A Case Study on the Local Organization of Two Mathematically Gifted Seventh-Grade Students

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This study explores the performance of two mathematically gifted Korean 7th-grade students in tasks involving local organization in geometry. The students understood the necessity of definitions and starting points in defining terms and organizing geometrical properties. They improved the clarity of their definitions and arranged the properties systematically with the belief that the properties of the geometric figures would be discussed only after the defining work was completed. Through these activities, they understood an axiomatic method and its importance in geometry. The results suggest that mathematically gifted lower secondary students can be encouraged to advance into axiomatic geometry through local organization activities.

Introduction

In light of the outstanding work of Krutetskii (1976), which synthetically discusses the characteristics of gifted students in mathematics, many studies on the features of mathematically gifted students' thinking processes have been conducted, focusing on problem solving, generalization, justification, and visualization (e.g., Heid, 1983; Lee, 2005; Na, Han, Lee, & Song, 2007; Presmeg, 1986; Ryu, Chong, & Song, 2007; Sriraman, 2003, 2004; Tretter, 2005). This study analyzed the thought patterns of mathematically gifted students as evidenced in tasks involving local organization in geometry.

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In the process of local organization, students have to construct mathematical definitions and organize geometrical properties. Students' construction of mathematical definitions and organization of mathematical properties, along with problem solving, generalization, formalization, and proving, have been highlighted in many studies (De Villiers, 1998; Fawcett, 1995; Freudenthal, 1973; Hanna & Jahnke, 2002; Mariotti & Fischbein, 1997; Ouvrier-Buffet, 2004, 2006). In contrast, few investigations regarding students' local organization process in geometry education have been carried out. In the studies on building definitions (Mariotti & Fischbein, 1997; Ouvrier-Buffet, 2004, 2006) and teaching proof (Blum & Kirsch, 1991; Fischbein, 1982, 1987; Leron, 1982, 1983; Miyazaki, 1991; Movshovitz-Hadar, 1988; Semadeni, 1984), the activities of students in organizing geometrical concepts and properties have been discussed only in part.

Fawcett (1995) explored how students organize geometry locally. In his study, some upper secondary students created elementary definitions and propositions, modified them, and accepted them as assumptions under the guidance of teachers. On the basis of their definitions and assumptions, the students organized proofs of geometrical propositions and developed a theory of geometry.

In contrast, the average lower secondary students experience difficulties in performing tasks related to local organization, which includes the construction of geometrical properties through proofs, due to the lack of the abilities required to prove a geometrical proposition (Usiskin, 1987). Be that as it may, mathematically gifted lower secondary students could be different. Assuming that they might be able to perform local organization, this study was conducted to investigate the locally organizing activities of mathematically gifted seventh-grade students. The research questions were as follows: (1) How do mathematically gifted lower secondary students construct definitions? (2) How do they organize the mathematical properties?

Review of Literature: Local Organization

Freudenthal (1973, p. 458) suggested that students in introductory geometry could be led to learn to organize shapes and phenomena in space by means of geometrical concepts and their properties. At higher levels these concepts and their properties can be organized by means of logical relations. Students at this level may be able to locally organize a mathematizable field of reality, although they may not globally organize the whole scope of geometrical concepts and properties. It would be unreasonable to expect that students at introductory levels can regard geometry as an axiomatic-deductive system. In reality, through the process of local organization, students can experience systematizing theories.

Global organization is the deductive systematization to comprehensively establish an axiomatic system over a field of mathematics. Local organization is organizing to a larger or smaller extent piecewise rather than as a whole from the facts assumed as true. To experience local organization, students need to ask themselves again and again what they presuppose in the process of proof. Without such experiences they cannot understand the meaning of axiomatics (Freudenthal, 1973, p. 151).

According to Hanna and Jahnke (2002, pp. 2–3), organizing locally means working on a "small" theory. By proving geometrical propositions, students can experience establishing a small theory. The proving activities, grounded on students' background knowledge, would be comprised of defining necessary terms, clarifying geometric properties in explicit terms that may serve as assumptions, and writing down the definitions and propositions they made in rigorous terms through a deductive process. These are the defining and organizing activities that this study focuses on.

Although it may lack mathematical rigor to some extent, local organization is essentially a theoretical activity. In his theory of levels of thinking, Van Hiele (1986) divided mathematical thinking into five levels. In the third level, concepts and propositions are the objects of study, and logical relations are the means of organizing them. Local organization, which arranges concepts and properties based on their logical relations, belongs to this level.

Students involved in local organizing tasks are expected to understand the necessity of clarifying definitions and undefined terms and to organize geometrical properties with questions such as, "How can we organize these factors to construct a theory acceptable to other people?," "What are the premises these factors are based upon?," and

"Under what assumptions are these factors true?" The local organization experience could help students to understand the global organization of geometry (Freudenthal, 1973, pp. 150–151).

Methodology

Participants

The students participating in this study were two 13-year-old seventh-grade boys (JH and IS)[1] living in Gyeonggi province, near Seoul, Republic of Korea. For 2 years, they had taken special education courses for gifted students in an academy affiliated with a university. The academy selected their students through written tests and in-depth interviews. The tests and interviews, designed to examine high intellectual abilities—including creativity and task commitment proposed by Renzulli and Reis (1986) as the defining elements of giftedness—were administered by professors in the Department of Mathematics at the university. The participants in this study were at the highest rank in the academy and were identified as highly capable in mathematics with the potential for achieving mathematical excellence (see Gagné, 1991). The academy was running a special mentoring program for the highest ranked students called the Research and Education program. The students formed a research team with an expert, and the two participants in this study were on the same research team. They learned elementary Euclidean geometry prior to this study and were familiar with geometrical proofs, but neither of them had had any experience in local organization.

Data Collection and Analysis

Preinterviews were carried out to investigate the participants' prior knowledge and views about geometry. In the preinterviews, a researcher asked questions such as, "What have you learned in geometry up to now?," "What kind of books have you read about geometry?," and "What do you think geometry means?" Through self-instruction and the help of tutors, the participants had mastered the 10th-grade level of the national mathematics curriculum in Korea. Moreover, the two participants thought geometry was the science for grasping properties of geometric figures.

Every Monday for 3 weeks during October and November of 2006, a member of the research team guided the participants through tasks. Each session lasted approximately 4 hours. The tasks were presented to the participants in a set order: Activity 1-1 and Activity 1-2 for the first week; Activity 1-3, Activity 2-1, and Activity 2-2 for the second week; and Activity 3-1 and Activity 3-2 for the final week. The participants were asked to find the necessary conditions for proving propositions from the previous sessions. It was the responsibility of the teacher/researcher to guide the discussion in such a way that the participants' attention was focused on the tasks and the statements of the participants were clear.

When we began our investigation, we could not find prior research that described students' characteristics in local organization. So, our analysis began informally in the field by identifying characteristics that demonstrate local organization in geometry. We recognized that the participants not only began to understand the necessity of defining geometric terms, they also experienced difficulties in discussing the properties of figures because of their desire to define terms unambiguously. It took a long time for the participants to begin discussing the properties of figures because of their laborious process for defining terms in propositions clearly. Therefore, the participants' local organization activities were classified into two categories: defining geometric terms and organizing geometric properties.

As the data were being collected and transcribed, we focused on identifying those characteristics of the participants that indicated they were defining the terms and discussing the properties of figures more precisely. After the sessions were completed, field notes, video transcriptions, worksheets, and observation records were analyzed. We observed several characteristics of their thought processes that were considered important for local organization activities. Consequently, we considered the following items as the attributes of local organization activities: understanding the necessity of definitions, defining terms in a hierarchical order, understanding the necessity of starting points, pursuing clearer definitions, discussing the properties of figures after defining the terms, drawing out grounding properties, con-

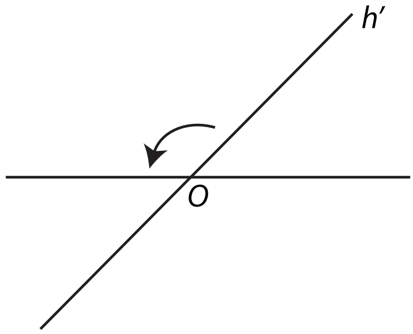

Figure 1. A rotating line around one fixed point.

ceiving the necessity of starting points, and experiencing difficulties recognizing nongeometric assumptions. We grouped the first four properties under the theme student thought processes for defining geometric terms and the second group of four properties under the theme student thought processes in organizing geometric properties.

Tasks

The tasks given to the students were composed of three parts: defining terms (Activity 1-1, 1-2), describing geometrical properties in a specific situation (Activity 1-3, 2-1, 2-2), and writing down the necessary factors for proving a geometrical proposition (Activity 3-1, 3-2). The tasks, developed on the basis of Fawcett's (1995) study, were not part of the gifted education courses or in the national curriculum. All of the tasks were provided to the students in written form. The following are samples of the tasks:

> [Activity 1-1] Today, we are going to build a theory on space. Where shall we start?
>
> [Activity 2-1] Describe all the mathematical properties that you can find in the following situation: The line h' rotates around the point O counterclockwise (see Figure 1).
>
> [Activity 3-1/3-2] We have been doing the "building-a-theory-on-space" task for the past 2 weeks. Can we draw the following conclusions from the theory that we have developed?

- Proposition 1. When parallel lines h and h' are intersected by another line, the sum of the interior angles on the same side is 180°. [Activity 3-1]
- Proposition 2. The sum of the interior angles of a triangle is 180°. [Activity 3-2]

Would we need anything to draw these propositions? If so, what would it be? Is there anything we may need to change in our theory?

Results

Students' Thought Process in Defining Geometric Terms

Four features were observed in the defining activities: understanding the necessity of definitions, defining terms in a hierarchical order, understanding the necessity of starting points, and pursuing clearer definitions.

Understanding the necessity of definitions. For the starting point of the task of building a theory on space, IS suggested, "Because space is the extension of a plane, we should start with a plane." JH, the other participant, also said, "Because space is the continuation of a plane, we should start with a plane." After a discussion on the meaning of the two terms *extension* and *continuation*, they concluded that the two terms must be used differently. Their definitions of those terms were as follows:

> IS: Continuation means laying things in succession, and extension means making something bigger in whatever way possible.
>
> JH: Continuation means carrying on something, and extension means expanding an object to obtain more general or higher dimensional knowledge.

The participants also tried to define the terms *space*, *plane*, *line*, *point*, and so forth. The participants tended to make a definition using more basic terms in the taxonomy they had built.

The teacher began the second session with a request for the students to identify all of the possible relations between two straight lines. JH responded that the term *straight line* had to be defined at the outset of the activity as follows:

> JH: What is a straight line, by the way? What we've defined is only *line*. Straight line hasn't been defined.

They discussed the definition of straight line, and eventually agreed that a straight line referred to "an arrangement of points." IS also argued that the word *arrangement* needed to be defined:

> Teacher: A straight line is an arrangement of points. Is this definition acceptable?
>
> IS: We have to clarify the term "arrangement."

Throughout the sessions, JH and IS put a lot of effort into defining new terms.

Defining terms in a hierarchical order. The two students proposed that building a theory on space must start from a theory of a plane, because they defined space as "an extension of a plane," or a figure formed by moving a plane. The participants were aware that geometric objects are related to each other. Thus, they wanted to define terms hierarchically, and this inclination was sustained in all of the other tasks. Finally, at the end of the session, it was concluded that a "plane is the extension of a line" and a "line is the extension of a point."

> Teacher: Is a plane an extension of a line?
>
> IS: I suppose so. It has to be regarded in that way.
>
> Teacher: Then, what would be the definition of a line?
>
> IS: The extension of a point.

In the second half of the final session, the teacher recommended the participants define *angle*. To comply, JH and IS also hierarchically constructed all of the definitions needed to define angle:

> JH: A figure surrounded by two half lines that start from a point. . . . And a half line is a half of a straight line.

Understanding the necessity of starting points. In the first session, after defining a line, the participants were asked to clarify the new terms that appeared, as in the following dialogue:

> Teacher: What is a point?
>
> JH: A thing that has no size, width, or breadth but only position.
>
> Teacher: Then, what are size, width, breadth, and position?

The teacher continually asked for definitions of new terms, and the participants eventually realized that they needed to stop somewhere, which was the point in this case.

> JH: Doing it this way, we'll continue endlessly, circling around a number of terms.
>
> Teacher: Do you think so? That could be a problem.
>
> JH: So, we have to stop at some terms accepted as commonly understood.
>
> Teacher: Which terms do you want to stop at?
>
> JH: I suggest "point."
>
> IS: I agree. Point may be the start.

The fact that the participants chose *point* as the starting point rather than *size* or *position*, indicates that they considered a point to be the most basic geometric figure that can be a starting point in developing a theory on space. After the above dialogue, the students set up a taxonomy of geometric figures such as point, line, plane, and space.

Pursuing clearer definitions. The participants made efforts to improve the lucidity of their definitions. For example, when the term extension was issued, they comprehensively defined it as "making something bigger in a way" or "expanding an object to obtain more general or higher dimensional knowledge." However, IS was not satisfied with these definitions, saying, "They seem to be valid, but not sufficiently specific." Consequently, the participants tried to explain extension in detail, using the illustration of making a solid by

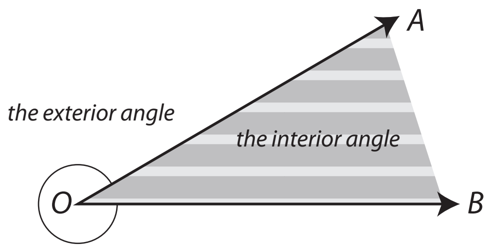

Figure 2. The exterior angle and the interior angle.

revolving a plane figure and making a slant cone through the image of spreading light passing through a tiny hole. In spite of these efforts, such explanations were regarded as limited, failing to encompass all of the meanings of extension in the participants' minds. In the end, they found a problem in their method of explanation: They could not show how a shape like an amoeba was formed since a ray of light is straightforward.

Perplexed with the difficulties, they did not say anything for a while. Then, the teacher encouraged the participants to continue their task of defining extension.

> Teacher: What is your definition of extension?
>
> JH: Raising dimension.

IS agreed to JH's definition. After choosing this definition, the participants decided that *dimension* and *raise* must be taken as starting points.

Their inclination to construct clearer definitions also was shown in defining angle. Initially, they defined it as a figure made with two half lines, but they were not satisfied with this definition because it was not clear whether the definition referred to interior angle or exterior angle (see Figure 2).

> JH: When you say angle AOB, do you mean the interior angle or the exterior angle?
>
> Teacher: Angle AOB?
>
> JH: Yes. Would you define either the interior part or the exterior part of the angle AOB? When people refer to the

angle AOB, which part do you think they mean, the interior or the exterior?

Teacher: (Indicating the interior angle) I always thought it referred to this part.

JH: I thought it naturally referred to that part. Yet, this (exterior) part also exists.

As demonstrated in this dialogue, whenever they felt there was a lack of clarity in their explanation or definition of a term, they were willing to revise it to improve lucidity.

Students' Thought Process in Organizing Geometrical Properties

Four features appeared in students' process of organizing geometric properties: discussing the properties of figures only after defining the terms, drawing out grounding properties, conceiving the necessity of starting points, and experiencing difficulties recognizing nongeometric assumptions.

Discussing the properties of figures only after defining the terms. As the participants understood the necessity of defining terms, they showed an inclination to define the terms even in situations where it was not required. In some cases, the inclination interfered with them organizing geometrical properties. For instance, when the teacher asked the participants to think about the relationship between straight lines, IS pointed out that the term *intersect* needed to be defined before investigation the relationship.

Teacher: Two lines intersect each other unless they meet at a point. Would this be a sufficient answer? Any opinions or questions?

IS: A definition for "intersect" is needed.

The teacher expected the participants to arrange the relationship between two straight lines on a plane, such as (1) coincide, (2) parallel, and (3) intersect at a point. However, IS adhered to defining the new term intersect.

JH also showed such an inclination. Upon the teacher casually saying, "Since a straight angle is 180° for every straight line . . ." JH

Table 1

The Necessary Elements for Proving a Theorem

Necessary assumption	The degrees of vertical angles are the same. When there are two parallel straight lines, the degrees of corresponding angles formed by a straight line that is not parallel to them are the same.
Terms to be defined	parallel, half line, angle, interior angle, corresponding angle, degrees of an angle, the same side, segment, 180°
Undefined terms	straight line, same, rotation, point

interrupted, "We cannot be sure of it yet." He objected to using the nondefined term *180°*. He argued that the definition of 180° must be determined prior to other investigations on straight lines. Similar incidents were observed in overall sessions.

Drawing out grounding properties. The participants were able to expose tacitly assumed grounding properties when discussing whether there were only two kinds of relationships between two straight lines sharing a point. They discovered a tacit assumption, "there is only one straight line that passes through two points."

> Teacher: There are only two cases: Either they intersect at one point or they meet at all the points. Why do you think there exist only two cases?
>
> IS: If they meet at two points, they coincide with each other.
>
> JH: Then we need to explain there is only one straight line that passes through two points.

The participants indicated that to explain the statement, "if two straight lines meet at two points, they coincide with each other," the assumption "there is only one straight line that passes through the two points" was necessary.

In proving the theorem, "the sum of interior angles of a triangle is 180°," the participants identified necessary assumptions, terms to be defined, and terms to be accepted with no definition. The assumptions and terms are listed in Table 1.

> Teacher: What assumptions do you think are necessary?
>
> IS: The degrees of vertical angles are the same.

(The teacher wrote it down.)

IS: Let's assume that there are two parallel straight lines. If another line intersects them, the degrees of corresponding angles formed by the intersecting line and the given parallel lines at the same position are the same.

Conceiving the necessity of starting points. Immediately after Activity 1-3, an assignment looking for the relationships between two lines, was given, JH told the teacher that there seemed to be some properties that could be used without proofs.

JH: I think we need a property that can be used without being defined (murmured some words to himself), such that the internal angle is the same for all straight lines.

Although he used the wrong expression, "a property that can be used without being defined," JH thought that there must be a property serving as a starting point. IS also recognized that there were some properties that could not be proved. He attempted to prove the proposition, "the corresponding angles (the degrees of corresponding angles) formed by two parallel straight lines and a straight line that is not parallel to them are the same" but failed. After thinking on a range of standpoints, IS concluded, "This theorem seems to be unprovable." The expressions the participants used, such as "properties that can be used without being defined" and "unprovable," indicate their sense that there must be some properties serving as starting points that could not be proved.

Experiencing difficulties recognizing nongeometric assumptions. In the discussion about the properties of geometric figures, the participants did not make any attempts to clarify necessary nongeometric assumptions, such as "when equals are subtracted from equals, the results are the same." We expected they might explicitly state this in the last session while extracting necessary assumptions, but they did not. The discussion on the statement, "if equals are subtracted from equals, the results are the same" proceeded as follows:

Teacher: All straight lines have the same interior angle.

JH: Then we can say that vertical angles are the same, which is quite interesting.

Teacher: Can you explain why vertical angles are the same?

JH: Because if an equal portion is subtracted from equals, the results are the same.

JH did not regard the statement, "if an equal portion is subtracted from equals, the results are the same" as an axiom or an assumption, whereas he perceived the necessity of geometric axioms for proving the proposition on vertical angles. It might be difficult for some students to identify general assumptions explicitly while they are involved in the investigation of geometrical properties.

Discussion

The two mathematically gifted lower secondary students who participated in this study were generally successful in creating definitions and organizing geometrical properties. They understood the necessity of definitions and starting points. They defined the terms hierarchically beginning with the starting points, stating them as clearly as they could. Whenever new terms were introduced, they attempted to construct clear definitions for the terms. They successfully found the assumptions necessary to claim a mathematical property and perceived the necessity of starting points for geometric properties.

Many studies on geometry education report students' lack of ability in proving and suggest more intuitive and informal methods for learning proofs (Bell, 1976; Blum & Kirsch, 1991; Fischbein, 1982, 1987; Leron, 1982, 1983; Miyazaki, 1991; Movshovitz-Hadar, 1988; Semadeni, 1984). In these studies, it is reported that secondary school students have difficulties learning axiomatic geometry formally. The results of this study show the plausibility of mathematically gifted students learning axiomatic geometry. Further research is needed on students' achievements in learning axiomatic geometry according to their ability to identify the geometrical activities each group can perform. Local organization can encourage mathematically gifted students to relate the geometrical knowledge they already know, to discover and compare the geometrical properties by themselves, to experience logical and critical thinking, and thereby be prepared to advance into axiomatic geometry.

This study suggests that local organization can provide an environment where definitions are made and improved. Ouvrier-Buffet (2004), discussing the process of constructing mathematical definitions, proposes SDCs (situations of defining process) where zero-definition is constructed and then either rejected or improved by students. Ouvrier-Buffet also suggested that SDCs can enrich and develop students' concepts of definitions in mathematics. Because the participants continually improved the clarity of their definitions, local organization may serve as SDCs to mathematically gifted students.

The participants in this study pursued the characteristics of mathematical definition, distinguished by Zaslavsky and Shir (2005) such as noncontradiction (i.e., all conditions of a definition should coexist) and unambiguouity (i.e., its meaning should be uniquely interpreted; p. 319). They increased the clarity of their definitions to avoid any ambiguities as much as they could. Zaslavsky and Shir also asserted that "being hierarchical" (p. 319) is one of the characteristics of mathematical definition that has to be pursued wherever it is possible. The participants in this study made efforts to define terms hierarchically and perceived the necessity of undefined terms that could serve as starting points.

Another important aspect of local organization is that students can understand the relationships between geometrical properties and basic premises, such as definitions, assumptions, and undefined terms, which are necessary to prove a certain property. The experience of encountering a never-ending cycle of defining new terms, a conundrum that cannot be resolved by logical judgment, made them acknowledge the necessity of undefined terms.

In the preinterviews carried out before the main sessions, JH and IS stated that geometry is something like "solving a problem by drawing auxiliary lines" and "solving it by drawing figures." These statements indicate that these students viewed geometry as being characterized by using drawings to solve problems. But the experience of local organization changed their conceptions of geometry. At the end of the last session, the teacher asked what they would do if they had to keep on making theories on space. They replied, "There are tons of things to do." The teacher told them about the work of mathematicians such as Hilbert (1862–1943) who constructed geometry systematically. To this JH said, "Oh, it must have taken decades." It seems that the students construed the huge scope of geometry and the difficulties inherent in organizing an axiomatic geometry.

After experiencing the process of making clear definitions, the students claimed that discussing the properties of geometric figures is possible only after completing the defining work. It is notable that their adherence to the defining work interfered with their activity of organizing properties. Further research is needed to explore how the interference between defining terms and organizing properties can be minimized.

Conclusion

This study investigated the plausibility of lower secondary students' learning of theoretical geometry through local organization activities. The mathematically gifted seventh-grade participants were able to organize geometric concepts and properties. Four features were found in their defining work: understanding the necessity of definitions, defining terms in a hierarchical order, understanding the necessity of starting points, and pursuing clearer definitions. In their organizing of geometrical properties, four other features were observed: discussing the properties of figures only after defining the terms, drawing out grounding properties, conceiving the necessity of starting points, and experiencing difficulties recognizing nongeometric assumptions.

The findings of this study indicate that local organization can serve as an intermediary for mathematically gifted lower secondary students learning a theoretical axiomatic geometry. Through local organization activities, the participants experienced the significance of theoretical inquiries of geometry and the fundamentals of axiomatic geometry. One limitation of this study is that it was conducted with two top students. Thus, further research on mathematically gifted lower secondary students' performances in local organization is required.

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End Notes

1 All names are pseudonyms.