Where is the rate in the rule?

Sandra Herbert
University of Ballarat
<s.herbert@ballarat.edu.au>

Introduction

A well-developed understanding of rate is foundational to conceptual understanding of introductory calculus. Many students achieve procedural competence with the application of rules for differentiation without developing an awareness of the connection between derivative and rate. In addition, rate-related reasoning is needed to make informed decisions in many everyday applications of rate. This paper reports on additional data collected during interviews for a project investigating the different ways rate may be experienced by pre-calculus students.

Many researchers (for example Kaput, 1999) have suggested that the conceptual understanding of function may be enhanced through the presentation and exploration of multiple representations of a variety of functions. In this paper, one section of each interview is considered in detail to evaluate the participants' understanding in a specific rate context. Participants were asked to discuss a dynamic geometry simulation of a blind on two different windows, one rectangular and the other not. Detailed analysis of the video-record of each participant's interview provides insights into their perceptions of rate in several different representations.

In the sections below, the conceptual framework is described; details of the interviews and the computer-based simulation are provided; and the analysis of the data is discussed.

Rate

This section presents the Victorian Essential Learning Standards (VELS) expectations with respect to the concept of rate; the importance of rate as a foundational concept for calculus; and the connection between the concepts of rate and proportion.

In Victorian schools rate is usually studied in Year 8 (VCAA, 2005), in conjunction with proportion and percentages, and appears in text books for

that level (see for example Bull et al., 2003). In Bull et al. (2003), a typical Year 8 text used in Victorian schools, rate is described as "measure of how one quantity changes with respect to another"(p. 138) and arithmetic calculations of average rate, such as growth rate of a tree, are included in the exercises for students to complete. Later, an extensive study of linear functions is undertaken (VCAA, 2005) where linear functions are explored in mathematical representations of tables, graphs and algebraic rules. Connections are made between these representations and they are used to solve linear equations. In addition, by the end of Year 10, students will also have undertaken a study of other functions where rate varies at some stage. So, it may be expected that the participants of this project will be familiar with functions and their multiple representations and also have had some experience with rate. However, the word 'rate' may not have been heard in mathematics classes for some time and, perhaps, never connected to the notion of gradient.

Textbooks for older students usually present more formal definitions for rate. For example, Stewart (2002, p. 146) defines the "average rate of change of y with respect to x over the interval $[x_1,x_2]$ [to be the difference quotient]

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\text{''}.$$

This difference quotient is the underpinning of the formal definition of derivative

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

where such a limit exists (Stewart, 2002, p. 150).

Confrey and Smith (1994) suggest a multiplicative view of rate, encouraged by an exploration of exponential functions, as an entry point to a covariational approach to functions. They consider a covariational approach to be more effective than the conventional correspondence approach. They propose that a narrow, abstract view of function "ignores the richness and viability in the student conceptions" (p. 137). This project seeks to reveal the participants' conceptions of rate with respect to the algebraic rule, in order for teachers to support the development of a covariational view of functions and build a conceptual understanding of derivative.

The initial steps to a covariational view of function are taken when rate and proportion are first presented to students. The concept of proportion is closely related to constant rate and may be viewed as a first experience of rate. It may be that early experiences of proportion inhibit the transfer of the concept of constant rate in the calculation to constant rate in graphs and rules. Indeed, many researchers have reported difficulties that students have with the concept of proportion (Tourniaire & Pulos, 1985; Carlson et al., 2002), so if proportion is a stepping stone to constant rate then it is not surprising that students also find rate a difficult concept to master.

Researchers investigating calculus students' understanding of rate consider it to be a significant foundation for a conceptual understanding of derivative (Tall, 1985, 1990; White & Mitchelmore, 1996; Hauger, 1997). Many calculus students achieve procedural competence in the algebraic manipulation of the definition of derivative given above, and can accurately find the algebraic representation of the derivative. However, some students

may be unaware of its connection to the earlier study of rate specified in VELS.

Students' awareness of the interconnectedness of mathematics may be facilitated by explicitly connecting new concepts to their understanding of earlier concepts (Hiebert & Carpenter, 1992), which, for differentiation, means explicitly connecting derivative to rate. So a teacher of calculus needs to be informed about students' prior experience of rate. Indeed, Ramsden (1988) advocates that teachers need to be aware of the understanding that students already have and emphasises "the importance of helping students change their ways of thinking and understanding." Many calculus textbooks (see for example Stewart, 2002) appear to regard speed as suitable foundation for an understanding of derivative. However, this reliance on speed may inhibit the transfer of the concept of derivative to non-motion contexts. This project investigates pre-calculus students' thinking about rate, in a non-motion context, through the detailed analysis of the video-record of one-to-one interviews.

The next section discusses the reasons for choosing dynamic geometry software, with its ready access to multiple representations, as a vehicle for fostering participants' discussions of rate.

Multiple representations

It has been advocated (Kaput, 1999; Borba, 1994; Kendal & Stacey, 2000) that the use of multiple representations: enables students to view and explore mathematical concepts in a variety of ways; and emphasises that there is not necessarily only one mathematical process which results in the successful solution of a problem. It is suggested that understandings developed in one representation support understandings in other representations and, together, the representations provide a more complete concept image (Tall & Vinner, 1981). In particular, the concepts of function and rate can be expressed in several different representations, such as diagrams and animations (Nemirovsky & Tierney, 2001). In addition, these concepts may also be expressed in the standard mathematical representations of tables, graphs and rules.

The scenarios, of blinds covering two windows of different shapes, were simulated using the dynamic geometry software, Geometers' Sketchpad (GSP). The simulations provided a visual starter for the interviews (Lorentzson & Trell, 1999) in order to tap into participants' existing conceptions about windows, tables, graphs and rules, thus connecting their existing understandings of rate with the scenarios. The scenarios provided the participants with concrete examples of rate to facilitate discussion. GSP was particularly well-suited for this purpose as it provides easy access to the mathematical representations of table, graph and rule, so participants' conceptions of rate could be explored more broadly.

The next section describes this use of GSP in more detail and outlines the method data collection.

Method

The participants of this project were twenty-five Year 10 students from five different secondary schools. They were interviewed about a variety of different aspects related to the concept of rate. These students were selected by their teachers to represent a diversity of mathematical aptitude and their willingness to discuss the rate-related animations. A Geometers' Sketchpad (GSP) file simulating two windows with blinds (Figure 1) was prepared. For both windows, the blinds could be raised by dragging point P.

Two different windows are included in the GSP simulation. One window was entirely rectangular to facilitate discussion of the participants' understanding of constant rate, whilst the other window constituted three different sections enabling the exploration of both constant and variable rate.

The main focus of the interview was to explore the participants' understanding of the change in area with the change in height the blind was raised. In addition, participants' understanding of rate was also explored using associated tables, graphs and rules. In particular, this paper reports on participants' responses with respect to the rules.

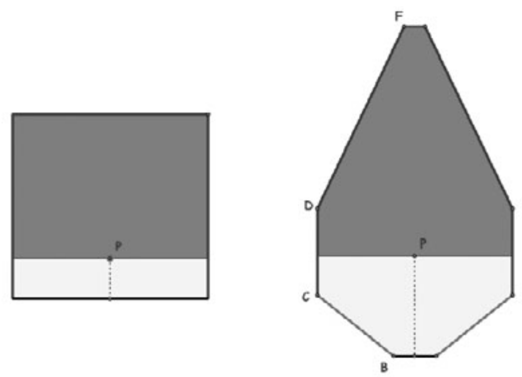

Figure 1. GSP simulation of windows.

The simulation was used to stimulate discussion of the rates involved and provide insights into the participants' understanding of constant and variable rate. The rectangular window afforded a focus on constant rate whilst the non-rectangular window facilitated discussion of the differences between constant and variable rate. So, the GSP simulation assisted in providing insights into the participants' understanding of constant and variable rate, in multiple representations.

Each interview was video-recorded in order to capture as much information about the participants' understanding of rate as possible (see Herbert & Pierce, 2007). A particular line of discussion was continued until the participant had nothing else to say about it (Akerlind, 2005). The audio-record of the interviews was transcribed to enable easy access to sections of interest in the videos. Participants were encouraged to explain their reasoning and think aloud as they were presented with different representational forms of rate: the simulation, table of values, graph and rule.

Findings and discussion

Analysis of the video evidence showed that the table provided participants information about constant rate and even some understanding of variable rate. The graph was often used to determine co-ordinate points along the line to calculate constant rate, whilst only three participants could discuss variable rate from this representation. Only one participant referred to the gradient of the line and its relationship to rate. While many of the participants were able to communicate ideas about rate in the form of tables and graphs, no participant could link the rule to rate. The following quotes, in response to the question, "What does the rule tell you about the rate?" show the rule's lack of rate-related meaning, even for the linear function, A=6.4h, associated with the rectangular window.

R1: Um, I don't know.
R3: Not sure.
R10: Nothing.
R14: I don't know, I don't understand.

Other students responded to this same question by translating the rule representation into words. The following quotes illustrate this translation.

R2: It would be 6.4 times the height from the bottom.
R12: The area of sunlight is 6.4 times the height.
R16: Area it's 6.4 times the height.

Some students went further by describing the process of substitution of values of one variable into the rule to calculate the corresponding values of the other rate-related variable.

R6: Um, the, for every area of sunlight It's 6.4 of the H, so for example 1.5 the height from the bottom times it 6.4 and gives how much area of sunlight area there is. I don't know it's [the rule] just kind of representing that, every time you lift it up you find the area of the thing, of sunlight increases by 6.4 times the height, so it's just a matter of the mathematical formula at the end of how to work out how to do that.
R11: Um, that whatever amount H is A would be six point four times the amount of H.

Other responses to the question "What does the rule tell you about the rate?" demonstrated confusion between rate and the changes in the variables. For example:

R5: That the rate of them, the height gets higher as the area of sunlight does.
R7: Because it gets higher, when there's less blind, the rate of sunlight is higher.

R13: It changes with the height of the blind.

R15: The area measurement is, each time the height goes up by 6.4, the area increases?

So the rule did not convey to students anything about rate in the simulated real-world situation of a blind on a rectangular window. This is surprising as it is likely that the participants' prior mathematical background would have included an extensive study of linear functions.

When considering the non-rectangular window, participants demonstrated even less connection between the rule and their understanding of rate. This is not really surprising as piece-wise functions may be difficult to comprehend. However, it was expected that the rate for the constant section, at least, may have been discerned. One participant gave a translation of the symbols into words.

I: What do you think these rules have to do with the rate that the area's getting bigger?

R11: It's, umm, 9.5 height squared plus 11 height minus 16.3. But it, well for every 0.5 meters squared you add 11 and minus 16.3.

All other participants did not express any understanding of the connection between the rate and the rule.

I: What do the rules tell you about the rate that the area is changing?

R3: I don't know.

I: What do the rules tell you about the rate the area of sunlight is changing?

R4: Not much.

The following exchange illustrates that the connection between the rule and the shape of the window has not been observed. It indicates that the participant considers each rule to be connected to a different window, rather than the three different sections of the window shown in the simulation.

I: Why do you think there are three rules ?

R3: Three different ways of explaining, oh maybe needing one rule for a wider window and one for a narrower window and one for a window that's a normal size.

The following exchanges indicate awareness that the three different rules are connected to the three different sections of the window.

I: Okay, let's look at the rules, what do the rules tell you about the rate that the area is changing?

R6: Um, so, so the area of sunlight equals, umm, okay well it's got three different rules for three sections and, a fourth section, umm, well I don't really understand the rule, um m its saying maybe the height of the blind, as it grows, I'm not really sure.

However, no interview reflected any understanding of the connection between the rules and the rate the area is changing. The following exchange demonstrates that this participant has discerned that each rule is related to one of the sections of the window, but the discussion indicates that the participant is reflecting on the change in area rather than the rate.

I: So what does the rule tell you about the rate that the area is changing?
R10: Um well it's, I dunno, it's different, different parts of it show. Is that right? Is it?
I: What do you think?
I10: I reckon, three different formulas. If it was all the same it would be one. But there's three, because of the shapes like, three times.
I: If we look at the first part what can you tell me about the rate that the area is changing in that first section?
R10: Well it goes from a little bit then it gets, it gets more as you go higher because it's increasing that way.
I: What about the rate that the middle section?
R10: Well it's the same as the window before because it's just a rectangle, and the other part — it becomes smaller.

These findings indicate that the participants did not move seamlessly between representations and that understandings demonstrated in one representation do not necessarily transfer to other representations. This is consistent with Amit and Fried's (2005) report that questions whether the potential of multiple representations, to enhance students' understanding of functions, is realised in the classroom. It appears that the participants of this project do not transfer their understanding of rate demonstrated in tables and graphs to the corresponding algebraic rule.

I: What would the rules tell you about the rate that the area is changing?
T: Probably not as much as the graph would.

Conclusion

In this project, dynamic geometry, with its ready access to multiple representations, proved to be an effective stimulus for discussion about rate and assisted in revealing participants' understanding of rate.

Findings reveal that rate was not seen, by the participants, in the algebraic form of the functions resulting from the GSP simulation. This suggests that explicit connections between tables, graphs and rules are required to enable students to transfer understandings of rate from one representation to another since the ability to move seamlessly between representations is dependant on extensive experience with each representation individually and, in particular, movement between them. In addition, care should be taken to ensure that the context of problem-solving be experientially real for students to avoid excessive cognitive demand. It may be that the cognitive

demand required of the participants to make sense of the non-rectangular window was too high, thus concealing the connection between the sections of the window and the rules, and hence the rates.

Specifically, the findings show that algebraic rules hold almost no rate-related meaning for the participants of this project. This is of particular concern to the teachers of introductory calculus, especially if such an introduction relies heavily on the algebraic form of functions. So, approaches to the teaching and learning of functions need to include explicit, frequent and strong connections between multiple representations of functions with an emphasis on rate in order to build a conceptual bridge between functions and associated derivatives and integrals.

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