The Function concept in middle-years mathematics

Mal Shield
Queensland University of Technology
<m.shield@qut.edu.au>

The concept of function is now included in most mathematics curricula, usually starting in the early years. Study of the concept begins as students work with number patterns and sequences. It is intended that function ideas will be formalised gradually as the students move through the curriculum. For example, the *Statements of Learning for Mathematics* (Curriculum Corporation, 2006) has students in year 3 "recognise and describe simple relationships… determine and describe rules that apply and continue them" (p. 6) while students in year 9 "use words and symbols to represent variables and constants when writing expressions for algebraic relations and functions" (p. 15) as well as drawing graphs and solving equations.

Much has been written over the years about ways of making algebra more meaningful to learners. Kaput (1999) provided a useful summary of changes needed in algebra teaching and learning.

- begin early (in part, by building on students' informal knowledge),
- integrate the learning of algebra with the learning of other subject matter (by extending and applying mathematical knowledge),
- include several different forms of algebraic thinking (by applying mathematical knowledge),
- build on students' naturally occurring linguistic and cognitive powers (encouraging them at the same time to reflect on what they learn and to articulate what they know), and
- encourage active learning (and the construction of relationships) that puts a premium on sense making and understanding. (p. 134)

As well as addressing pedagogical issues, Kaput (1999) described five interrelated forms of algebraic reasoning. Two of these forms focus on the generalisation of patterns and the formalisation of structures derived from working with numbers. For example, students learn to recognise that $3 \times 4 = 4 \times 3$ and that $4 + 4 + 4 = 3 \times 4$. Such generalisations lead to the third form, "algebra as syntactically guided manipulations of (opaque) formalisms" (p. 139). This is the traditional secondary school notion of algebra as the manipulation of symbolic expressions and equations following rules derived from the study of arithmetic. The final two forms of algebraic reasoning described by Kaput involve the study of relations and functions and their use in modelling situations and phenomena. Kaput notes that “many would argue that modeling of situations is the primary reason for studying algebra” (p. 149).

In this article, I focus on possible ways to develop the concepts of joint variation and function through the upper primary and lower secondary years of education. The examples demonstrate the building of a range of important mathematical ideas by modelling life-related situations using students’ informal and intuitive knowledge. The aim is to gradually formalise the ideas over several years. In many mathematics programs, relationships between variables have been introduced by having students explore number patterns, often derived from geometric patterns such as those created with toothpicks. These patterns are usually easy to generalise recursively (rule based on the previous term) but are often more difficult for students to describe with a generalised position rule, thus limiting their value for developing the function concept.

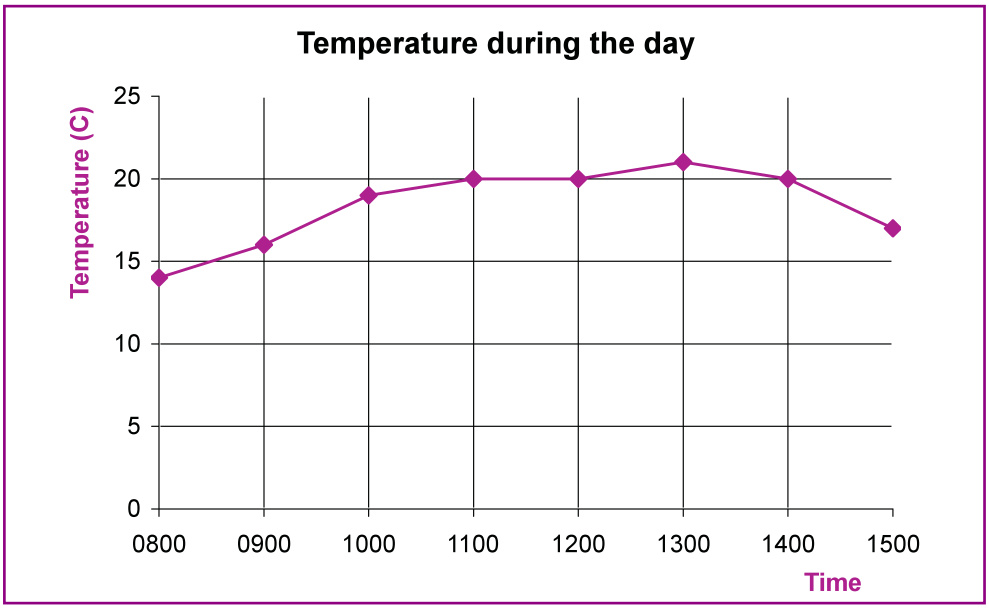
The life-related situations

Three situations are examined in detail, with a wide range of ideas illustrated. Each of the example situations is amenable to use at a range of school levels. In practice, the ideas highlighted, the language used and the formalism adopted depends on the current development of the target students. Similar situations can be used later in the program to continue the development of new mathematical ideas.

Example 1: Temperature and time of day

Students gather data on the air temperature every hour during the school day. This is a regular type of measurement and data activity in common use. I am suggesting that such an activity can be used to introduce modelling and function ideas in line with Kaput’s (1999) suggestion to integrate algebra learning with the learning of other subject matter. Students would represent the data in a table and with a graph.

Time	0800	0900	1000	1100	1200	1300	1400	1500
Temperature (C)	14	16	19	20	20	21	20	17



Typically, students will join the points on this graph and can be asked what that means in terms of the context. Discussion can raise the idea that even though the temperature was only measured each hour, there must have been a temperature at every instant between measurements. We also know from the context that the change would most likely have been reasonably uniform between measurements. Students are being asked to think about *continuity* and the formal terms can be introduced when the teacher feels it is appropriate. From a modelling perspective, students need to recognise that they are making assumptions about the behaviour of the variable temperature between the known measurements.

Students should be asked to describe and explain what the data and graph show about the temperature. A statement like this will result: "During the morning the temperature rose, was highest at 1300, and fell after that". By talking and writing about their observations, students begin to use the language of change and relationship. Highlighting the idea that temperature changes as the time of day changes assists the development of the concept of a *relationship between variables* and the natural language can lead to the ideas of *dependent* and *independent* variables. The temperature depends on the time of day chosen to measure it. Questions can also be asked about when the temperature was going up the fastest, when it was not going up, and when it was coming down. Informal associations can be made between rate and gradient. It is important that students at all levels work with some functions that are not uniform and that cannot be represented with a symbolic equation.

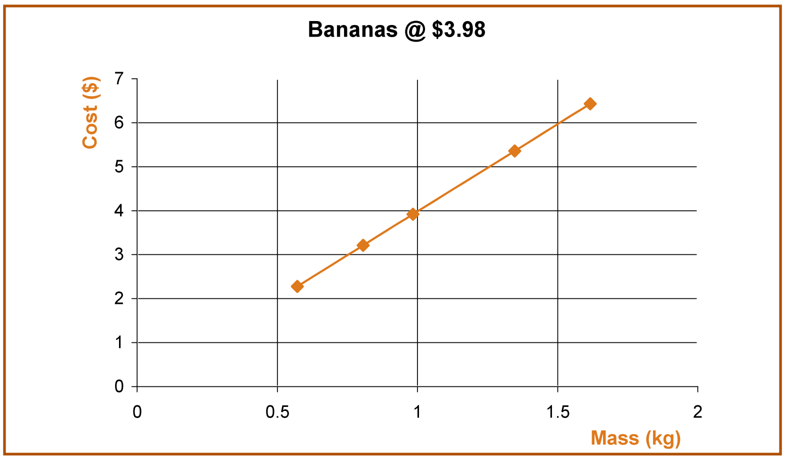
Example 2: Buying quantities at a set price

Money situations provide rich contexts for developing many mathematical ideas. In terms of the function concept, relationships between quantity and cost can be explored, beginning with cases involving simple numbers such as $2 per kilogram. The following situation involves realistic prices and quantities.

> Bananas are currently priced at $3.98 /kg. Students are asked to write down 5 typical masses that might be bought at the supermarket and calculate the costs.

This situation involves measurement ideas and is based on the idea of rate. Students should think about the sorts of masses and the numbers that are seen on the computer scales. They need to calculate with decimal numbers using a calculator and round the answers appropriately. The data from the calculations can be represented in a table and a graph.

Mass (kg)	Cost ($)
0.572	2.28
0.806	3.21
0.984	3.92
1.347	5.36
1.616	6.43



Again, justification of the joining of the points with a continuous line can be a subject of discussion. In this situation the discussion can be extended to include consideration of whether the line can be continued down to the origin. Students can speculate on the mass of the smallest possible banana. The mathematical idea of *domain*, the possible values of the independent variable, is being considered in an informal way when developing the model. The nature of the variables can also be explored: the buyer chooses the bunch of bananas (mass, the independent variable) and the cost depends on the mass chosen.

This is a constant rate situation which results in a linear function. Students can generalise the calculations they made to find the costs for various masses and eventually express this as a symbolic equation.

$$\text{Cost} = \text{mass} \times 3.98$$
$$C = m \times 3.98$$
$$C = 3.98m$$

Equation solving can be viewed as cases in which the value of one variable is known and the unknown value of the other variable is calculated. Solutions can be found graphically and from an equation. For example, if 1.254 kg are being bought, the cost can be found from the equation $C = 3.98 \times 1.254$. The question of the maximum mass of bananas that could be bought for \$5 can be answered by solving the equation $5 = 3.98m$.

The situation can be further extended by raising the issue that fruit prices change from time to time. Students can calculate values and draw the graph for another price such as \$4.98 per kg.

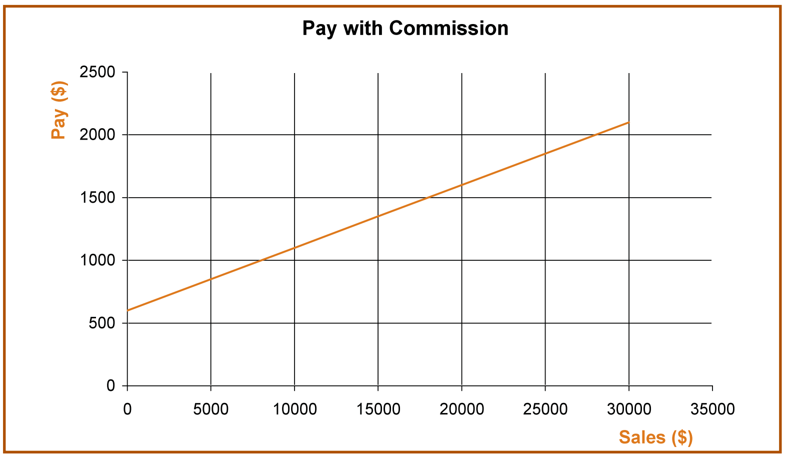
The important link between rate and gradient can be built and students can observe the association between the graphical representation and the symbolic equation, comparing $C = 3.98m$ with $C = 4.98m$.

Example 3

In lower secondary mathematics, again when working with money, students solve problems involving commission.

> A sales person is paid \$600 per week and receives 5% commission on their total sales for the week. How much would they be paid for a week with sales of \$22 346? How much would they need to sell in a week to earn \$1000?

A problem like this can be extended by developing a mathematical model for the situation. If necessary, students can calculate a table of values to draw the graph.



The mathematical ideas developed with simple cost-quantity linear functions can be consolidated and extended with this context. The original problem about particular cases of pay and sales can be answered by reading from the graph. Accurate answers can be achieved if appropriate technology is used. The functional model for the problem can also be represented by the equation $P = 600 + 0.05s$, with equation-solving strategies being further developed. The model can also be explored in terms of changing the parameters, the retainer ($600) and the commission (5%), exploring the ideas of y-intercept and gradient.

Conclusion

The three examples demonstrate ways in which many important mathematical ideas related to the study of algebra can be developed from students' intuitive understanding of life-related situations. The learning of algebra is integrated with other areas of the mathematics curriculum and could be linked with investigations of scientific phenomena such as distance–time relationships. The emphasis is on students making sense of situations and using the models to generalise and predict. Over several years, ideas such as dependency, continuity, gradient and function can be gradually formalised. Students are modelling life-related situations with functional relationships and representing the relationships in a variety of forms — tables, graphs, equations — and in everyday language. As students work with more complex functions, the need for syntactic manipulations of terms in symbolic equations arises and can be introduced in the context of the situation being modelled. Students can also be introduced to the use of technology in the forms of graphics calculators, CAS calculators and spreadsheets as they work with functional models.

References

Curriculum Corporation (2006). *Statements of Learning for Mathematics*. Retrieved 14 February 2007 from http://www.curriculum.edu.au/ccsite/default.asp?id=17706

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema & T. A. Romberg (Eds), *Mathematics Classrooms that Promote Understanding* (pp. 133–155). Mahwah, NJ: Lawrence Erlbaum.

From Helen Prochazka's Scrapbook

Let no one who is not a mathematician read my works.
Leonardo da Vinci (15th century)

Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.
W. S. Anglin in "Mathematics: A Concise History and Philosophy" (1994)