From graphical to mathematical: The spiral of golden proportion

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The diagram in Figure 1, formed initially by abutting two unit squares and then abutting further squares, of size according to the elements of the Fibonacci sequence, in an anticlockwise manner, has been around for a long time.

In addition to the squares, inherent in the diagram are rectangles, increasing in size and whose sides are in the ratio of successive elements of the Fibonacci sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5} \dots \frac{F_n}{F_{n-1}}$$

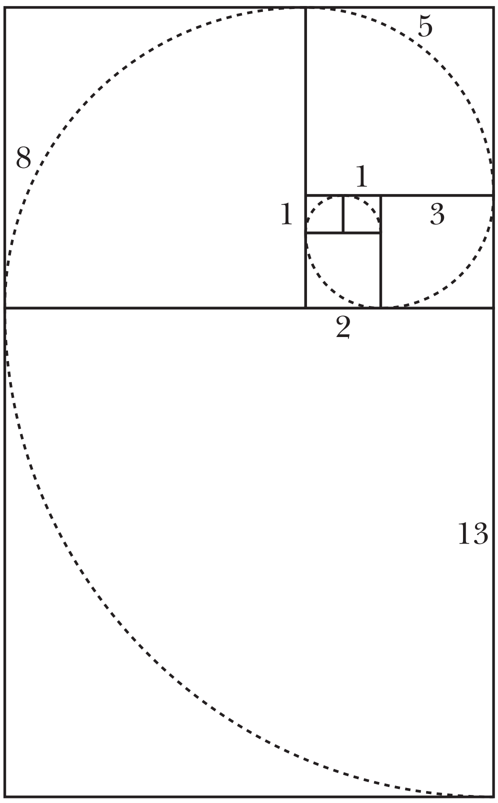

Figure 1

It is well known that as n gets large this ratio approaches what is often referred to as the golden ratio and is assigned the value ϕ where

$$\phi = \frac{\sqrt{5}+1}{2} = 1.618\dots$$

It has also been known for a long time that rectangles with golden proportion are more aesthetic to the eye than others. In fact there is a plethora of material written about the occurrence of ϕ in art, nature and music and its pleasing qualities to the eye and ear.

Artists, graphic designers, wrought iron craftsmen, and others have used the diagram in Figure 1 to, among other things, create a spiral of golden proportion. This is done graphically by drawing successive quarter-circles of increasing radii according to the elements of the Fibonacci sequence as shown dotted in Figure 1. The ensuing spiral (albeit piece-wise) is tangential to the sides of the rectangles. In articles associated with such diagrams, it is often stated, that the curve thus obtained approximates a logarithmic spiral of golden proportion.

There has been a lot of material written about logarithmic spirals of golden proportion but I have never come across an article which states the exact equation of the spiral which ultimately spirals tangentially to the sides of the rectangles as in Figure 1.

In this article I intend to develop such an equation.

The general equation of a logarithmic spiral, attributed to Descartes, is given in polar form by

$$r = Ae^{c\theta} \tag{1}$$

where A and c are positive constants.

In order to find c, use is made of a diagram in which part of a logarithmic spiral is enclosed by a rectangle of golden proportion; the sides of which are tangential to the spiral (see Figure 2). The origin for this graph is at the accumulation point of the spiral. Coordinates of points on the spiral are given by $x = r\cos\theta$ and $y = r\sin\theta$.

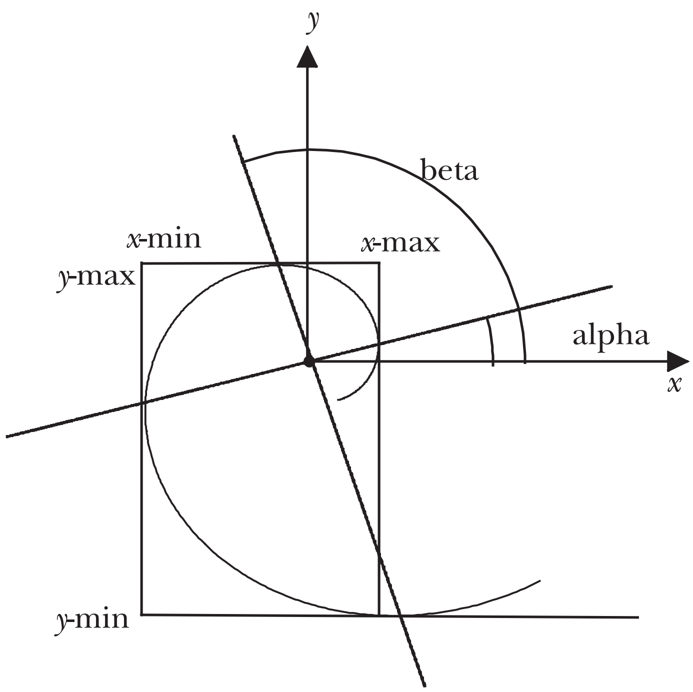

Figure 2

Hence $x = Ae^{c\theta}\cos\theta$ and $\dfrac{dx}{d\theta} = Ace^{c\theta}\cos\theta - Ae^{c\theta}\sin\theta$. Putting this derivative equal to zero yields $\tan\theta = c$ and hence for $x_{\max}$,

$$\theta = \alpha = Ar\tan c \tag{2}$$

Hence
$$\cos\alpha = \frac{1}{\sqrt{1+c^2}}$$

and this gives $$x_{\max} = \frac{A}{\sqrt{1+c^2}} e^{cAr\tan c}$$

$x_{\min}$ will also occur when $\tan\theta = c$ but with θ in the third quadrant $\theta = \pi + \alpha$

This gives $$x_{\min} = \frac{-A}{\sqrt{1+c^2}} e^{c(\pi + Ar\tan c)}$$

The width of the rectangle is given by $x_{\max} - x_{\min}$ which simplifies to

$$\text{Width} = \frac{A}{\sqrt{1+c^2}} e^{cAr\tan c}\left(1+e^{c\pi}\right) \tag{3}$$

Now the y coordinate of the spiral is given by $y = Ae^{c\theta}\sin\theta$ and hence

$$\frac{dy}{d\theta} = Ace^{c\theta}\sin\theta + Ae^{c\theta}\cos\theta$$

and putting this equal to 0 yields

$$\tan\theta = -\frac{1}{c}$$

For $y_{\max}$, $$\theta = \beta = \pi - Ar\tan\frac{1}{c} \tag{4}$$

hence $$\sin\beta = \frac{1}{\sqrt{1+c^2}}$$

and $$y_{\max} = Ae^{c\beta}\sin\beta = \frac{A}{\sqrt{1+c^2}} e^{c\left(\pi - Ar\tan\left(\frac{1}{c}\right)\right)}$$

$y_{\min}$ will also occur when $\tan\theta = -\frac{1}{c}$ but θ will be in the fourth quadrant:

$$\theta = \pi + \beta = 2\pi - Ar\tan\frac{1}{c}$$

$$y_{\min} = -\frac{A}{\sqrt{1+c^2}} e^{c\left(2\pi - Ar\tan\left(\frac{1}{c}\right)\right)}$$

The length of the rectangle is given by $y_{\max} - y_{\min}$ and this simplifies to

$$\text{Length} = \frac{A}{\sqrt{1+c^2}} e^{c\pi} e^{-cAr\tan\left(\frac{1}{c}\right)}\left(1+e^{c\pi}\right) \tag{5}$$

The Golden Ratio can now be established using equations (3) and (5):

$$\frac{\text{length}}{\text{width}} = \phi$$

and this simplifies to $$e^{c\pi} e^{-c\left(Ar\tan c + Ar\tan\left(\frac{1}{c}\right)\right)} = \phi$$

but, for $c > 0$, $$Ar\tan c + Ar\tan\frac{1}{c} = \frac{\pi}{2}$$

which leads to $$e^{\frac{c\pi}{2}} = \phi$$

from which $$c = \frac{2}{\pi}\ln\phi \qquad (6)$$

The next task in finding the complete polar equation of the spiral is to find its accumulation point. This can be referred to as the point where the spiral begins. In Figure 1 it can be referred to as the point about which the squares, of increasing size, rotate.

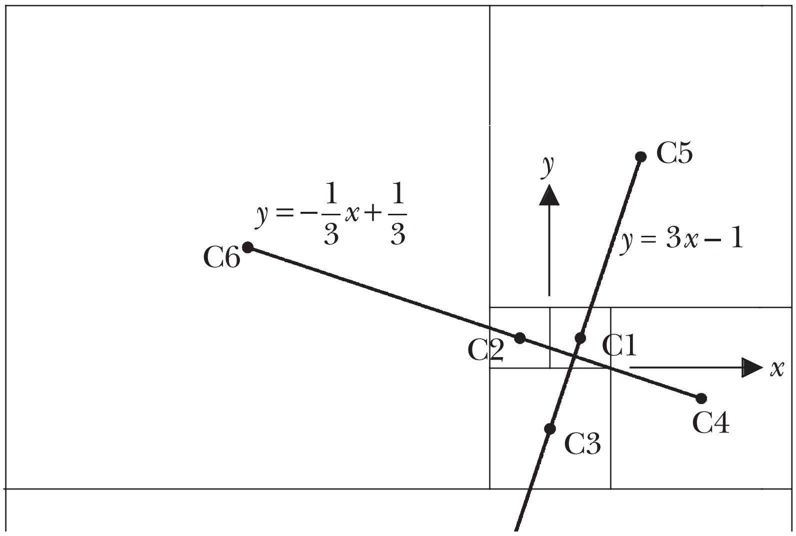

Figure 3

In Figure 3 the centres of the squares have been marked C1, C2, C3, C4, C5… as they occur in an anticlockwise manner. As can be seen, the odd numbered centres are collinear. The equation of this line is

$$y = 3x - 1.$$

The even numbered centres are also collinear on the line

$$y = -\frac{1}{3}x + \frac{1}{3}$$

Where these lines meet, at the point

$$\left(\frac{2}{5}, \frac{1}{5}\right)$$

is obviously the centre about which all of the squares can be considered rotating and is hence the accumulation point of the spiral. It is interesting to note that these lines through respective centres are mutually at right-angles. This further enhances the concept of rotational symmetry of the squares about this point.

Return to equation (1) the equation of the logarithmic spiral $r = Ae^{c\theta}$. By letting $A = e^k$ this equation can be written in an equivalent form which better suits further calculation:

$$r = e^{c\theta + k} \qquad (7)$$

The final task now is to find a value for the constant k.
Since c is known, what is required is to find a value for r at a given θ in order to find k.

However it must be remembered that the rectangles shown in Figure 1, unlike the one shown in Figure 2, are not of golden proportion. Their proportion approximates and approaches ϕ as the number of rectangles approaches ∞. Despite this, a value for k can be found as follows.

Previous work has shown that the angle θ makes to the x-axis, when the spiral is tangent to the golden rectangle in the first quadrant, is $\alpha = Ar\tan c$ (see equation (2)).

In Figure 4, notice that the origin has been moved to the accumulation point of the proposed spiral. The dotted line shows the angle α to the x-axis.

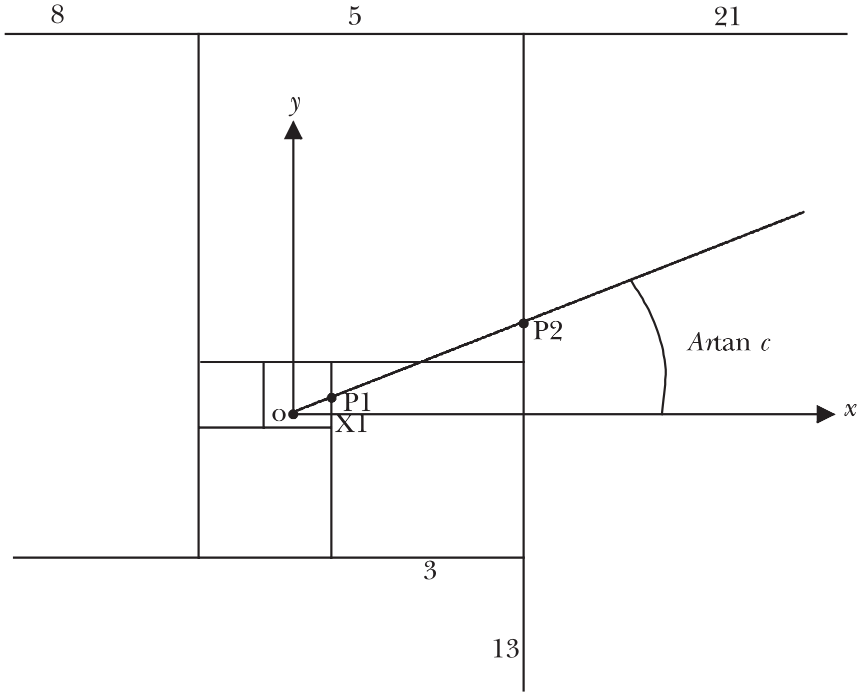

Figure 4

The points P1, P2, P3 (not shown)… Pn are points close to, but not on, the intended spiral. This is because, as stated earlier, the rectangles do not have golden proportion. Only as n approaches infinity will the points Pn be on the spiral and occur where the rectangles are tangent to the spiral.

Figure 4 Shows that $r_1 = x_1 \sec \alpha$ where x_1 is the x-coordinate of P1 and hence

$$r_n = x_n \sec (Ar\tan c) \tag{8}$$

Expressions now have to be found for x_n and θ_n so that they can be substituted into a modified equation (7); i.e.,

$$r_n = e^{c\theta n + k} \tag{9}$$

For P1	$X_1 = \frac{3}{5}$	$\theta 1 = Ar\tan c$
For P2	$X_2 = 3 + \frac{3}{5}$	$\theta 2 = 2\pi + Ar\tan c$
For P3	$X_3 = 21 + 3 + \frac{3}{5}$	$\theta 3 = 4\pi + Ar\tan c$
For P4	$X_4 = F_{12} + F_8 + F_4 + \frac{3}{5}$	$\theta 4 = 6\pi + Ar\tan c$
For Pn	$X_n = F4_{(n-1)} + F4_{(n-2)} + F4_{(n-3)} + \ldots + \frac{3}{5}$	$\theta n = 2(n-1)\pi + Ar\tan c$

From these patterns the expression for θ_n in terms of n is quite clear.
In order to simplify the expression for x_n, the fact that n approaches ∞ has to be used at this stage.

As $n \to \infty$ $\qquad \dfrac{F_n}{F_{n-1}} \to \phi$ and $\dfrac{F_{4(n-1)}}{F_{4(n-2)}} \to \phi^4$

For P_n then $$X_n = F_{4(n-1)} + \frac{F_{4(n-1)}}{\phi^4} + \frac{F_{4(n-1)}}{\phi^8} + \frac{F_{4(n-1)}}{\phi^{12}} + \ldots$$
$$X_n = F_{4(n-1)}\left(1 + \frac{1}{\phi^4} + \frac{1}{\phi^8} + \frac{1}{\phi^{12}} + \ldots\right)$$

The value of a geometric series where the first term is 1 and the common ratio is $\dfrac{1}{\phi^4}$ is $\dfrac{\phi^4}{\phi^4 - 1}$

This simplifies to $$\frac{3\phi+2}{3\phi+1}$$

and further simplifies to $$\frac{3\phi+1}{5}$$

Hence $$X_n = \frac{3\phi+1}{5} F_{4(n-1)}$$

In order to simplify this expression further, use of Binet's Formula is necessary.
Binet's Formula states that as $n \to \infty$, $F_n \to \dfrac{\phi^n}{\sqrt{5}}$

Hence as $n \to \infty$ $$F_{4(n-1)} \to \frac{\phi^{4(n-1)}}{\sqrt{5}}$$

and therefore $$X_n = \frac{(3\phi+1)\phi^{4(n-1)}}{5\sqrt{5}}$$

If this expression for X_n is substituted into equation (8) an expression for r_n is obtained.
$$r_n = \frac{(3\phi+1)\phi^{4(n-1)}}{5\sqrt{5}} \sec(Ar\tan c)$$

This simplifies to $$r_n = \frac{(\phi+1)\phi^{4(n-1)}}{5} \sec(Ar\tan c)$$

From equation (9) $$e^k = \frac{r_n}{e^{c\theta n}}$$

and expressions for r_n and θ_n can now be substituted:
$$e^k = \frac{(\phi+1)\phi^{4(n-1)} \sec(Ar\tan c)}{5e^{c[2\pi(n-1)+Ar\tan c]}}$$

which simplifies to $$e^k = \frac{(\phi+1)\sec(Ar\tan c)}{5e^{cAr\tan c}}$$

from which finally k is obtained:

$$k=\ln\left[\frac{\phi+1}{5}\sec(Ar\tan c)\right]-cAr\tan c$$
$$=-0.693231369704738\ldots$$

Alternatively this can be written as

$$k=\ln\left[\frac{(\phi+1)\sqrt{1+c^2}}{5}\right]-cAr\tan c$$

Now that the equation for the spiral has been obtained it can be plotted on the diagram of squares with centre at (0.4, 0.2) as determined in Figure 3. The result of this can be seen in Figure 5. Notice how the spiral is oscillatory convergent to being tangent to the sides of the rectangles. This is not surprising because the ratio of successive elements of the Fibonacci sequence is also oscillatory convergent to ϕ.

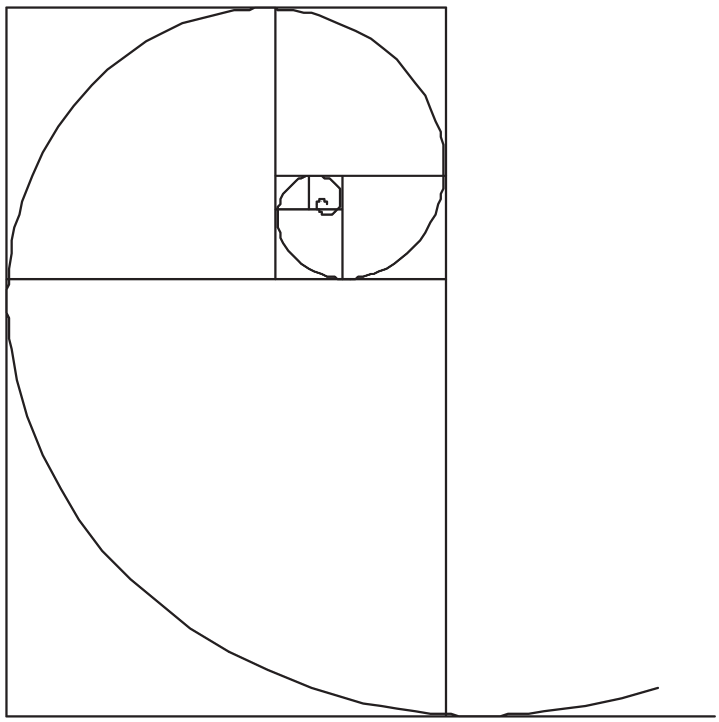

Figure 5

I present this article here in that it is an example of a problem solving exercise in which a good variety of secondary level mathematics is required. There are many sub-problems within this report that teachers can extract for their students, for example:

1. Given that $\phi=\frac{\sqrt{5}+1}{2}$, show that $\frac{3\phi+2}{3\phi+1}=\frac{3\phi+1}{5}$ (Year 11, 12 Surds).
2. Simplify $Ar\tan c+Ar\tan\frac{1}{c}$ (Year 12 circular functions).
3. Find the derivative of $Ae^{c\theta}\cos\theta$ with respect to θ (Year 12 differential calculus).