Convolution of two series

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Abstract

In this note, we introduce and discuss convolutions of two series. The idea is simple and can be introduced to higher secondary school classes, and has the potential of providing a good background for the well known convolution of function.

Introduction

Undergraduate mathematics/engineering/science students usually come to know the term "convolution" in the second or third year at the university. This is in mathematics courses containing topics such as Laplace transforms, where an integral of convolution of two functions is evaluated for some specific functions (HREF1). Students get to know more of the power and application of convolution as they advance their studies in various fields such as Physics, Computer Science, Statistics and Engineering. As noted in HREF2, convolution and related operations are found in many applications of engineering and mathematics. Some cited examples in the site are:

- In statistics, a weighted moving average is a convolution. The probability distribution of the sum of two random variables is the convolution of each of their distributions.
- In optics, many kinds of "blur" are described by convolutions. A shadow (e.g., the shadow on the table when you hold your hand between the table and a light source) is the convolution of the shape of the light source that is casting the shadow and the object whose shadow is being cast. An out-of-focus photograph is the convolution of the sharp image with the blur circle formed by the iris diaphragm.
- In acoustics, an echo is the convolution of the original sound with a function representing the various objects that are reflecting it.
- In electrical engineering and other disciplines, the output (response) of a (stationary, or time- or space-invariant) linear system is the convo-

lution of the input (excitation) with the system's response to an impulse or Dirac's delta function.
- In time-resolved fluorescence spectroscopy, the excitation signal can be treated as a chain of delta pulses, and the measured fluorescence is sum of exponential decays from each delta pulse.
- In physics, wherever there is a linear system with a "superposition" principle, a convolution operation makes an appearance.

However, we feel that "convolution," if restricted to series instead of functions could be introduced at higher secondary school or "A level" without any difficulty. Our aim is to show how this could be achieved. This will not only help students to anchor the concept firmly, but will also serve as a bridge between the lower and the higher concept.

2. Arithmetic progression series

The concept of sequences and series are well known and usually introduced in the higher secondary schools or A level syllabus. The most common use of this is in the arithmetic or geometric progressions. We intend to show that after introducing the concept of sequence and series, a teacher can introduce convolution of these series. Students' interest can be stimulated by indicating that convolution has far reaching application in higher mathematics, sciences and engineering.

In general, a sequence (finite or infinite) is called an *arithmetic progression* if the difference of any term from its preceding term is constant. This constant is usually denoted by d and is called the common difference. The sum of the first n terms in an arithmetic progression is sometimes called an arithmetic series. The formula for finding the nth term of arithmetic progression, and the sum of the first n terms of Arithmetic series are available. An often-told story is that Gauss discovered this formula when his teacher asked him to find the sum of the first 100 numbers, and instantly computed the answer, 5050. It is worth noting here that Gauss used a kind of convolution to get this answer.

Now to introduce convolution of the arithmetic series, let

$$S_1 = a_1 + a_2 + a_3 + \dots + a_n$$

and

$$S_2 = b_1 + b_2 + b_3 + \dots + b_n.$$

Then the *convolution* series of S_1 and S_2 is denoted by and defined as

$$C = C(S_1, S_2) = \sum_{i=1}^{n} a_i b_{n+1-i} = a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \dots + a_{n-1} b_2 + a_n b_1 \quad (2.1)$$

If $a_i = b_i$ (for all i) then $S_1 \equiv S_2$ and in this case we shall refer to C as the *self-convolution* of S_1. The following results can be easily deduced using induction. Also, they can be given to students as challenging exercises. The first result is well known while the second result about the self-convolution of the (first n) natural numbers is not so well known.

Result 2.1 (Recall) $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

Result 2.2 $$\sum_{i=1}^{n} i(n+1-i) = \binom{n+2}{3}$$

Proof. The proof is by induction. The induction step is:

$$\begin{aligned}\sum_{i=1}^{n+1} i(n+2-i) &= \sum_{i=1}^{n+1} i(n+1-i) + \sum_{i=1}^{n+1} i \\ &= \sum_{i=1}^{n} i(n+1-i) + \sum_{i=1}^{n+1} i \\ &= \binom{n+2}{3} + \binom{n+2}{2} \\ &= \binom{n+3}{3}\end{aligned}$$

Result 2.3. Let

$$S_1 = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d)$$
$$S_2 = a' + (a'+d') + (a'+2d') + \ldots + (a'+(n'-1)d')$$

be two arithmetic progression series. Then the sum of their convolution series is:

$$C = naa' + (ad' + a'd)\binom{n}{2} + dd'\binom{n}{3}$$

Proof. By equation 2.1 we see that:

$$C = a(a'+(n-1)d') + (a+d)(a'+(n-2)d') + (a+2d)(a'+(n-3)d') + \\ \ldots + (a+(n-2)d)(a'+d') + (a+(n-1)d)a'$$

By expanding the brackets in the above expression, we have:

$$C = (aa' + a(n-1)d') + (aa' + a(n-2)d' + a'd + d(n-2)d') \\ +(a+2d)(a'+(n-3)d') + \ldots + (a+(n-2)d)(a'+d') + (a+(n-1)d)a'$$

Now by collecting like terms, the expression can be rearranged to:

$$C = aa'n + \left(ad'\left((n-1)+(n-2)+\ldots+2+1\right) + a'd\left(1+2+\ldots+(n-2)+(n-1)\right)\right) \\ + dd'\left(1.(n-2)+2.(n-3)+3.(n-4)+\ldots+(n-3).2+(n-2).1\right)$$

It is now not difficult to see how the above expression is reduced to

$$C = aa'n + (ad' + a'd)\binom{n}{2} + dd'\binom{n}{3}$$

In addition, if $a = a' = 1 = d = d'$, then using similar argument, students can prove Result 2.2:

$$C = \sum_{i=1}^{n} i(n+1-i) = \binom{n+2}{3}$$

3. Geometric progression series

The geometric progression is a sequence of numbers such that the quotient of any two successive members of the sequence is a constant. This constant is usually referred to as the common ratio of the sequence. As in the arithmetic progression, formula for the nth term and the sum of the first n terms of a geometric progression are known. Here, we intend to introduce the convolution of two geometric series. For this, let

$$S_1 = a + ar + ar^2 + \ldots + ar^{n-1} \text{ and } S_2 = b + bt + bt^2 + \ldots + bt^{n-1}$$

be two geometric progression series. Then the convolution of the two series is given as:

$$\begin{aligned} C &= a\left(bt^{n-1}\right)+\left(ar\right)\left(bt^{n-2}\right)+\left(ar^2\right)\left(bt^{n-3}\right)+\ldots+\left(ar^{n-2}\right)\left(bt\right)+\left(ar^{n-1}\right)b \\ &= abt^{n-1}+abrt^{n-2}+abr^2t^{n-3}+\ldots+abr^{n-2}t+abr^{n-1} \end{aligned} \tag{3.1}$$

With $t \neq 0$ (which is mostly the case), equation 3.1 can be transformed to:

$$\frac{r}{t}C = abrt^{n-2}+abr^2t^{n-3}+abr^3t^{n-4}+\ldots+abr^{n-1}+ab\frac{r^n}{t} \tag{3.2}$$

Now subtracting (3.2) from (3.1) gives:

$$\left(1-\frac{r}{t}\right)C = abt^{n-1}-ab\frac{r^n}{t} \Leftrightarrow \left(\frac{t-r}{t}\right)C_n = ab\left(t^{n-1}-\frac{r^n}{t}\right) \Leftrightarrow C_n = \frac{ab}{t-r}\left(t^n-r^n\right),\left(r \neq t\right)$$

If $r = t$ then from equation 3.2 we deduce that:

$$C = abr^{n-1} + abr^{n-1} + abr^{n-1} + \ldots + abr^{n-1} = nabr^{n-1}$$

Thus we have proved the following result:

Result 3.1. Let $S1 = a + ar + ar^2 + \ldots + ar^{n-1}$, $S_2 = b + bt + bt^2 + \ldots + bt^{n-1}$ be two geometric progression series. Then

(1) $$C = \frac{ab}{t-r}\left(t^n-r^n\right) \qquad \text{(if } r \neq t\text{);}$$

(2) $$C = nabr^{n-1} \qquad \text{(if } r \neq t\text{).}$$

4. Sum of consecutive powers of integers

Let
$$S\left(n,m\right) = \sum_{r=0}^{n} r^m = 1^m + 2^m + 3^m + \ldots + n^m \tag{4.1}$$

be the sum of the mth powers of the first n natural numbers. This series has attracted considerable attention in the last few decades (e.g., see Mackiw, 2000 and its references). Many higher secondary or "A level" mathematics texts (e.g., Blackey, 1960) devote a section outlining an iterative procedure on how to compute closed formulae for $S(n,m)$. In particular, the cases $m = 1, 2, 3, 4$ are usually covered. However, its self-convolution series has not attracted similar attention. We intend to deduce some results in this direction. We note that the convolution

$$
\begin{aligned}
C = C(m,n) &= \sum_{r=1}^{n} r^{m}\left((n+1)-r\right)^{m} \\
&= \sum_{r=1}^{n}\ \sum_{k=0}^{m} (-1)^{m} (n+1)^{m-k} \binom{m}{k} r^{m+k} \\
&= \sum_{k=0}^{m} (-1)^{k} (n+1)^{m-k} \binom{m}{k} \sum_{r=1}^{n} r^{m+k} \\
&= \sum_{k=0}^{m} (-1)^{k} (n+1)^{m-k} \binom{m}{k} S(n,m+k) \qquad (4.2)
\end{aligned}
$$

The following results (summarised in Proposition 4.2 below) can be proved by induction or directly from (4.2) and Lemma 4.1. The results in Lemma 4.1 can be obtained from the iterative procedure mentioned in Comtet (1974, p. 155).

Lemma 4.1.

(a) $$S(n,2) = \frac{1}{3} n(n+1)\left(n+\frac{1}{2}\right)$$

(b) $$S(n,3) = \frac{1}{4} n^{2}(n+1)^{2}$$

(c) $$S(n,4) = \frac{1}{5} n(n+1)\left(n+\frac{1}{2}\right)\left(n^{2}+n-\frac{1}{3}\right)$$

(d) $$S(n,5) = \frac{1}{6} n^{2}(n+1)^{2}\left(n^{2}+n-\frac{1}{2}\right)$$

Proposition 4.2. Let *S(n,m)* be as defined in (4.1) and let *C(n, m)* be the self-convolution series of *S(n,m)*. Then

1. $$C(n,1) = \binom{n+2}{3}$$

2. $$C(n,2) = \binom{n+2}{3}\left(\frac{n(n+2)=2}{5}\right)$$

Some selected values of these sequences are presented in the table below

n	1	2	3	4	5	6	7	8	9	10
$C(n,1)$	1	4	10	20	35	56	84	120	165	220
$C(n,2)$	1	8	34	104	259	560	1092	1968	3333	5368

Remark. The sequences $\{C(n,1)\}$ and $\{C(n,1)\}$ have been recorded as A000292 and A033455, respectively, in HREF2.

Conclusion

In this paper we have shown that if we restrict convolution to series in place of function, the concept can be introduced to higher secondary school or "A level" students. This will not only give the students a good introduction that will help them to anchor the concept firmly, but will also serve as a bridge between the lower and the higher level mathematical concepts. Furthermore, it will help students to see other applications of series. Here we have succeeded not only in exploring convolution of two series but also of deducing some results that can be given to students as challenging exercises.

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HREF1: *The On-line Encyclopaedia of Integer Sequences* http://www.research.att.com/~njas/sequences/

HREF2: http://www.physicsdaily.com/physics/Convolution

HREF3: http://www.ilovemaths.com/3gp.htm

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