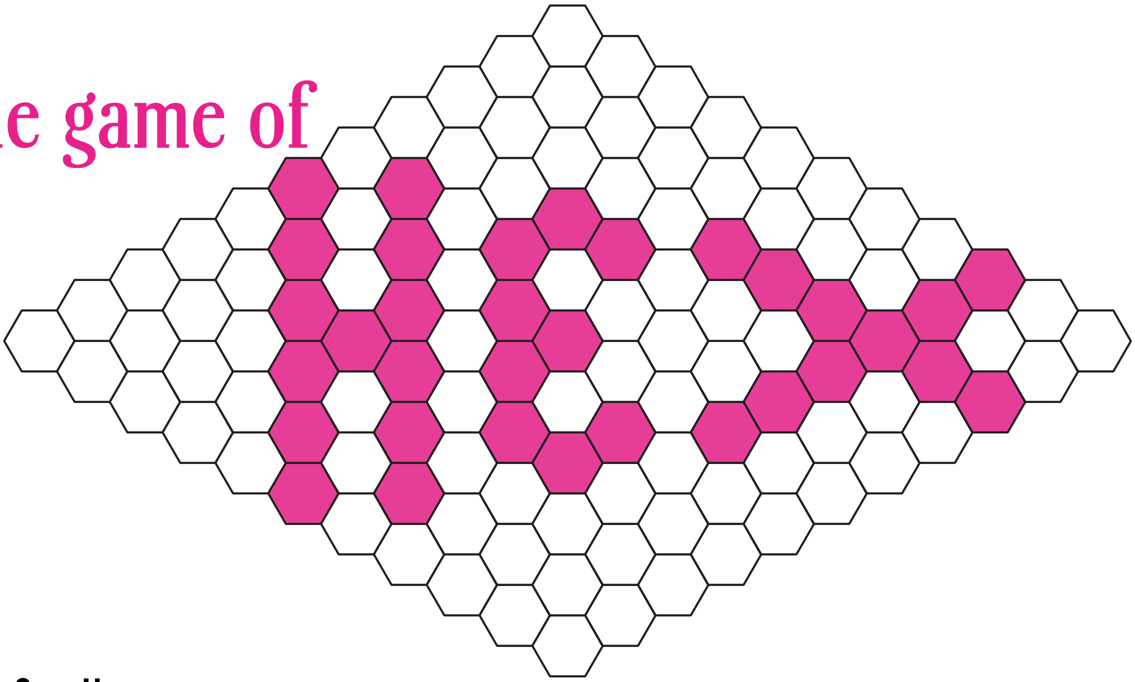
The game of



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The game

Many years ago I remember being fascinated with the game of Hex. This game seems to have gone out of fashion, which is a pity, because it is interesting to play, and provides some thoughtful mathematical analysis.

Like all good games, the rules are very simple. Hex is played on a diamond shaped board made up of hexagons. It can be of any size, but an 11 × 11 board makes for a good game. Two opposite sides of the diamond are labelled “red,” the other two sides “blue” (or, often, “black” and “white”). The hexagons at the vertices of the diamond belong to either side. The game is for two players. One has a supply of red pieces, the other a supply of blue pieces. If the board is of suitable size, draughtsmen make good pieces. The players alternately place their pieces on any of the hexagons, providing that hexagon is unoccupied. Each player tries to construct an unbroken chain of pieces from one side to the opposite side: red to red, or blue to blue.

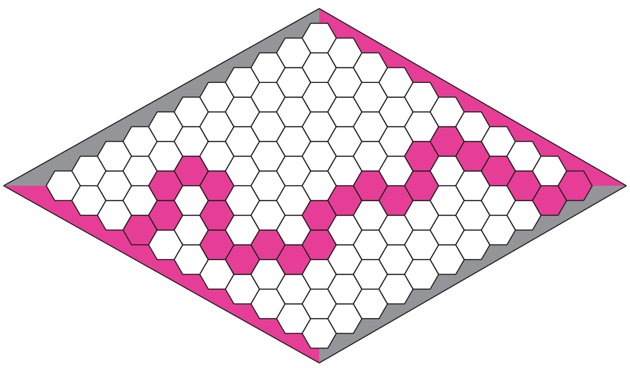


Figure 1

A chain can twist and turn. An example of a winning red chain is shown in Figure 1. It is clear that there can be no draw in this game: for example, a red winning chain will prevent blue from winning, and in fact this is the only way red can stop blue constructing a winning chain.

These days it is easiest to play Hex on the web; there are a number of possible sites, for example, the Mazeworks site given in the bibliography below.

Some history

Hex was invented by Piet Hein, a remarkable Dane who worked in mathematics, engineering, and theoretical physics. He gave a lecture on Hex in 1942 to students at the Institute for Theoretical Physics, and the game soon became immensely popular in Denmark, where it appeared under the name “Polygon.” The game was played using pencil and printed pads, and for many months the publication *Politiken* ran a series of Polygon puzzles. The name “Hex” was not given until 1952 when the firm Parker

Figure 2. Piet Hein.

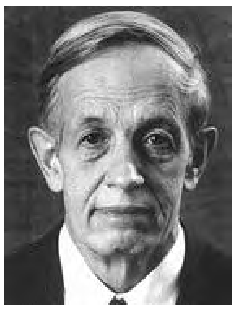


Figure 3. John Nash.

Brothers produced a version of the game under this title.

The game was independently re-invented (re-discovered?) in 1948 by Princeton university graduate John Nash, and quickly became popular there. The game was commonly called *Nash* or *John* — the latter name alluding to the fact that it was often played on the hexagonal tiles of bathroom floors!

Strategies

To understand the principles of Hex, it is a good idea to play the game on boards of various sizes, beginning with the very small.

- Let us begin with a 2 × 2 board (Figure 4). To win, would you prefer to be the first player, or the second?

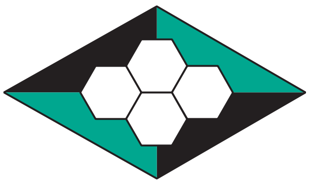


Figure 4

- Now a 3 × 3 board (Figure 5): would you prefer to be the first player or the second? If you chose to be the first player, where would you place your first counter?

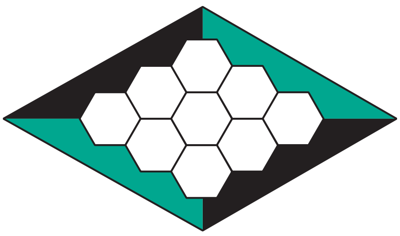


Figure 5

- Next consider a 4 × 4 board (Figure 6), and answer the same questions.

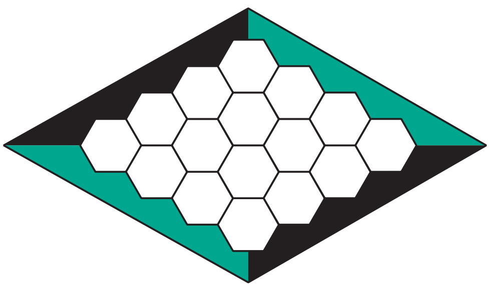


Figure 6

It is clear that working out a strategy gets more and more difficult, the larger the board. The standard 11 × 11 board provides so many possibilities that a complete analysis is likely to remain out of reach.

In the above examples, in the 2 × 2 case, the first player obviously always wins by choosing one of the cells of the vertical diagonal. In the 3 × 3 case, an obvious plan would be for the first player to choose the central cell. However, choosing any of the cells in the vertical diagonal puts the first player in a winning position. For then at each stage, for his next turn the first player has a choice of two useful cells, both of which cannot be taken by the opposing player. Similarly for the 4 × 4 board, the first player can always win by choosing one of the cells on the vertical diagonal.

From these considerations it seems likely that the first player has some advantage, and there is an "existence proof" that the first player can always win. This was worked out by John Nash in 1949. Briefly, the argument goes like this:

Since Hex is a finite game which cannot end in a tie, we conclude that either the first or second player has a winning strategy. We note that an extra move for either player in any position can only improve that player's position. Therefore, if the second player has a winning strategy, the first player could steal it by making an irrelevant move and then follow the second player's strategy. In effect, the first player becomes the second player. If the strategy ever called for moving on to a cell already chosen, the first player makes another random move. This ensures a first player win. We deduce there must be a winning strategy for the first player.

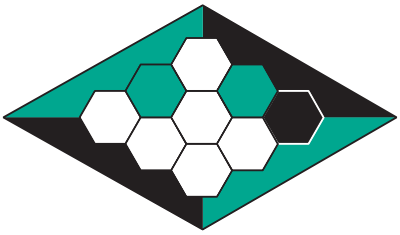
We should emphasise that this is a theoretical existence proof — not a practical technique!

Hex problems

Like the game of chess, certain hex problems have been posed for solution. You may like to try your hand at solving the following three which were devised by Piet Hein.

In each case, it is black's move. You are asked to give the next move for black which will guarantee a win for black.

1.



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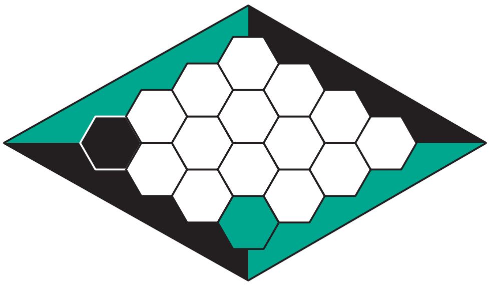

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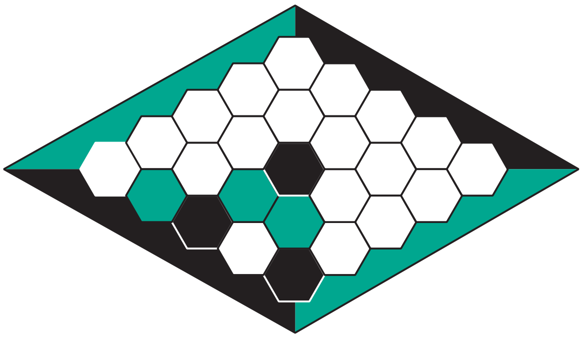

Bibliography

Gardner, M. (1961). *Mathematical Puzzles and Diversions.* G. Bell and Sons.

Mazeworks. *Hex.* www.mazeworks.com/hex7

Wikipedia. *Hex (board game).* http://en.wikipedia.org/wiki/Hex_(board_game)

Maarup, T. *Hex.* http://maarup.net/thomas/hex/

From Helen Prochazka's

Scrapbook

An apology

In A Mathematician's Apology, (Cambridge University Press, first edition 1940) English mathematician Godfrey Harold Hardy (1877–1947) used the word "apology" in the sense of a formal justification. He had felt the approach of old age and wanted to explain his mathematical philosophy to the next generation of mathematicians. One of Hardy's themes is that mathematics is a "young man's game." Here is an excerpt from the book:

If a man is in any sense a real mathematician, then it is a hundred to one that his mathematics will be far better than anything else he can do, and that he would be silly if he surrendered any decent opportunity of exercising his one talent in order to do undistinguished work in other fields. Such a sacrifice could be justified only by economic necessity or age.

I had better say something here about this question of age, since it is particularly important for mathematicians. No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game... We may consider, for example, the career of a man who was certainly one of the world's three greatest mathematicians. Newton gave up mathematics at fifty, and had lost his enthusiasm long before; he had recognised no doubt by the time he was forty that his greatest creative days were over. His greatest idea of all, fluxions and the law of gravitation, came to him about 1666, when he was twenty-four...

I do not know an instance of a major mathematical advance initiated by a man past fifty. If a man of mature age loses interest in and abandons mathematics, the loss is not likely to be very serious either for mathematics or for himself. On the other hand the gain is no more likely to be substantial: the later records of mathematicians are not particularly encouraging. Newton made a quite competent Master of the Mint (when he was not quarrelling with anybody). Painlevé was a not very successful Premier of France. Laplace's political career was highly discreditable, but he is hardly a fair instance since he was dishonest rather than incompetent, and never really "gave up" mathematics. It is very hard to find an instance of a first-rate mathematician who has abandoned mathematics and attained first-rate distinction in any other field.