MAGIC SQUARES

from primary classroom to postgraduate research in 10 simple exercises

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invites students and their teachers to engage with the mathematics of magic squares.

It is not often that one can introduce primary school students to a problem at the forefront of mathematics research, and have any expectation of understanding; but with magic squares, one can do exactly that. Magic squares are an ideal tool for the effective illustration of many mathematical concepts. This paper assumes little prior knowledge on the part of the student except for addition and multiplication, reflection and rotation; but, using questions and exercises throughout, finishes by posing a problem tough enough to test postgraduate mathematics students, to which no-one has yet managed to find a solution.

Who knows what a magic square is?

Some students may know: a square consisting of numbers, where the total of each row and each column is the same. Close! In fact, to be truly regarded as a magic square, each of the diagonals should sum to that same total, too. If they do not, the square is often called semi-magic.

All of the numbers used are generally required to be different. If this is not enforced, then we could have a magic square in which every entry was 1, for example — which tends to be rather boring (see Figure 1).

1	1	1
1	1	1
1	1	1

Figure 1. A very boring 3 by 3 magic square.

This article will show how magic squares can be used to illustrate many mathematical concepts.

4	3	8
9	5	1
2	7	6

Figure 2. A genuine 3 by 3 magic square.

8	3	4
1	5	9
6	7	2

Figure 3. The same square rotated through 90 degrees.

8	1	6
3	5	7
4	9	2

Figure 4. The same square reflected in the vertical axis.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Figure 5. A 4 by 4 magic square.

8	6	16
18	10	2
4	14	12

Figure 6. The 3 by 3 magic square, doubled.

Magic squares

Exercise 1. Draw a 3 by 3 grid. Without any clues, see if students can fill in the numbers 1 to 9 so that the resulting square is magic.

Some students will be able to find a solution. If not, you can give an additional hint: put the number 5 in the middle. Figure 2 shows one solution.

It should be noted that every row, and every column, and even the two main diagonals, sums to the same total; in this case, 15. This is called the magic constant.

Exercise 2. Rotate the square through 90 degrees. Is it still magic?

Exercise 3. Reflect the square vertically or horizontally. Is it still magic?

The answer to both questions is: yes. Figure 3 shows the same square rotated through 90 degrees and Figure 4 shows it again reflected about a vertical axis.

Exercise 4. Is it possible to construct another 3 by 3 magic square using the numbers 1 through 9 where the magic constant is any other value than 15?

No: since all of the numbers 1 through 9 sum to 45, each row and each column must be 45 ÷ 3, or 15. It has also been shown that 5 must always go in the centre square.

Larger magic squares are possible. A 4 by 4 magic square is illustrated in Figure 5. Of course, all rows and columns, as well as both diagonals, sums to the magic constant, which is the sum of the numbers 1 through 16, divided by 4, which equals 34.

Arithmetic and 3 by 3 squares

Looking at bigger squares can be very interesting, but for our purposes, let us concentrate on 3 by 3 squares.

Exercise 5. Try doubling every entry in the square. Is the square still magic?

Yes. Of course, the magic constant is doubled too, because we are no longer using the numbers 1 to 9, but rather, 2 to 18 (see Figure 6).

Exercise 6. Add a number of your choice — say 25 — to every entry. Is the square still magic?

Yes again. The magic constant is whatever it was before, plus three times the number we have just added (see Figure 7).

29	28	33
34	30	26
27	32	31

Figure 7. The 3 by 3 magic square, with 25 added.

Exercise 7. Take our original square (using the numbers 1 to 9) and square each entry; that is, multiply it by itself. Is the square still magic?

No. Unfortunately we end up with a square which is not magic at all! (see Figures 8 and 9).

So, we can multiply every entry, or add a particular number to each entry, and in both cases the square remains magic; but we cannot multiply each entry by itself!

4^2	3^2	8^2
9^2	5^2	1^2
2^2	7^2	6^2

Figure 8. Squaring all the numbers.

16	9	64
81	25	1
4	49	36

Figure 9. Not quite magic!

Prime 3 by 3 squares

A prime number is a number divisible only by itself and 1. So 17 and 19 are both primes, but 21 is not (since $21 = 3 \times 7$). Can we make a magic square using only prime numbers? The answer is yes!

Exercise 8. Provide the students with the nine numbers 1, 7, 13, 31, 37, 43, 61, 67, and 73. Can these numbers be arranged into a magic square?

The key to arranging the numbers correctly in any magic square is to realise that the middle number (in this case, 37) must always go in the centre (see Figure 10).

Although 1 may seem to conform to the rules of being a prime number, mathematicians restrict the term prime number to numbers greater than 1. Even if we do not allow the use of 1, we can still find a square consisting entirely of primes, though the numbers are a little larger (see Figure 11).

67	1	43
13	37	61
31	73	7

Figure 10. A square of primes, including 1.

Exercise 9. Provide the students with the nine numbers 5, 17, 29, 47, 59, 71, 89, 101, and 113. Can these numbers be arranged into a magic square?

Every row, column, and diagonal sums to 177.

17	89	71
113	59	5
47	29	101

Figure 11. A square of all primes.

Magic 3 by 3 square of squares

We have done nothing very advanced so far — certainly nothing that would test a mathematician! Yet we can already pose the puzzle to which no-one — and I do mean *no-one* — has ever — and I do mean *ever* — found an answer!

We have already seen that imposing what would appear to be a very severe restriction on the entries in our square — that they all be prime numbers — really does not cause much of a problem at all. The puzzle that is so tough is very similar: is it possible to have a 3 by 3 magic square in which all of the entries are square numbers, like 16, 25, etc.?

13	10	25
28	16	4
7	22	19

Figure 12. A magic square with three square entries.

373^2	289^2	565^2
360721	425^2	23^2
205^2	527^2	222121

Figure 13: a magic square with seven square entries

Exercise 10. Starting with our usual magic square, and knowing that you can add the same number to each entry, and the square remains magic, and you can multiply each entry by any number, and it still remains magic, can you construct a magic square with more than three square numbers in it?

For example, Figure 12 is a magic square with three squares (4, 16, and 25):

Made from our original square (Figure 2) by multiplying each entry by 3, and adding 1. Notice that this does not improve on our original square that had three square numbers too (1, 4, and 9)!

For interest, Figure 13 is the record so far: a magic square with seven square entries. The entries are very large!

If you (or your students) can beat this: congratulations! The task has beaten everyone so far; but a word of warning: any magic square with even eight square numbers will have entries that are very, very large. Computers have been used extensively in the search for such a square, and been found wanting.

Just in case you thought that magic squares were just amusing diversions and had no relevance to any other areas of mathematics, this very problem (construction of a 3 by 3 magic square all of whose entries are squares) has been shown to be highly relevant to problems in various domains, including arithmetic progressions, Pythagorean triangles, and elliptic curves.

Further reading

Boyer, C. (2006). *Magic Square of Squares*. Accessed at http://www.multimagie.com/indexengl.htm

Pickover, C. (2002). *The Zen of Magic Squares, Circles, and Stars: An Exhibition of Surprising Structures across Dimensions.* Princeton University Press

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Weisstein, E. W. (2002). *Magic Squares*. Accessed at http://mathworld.wolfram.com/MagicSquare.html

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