations. From this, a pedagogical problem arises: How can we teach our prospective secondary mathematics teachers to pose mathematical problems so that they, in turn, can teach their students to pose problems?

As a student of mathematics, I was never given the opportunity to pose problems, let alone interesting problems. Most of the problems that I solved came from the textbook and, on rare occasions, from the teacher. I was certainly content with this situation because I considered posing problems as a creative endeavor beyond my reach.

Calls for teachers to ask students to pose problems (e.g., NCTM, 1989) challenged me to find ways of teaching my students how to pose mathematical problems. Brown and Walter's (1990) *The Art of Problem Posing* motivated me to engage in creating problems and, as a result, I developed the problem-posing framework described here. This problem-posing framework calls for the systematic generation of problems using the following mathematical processes: proving, reversing, specializing, generalizing, and extending. These processes are essential means for discovering new mathematical patterns or relationships.

Posing and solving mathematical problems are worthwhile but challenging activities for prospective teachers. As with any other worthwhile mathematical activity, prospective teachers need to be engaged actively and reflectively in the problem-posing process so they can generate non-trivial, productive, and well-posed mathematical problems. I truly believe that all of us—mathematics educators, teachers, and students—should experience the joy of generating problems and discovering mathematical relationships, even if they are new only to us. In this process, we develop a better appreciation and understanding of the origin of mathematical problems.

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