Polya's legacy: Fully forgotten or getting a new perspective in theory and practice?

Tim Passmore
University of Southern Queensland
<passmore@usq.edu.au>

Problem solving and student-centred learning have received a great deal of attention in mathematics curricula for schools and in some universities. Much of this emphasis developed from the pioneering work of George Polya in heuristics, problem solving and mathematics education. Polya's work, and some of its later extensions, are reviewed in the light of current research findings. It is argued that, despite the attention it has received, problem solving remains a difficult skill, both to teach and to learn, and directions for future work are suggested.

Introduction

"The aim of mathematics in the high school curriculum should be to teach young people to think." So said the researcher, author and teacher George Polya who continued: "Such thinking may be identified… in first approximation, with problem solving," (Polya, 1965, p. 100). He already had a distinguished career in mathematical research when he said this. For example, he had published in analysis (Hardy et al., 1934; Polya & Szego, 1925), combinatorics (Polya, 1937) and mathematical physics (Polya & Szego, 1951). The book that made him world famous was How To Solve It (Polya, 1945) which is still in print sixty years after its publication; it has sold over a million copies and been translated into seventeen languages. This work and those which followed it established Polya as the father of the modern focus on problem solving in mathematics education:

"For mathematics education and for the world of problem solving [Polya's work] marked a line of demarcation between two eras, problem solving before and after Polya," (Schoenfeld, 1987a, p. 27).

He spent the last forty years of his life writing and teaching on the means and methods of problem solving, which he called heuristics. This article seeks to examine his legacy. What have we learned from his methods and are they still relevant to teaching mathematics today? How should we extend his ideas in the light of more recent research?

According to Polya,

> Mathematical thinking is not purely "formal"; it is not concerned only with axioms, definitions and strict proofs, but many other things belong to it: generalising from observed cases, inductive arguments, arguments from analogy, recognising a mathematical concept in, or extracting it from a concrete situation. The mathematics teacher has an excellent opportunity to acquaint his students with these highly informal thought processes… stated incompletely but concisely: let us teach proving by all means, but let us also teach guessing. (Polya, 1965, p. 100)

When the teacher proposes a problem to be solved, he should begin with letting the students guess the result. The students, impatient to know whether their guess will turn out right or not, will work afterwards with much more interest. Guessing hones both judgement and intuition, which are indispensable tools in constructing proofs and in research (Polya, 1979).

I urge the reader who is already familiar with these ideas to consider again how radical they seemed when Polya first proposed them. At the time, drill and practice on basic skills were the norm in elementary mathematics teaching. Instead, Polya said, the student should be plunged into situations which might require beginning from scratch, with perhaps a guess or some sort of analogy and argue towards a solution with false starts, blind alleys, mistakes and failures, till the problem became clearer and a solution emerged. In so doing, the student would experience mathematical thinking which is similar in kind, if not degree, to that of mathematical research.

Polya's legacy

To begin with some of Polya's concerns have a surprisingly modern ring to them. For example, underlying his thinking was the distinction between the way mathematics is formally presented and the very different way in which it is actually done — a key argument in the "Math Wars" curriculum debate in the United States in the last five years (Latterell, 2005). Secondly, he was passionately concerned about teacher education, giving many problem-solving courses to both pre-service and in-service teachers. His opinion that "the preparation of high school mathematics teachers is insufficient" (Polya, 1965, p. xi) is echoed by many today who are concerned with the quality of mathematics teaching in our schools. The recent National Strategic Review of Mathematical Sciences Research in Australia (Australian Academy of Science, 2006), for example, highlighted a decline both in the number of Year 12 students taking intermediate or advanced mathematics, and in the mathematical content knowledge of graduating pre-service teachers. Thirdly, Polya recommended that in tertiary mathematics courses for non-specialists, like engineers, different standards of proof should apply than those expected of professional mathematicians, so that proof by generic example could be considered adequate (Polya, 1954b, p.159). In this he anticipated more

recent research, to be discussed below, on the nature of proof and how students understand proof.

Most mathematics teachers today will have heard of the four steps to problem solving recommended in *How to Solve It*:

1. understand the problem;
2. devise a plan;
3. carry out the plan;
4. look back;

and perhaps a few of the heuristic strategies: Draw a Diagram, Work Backwards, or Find a Pattern. However, few teachers will have experience of problem solving in the way Polya intended. His method was as follows: the student should be presented with sequences of "worthwhile and interesting problems" (Polya, 1965, p. xvi) and "not merely routine problems but problems requiring some degree of independence, judgement, originality, creativity" (ibid, p. xi). These problems would form the basis of a Socratic dialogue with the student, by which the teacher would guide the student to explore various heuristic stratagems leading to a solution.

How to Solve It was devoted to Polya's exposition of heuristics with some problems. More interesting and worthwhile problems were published subsequently in two volumes of Mathematics and Plausible Reasoning (Polya, 1954a, 1954b), Volume I focusing on induction, analogy, generalisation and specialisation, and Volume II on patterns, the use of heuristics to solve problems, and the ways in which mathematicians become convinced that something is true before they try to prove it.

A good example of reasoning by analogy comes from Polya's account, in Volume I, of Euler's discovery of the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=\frac{\pi^2}{6}$$

the exact value of the sum being hitherto unknown. Euler had already shown by numerical techniques that the sum was 1.644934, to 7 significant figures, but he wanted the exact answer. He began with the equation sin $x = 0$, which has roots $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi\ldots$ Expanding the left-hand side as a Maclaurin series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=0$$

and dividing both sides by x, which would remove the root $x = 0$, Euler concluded that the equation

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots=0 \qquad (1)$$

must have roots $x = \pm\pi, \pm 2\pi, \pm 3\pi\ldots$ He then argued by analogy with the finite-dimensional case of a polynomial equation of the form

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n=0$$

If this equation has n distinct roots $\alpha_1, \alpha_2, \ldots \alpha_n$, then the left-hand side can be factorised as $a_n(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$. Euler factorised the left-hand side of equation (1) to be

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots$$

This was a risky step, as the left-hand side is an infinite series rather than a polynomial of degree n, and the convergence of infinite products like that on the right-hand side was a still open question in his day. Equating the coefficients of x^2 on both sides yields

$$\frac{1}{3!}=\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

"Euler knew very well that his conclusion was daring," writes Polya (ibid, p. 20), yet he had some reassurance because it agreed to the last place with his numerical estimate. Moreover, using this result and equating coefficients implies that

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

a result which also agreed exactly with Euler's numerical estimates. Finally, by applying a similar technique to the equation $1 - \sin x = 0$, Euler replicated a known result due to Leibniz that

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Still Euler kept doubting, continuing his numerical validations, looking at other cases, till eventually he constructed a different proof for the result which could be accepted as rigorous. The point, as far as Polya was concerned, was that Euler was prepared to be completely open about his process of discovery. "[A]mong old mathematicians, I was most influenced by Euler and mostly because Euler did something that no other great mathematician of his stature did. He explained how he found his results and I was deeply interested in that. It has to do with my interest in problem solving," (Alexanderson, 1979, p. 16).

Lastly, in *Mathematical Discovery* (Polya, 1965) he honed his ideas and addressed how mathematics teachers should approach their craft. Polya's exposition of heuristics was highly structured and the teacher, by choosing an appropriate sequence of worthwhile and interesting problems, could help the student systematically gain confidence in solving a variety of problems. Of course, the teacher must first have gained similar skills in order to teach them effectively.

Problems should be designed to deserve the attention of students; should entail recognising an essential mathematical concept in a concrete situation; should require formulation and exploration; may require students to guess an essential point of the solution, or use other heuristics which Polya described (ibid., volume II, pp. 105–112).

Criticism of Polya's heuristics

Schoenfeld (1987a) pointed out that despite the fact that Polya's books were enthusiastically received by mathematicians and mathematics educators (e.g., Suydam, 1987) who could recognise in them strategies that they themselves had learned to use but had never been taught, the results of heuristic training have proved disappointing. He was critical that, despite some serious effort, much of the problem-solving movement has been superficial — adding a few trivial word problems to the curriculum, or studying a few easy problem-solving techniques in isolation.

> …there was at best marginal evidence (if any) of improved problem-solving performance. Despite all the enthusiasm for the approach, there was no clear evidence that the students had actually learned more as a result of their heuristic instruction or that they had learned any general problem-solving skills that transferred to novel situations. (Schoenfeld, 1987a, p. 41)

The reason, Schoenfeld said, was not just poor implementation but the heuristic strategies themselves. They are too general and when any given strategy is applied to a problem its precise application depends on context: it needs to be applied in different ways in different situations. Lester (1994), summarising research findings till then, agreed that teaching students *about* problem solving strategies and heuristics (such as Polya's) does little to improve problem-solving ability in general. The learning takes place in doing, and it seems that problem-solving ability develops slowly over a prolonged period of time. In Polya's defence however, it is clear he always intended his heuristics to be combined with doing carefully constructed sequences of interesting problems.

Another difficulty comes from cognitive research (Desoete et al., 2003; Ginsburg-Block & Fantuzzo, 1998; Pape & Smith, 2002) which shows that, as well as training in the strategies and tactics of problem-solving, students need training in command and control — in self-regulation and mental resource allocation during the problem-solving process. For example, a hallmark of good problem solvers is that they do not get lost forever in pursuing their wrong guesses. Schoenfeld (1987b) showed that these metacognitive skills could be taught and successfully incorporated into problem-solving training. Lester (ibid) warned, however, that to be effective, metacognitive instruction should be given in the context of learning particular concepts and techniques. It is not readily generalised or transferred.

Extensions of Polya's work

In their book *Thinking Mathematically*, Mason, Burton and Stacey (1985) have taken into account these criticisms of Polya's work while acknowledging his and others' inspiration. The four steps have been expanded to seven phases, to try to make the identification of each phase more useful: getting started,

getting involved, mulling, keeping going, insight, being sceptical and contemplating. There is less emphasis on describing particular heuristic strategies and more emphasis on the metacognitive skills of goal-setting and monitoring and assessing the problem-solving process. Still present are the worthwhile and interesting problems and the thrill of engaging with them, as is the notion that solution will entail false starts, getting stuck and flashes of inspiration or intuition. They also include the idea of different levels of proof, that Polya suggested (Polya, 1954a, 1954b), here pithily reduced to: *convince yourself*, *convince a friend*, and *convince an enemy* — the first being the easiest (possibly) and the last the toughest.

Comparing two examples from Polya and Mason et al. will highlight differences of approach. Polya twice discusses the problem of finding a formula for the sum of squares of consecutive integers, $S = 1^2 + 2^2 + \ldots + n^2$. Having first dealt with the simpler case and established the result that

$$1+2+\ldots+n=\frac{n(n+1)}{2}$$

in Polya (1954a, p. 108), calculating both these sums for $n = 1, 2, \ldots 6$ and observing their ratio, he is led to conjecture that

$$\frac{1^2+2^2+\ldots+n^2}{1+2+\ldots+n}=\frac{2n+1}{3}$$

and, using the known result for the denominator, to a formula for S which can be proved by mathematical induction. In Polya (1965, Vol. 1, p. 62), discussing the same problem in the context of recursion, Polya begins with the identity $(k + 1)^3 - k^3 = 3k^2 + 3k + 1$ and sums it over values $k = 1, 2, \ldots n$. The left-hand side collapses as a telescopic sum, and on the right, use the result from the simpler case to obtain

$$(n+1)^3-1=3S+3\frac{n(n+1)}{2}+n$$

which yields the required formula after some algebraic rearrangement. Polya comments: "I shall be highly pleased with the reader who is displeased with the foregoing solution provided that he gives the right reason for his displeasure." Namely that the result appears *out of the blue*, like a rabbit pulled out of a hat. Proofs such as this have been shown to be less meaningful to students than those whose structure and reasoning is made explicit (Alibert & Thomas, 1991; Tall, 1979).

Compare the approach taken by Mason et al. in solving the problem of finding which numbers can be written as sums of consecutive positive integers, for example,

$$\begin{aligned} 9 &= 2 + 3 + 4 \\ 11 &= 5 + 6 \\ 18 &= 3 + 4 + 5 + 6 \end{aligned}$$

The authors open their discussion with general advice:

- Try lots of examples.
- Try changing the question, extending its scope in some way.
- Be systematic in your specialising and try several different systems.
- Look for patterns.

In the getting-started phase they say, “Begin by specialising. Two systematic approaches come to mind. Either take each number in turn and try to express it as a sum of consecutive numbers or be systematic and take sets of two, then three, then four consecutive numbers and find the sums.” The reader is then led through a series of conjectures, some true, some false, leading to subsidiary questions and arguments that require some algebra. The structure of the reasoning is emphasised and the connections between all conjectures explicitly mapped. Eventually, the conclusion is reached that powers of 2 cannot be written as sums of consecutive positive integers. Heuristic and metacognitive strategies are intermingled in the discussion and they also emphasise journaling or rubric writing as a way of recording progress.

The way ahead

It seems that problem solving is here to stay as far as primary and secondary school mathematics are concerned. It has been stressed in both the 1989 and 2000 recommendations of the NCTM (NCTM, 1989; 2000) and in the latter, more emphasis has been placed on teaching metacognitive skills and self-regulation (Desoete et al., 2003) consistent with research findings. (Interestingly, the NCTM process standard for problem solving requires that students grapple with “complex problems that involve a significant amount of effort.” What became of Polya’s “worthwhile and interesting” problems?)

In Australia the attempt to forge national curriculum guidelines foundered in 1993 with the collapse of Commonwealth–State talks in the Australian Education Council. However, the Curriculum Corporation survives as “an independent education support organisation owned by all Australian education ministers established to assist education systems in improving student learning outcomes. We do this in collaboration with education systems, responding to agreed national directions.” (Curriculum Corporation, 2007) A check of the mathematics curriculum guidelines in all States and Territories confirms problem solving is a key skill throughout Australia in Primary and Secondary mathematics. It is likely that, should recent attempts to restart discussions on a national curriculum be successful (ABC, 2007; Blair, 2007; Reid, 2005), the focus on problem solving will be maintained.

Polya’s ideas have been slower to impact undergraduate mathematics teaching. In the 1960s this was due to an emphasis in curricula on formal proof, axiomatic presentations of elementary algebra and the precise formulation of mathematics as a deductive system (Hanna, 1991). More recent views of mathematics, she says, have acknowledged the realities of mathematical practice, that proofs can have different degrees of formal validity and still

gain the same degree of acceptance. The emergence of problem solving seems to have mirrored what happened in schools: at first a somewhat uncritical enthusiasm — as part of calculus reform (Cohen et al., 1994) for example — followed by more sober reflection. Epp complains that:

> Enthusiasm for this more human view of mathematical thinking has led some to relegate proof to… an… often intuitive, somewhat pedantic justification for statements already known to be true… The view that intuitive understanding is separate from and precedes proof is sometimes given as a reason for presenting mathematics informally [at first], leaving proof to… senior courses.
> (Epp, 1994, p. 257)

Hanna (ibid, p. 61) concurs that formalism should not be seen as a side issue, but as an important tool for clarification, validation and understanding. One approach that has been used successfully at undergraduate level is group discussions of proofs, following a paradigm similar to Mason et al. (Alibert & Thomas, 1991).

Vithal, Christiansen and Skovsmose (1995) report on extensive Danish experience with problem-centred project work at tertiary level both in instruction and assessment. They conclude that project work in mathematics is feasible, if it is combined with lecture courses which support content, and if it is given appropriate weighting in assessment. However, it is expensive and labour intensive and, for both students and staff, there is a real tension between developing an overview of the discipline and delving into project topics in depth.

To summarise what has been learned about problem solving since Polya, heuristic training is valuable if it is done in the context of particular problem-solving domains rather than in general. It should be combined with explicit metacognitive training to encourage self-regulation and monitoring of the problem-solving process. It appears that the ability to solve problems develops slowly over a very long period of time and success depends on much more than just mathematical content knowledge, though expert problem solvers have sound schematic and semantic knowledge compared to novices.

Since heuristics are domain specific rather than general, future research has quite a way to go to specify them. Lester (ibid) laments that research interest in problem solving appears to be declining, partly because some believe the work has been done, but also because it is now clear that problem solving is more complex than was once thought. He identifies three areas that need more research: the role of the teacher in a problem-centred classroom, what actually takes place in a problem-centred classroom, and group and whole-class processes in a problem-centred classroom.

At university, the student must cope with more new concepts in less time than at school, leading to a greater concentration on a smaller number of inter-related topics, different from anything in earlier years, and increasing generalisation, abstraction and formalisation (Robert & Schwarzenberger, 1991, p. 128). More research is needed on: learning theories for advanced mathematical thinking; balancing the deep-learning, narrow-focus opportu-

nities afforded in problem-centred instruction with the acquisition of broad, discipline-wide, schematic and semantic knowledge, and the role of student choice and learning style in all of the above.

Problem solving will need to be informed by all this research before we can teach it better. Well-structured encounters with interesting and worthwhile problems which encourage self-reflection and monitoring are still the most effective means available to the teacher.

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