Add *another dimension* to your life

With a bonus recipe for making tesseracts

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Let me pose a simple question: What is Figure 1?

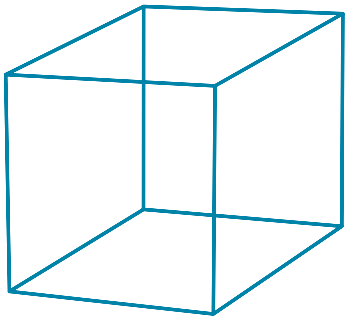

Figure 1

Did you answer "A cube"? Figure 1 certainly looks like a cube, and I suspect a good many of you identified it as such, but it is not actually a cube. A cube is a solid in three dimensions, with three mutually perpendicular right angles evident at the vertices. The thing in Figure 1 is not a solid: it is flat, and so it is not a cube. In fact, Figure 1 is only a two-dimensional representation of a cube, since a cube's three-dimensional nature cannot exist in the two-dimensional plane of a piece of paper. In drawing the cube in two dimensions (which is what we have done in Figure 1), we have to cheat and draw one of the right angles at less than 90 degrees.

This insistence that Figure 1 is not a cube but merely a representation of one may seem a little pedantic yet it is critical for understanding about shapes, and for allowing us to try to imagine the fourth dimension.

Understanding the space in which we live

Before we go in search of this extra dimension, let us review what we know about zero, one, two and three dimensions, and certain shapes in each of these. We will start with the first three cases, because we can draw things realistically for these. So, imagine a dot—with no width, no breadth, and no depth (which means that my picture of it in Figure 2 is actually not very realistic at all!). This is the 0-dimensional universe, where you have no place to go but where you already are! Here there are no directions: no ups nor downs, no lefts nor rights, no backs nor forths.

Now imagine that we can take the point and move it in one direction only, say horizontally from left to right across the page. If we allow the dot to move infinitely far to the left and the right, we will create an infinite straight line. This is the one-dimensional universe, where you have some freedom to move provided you only want to go left and right along the line. If we mark a point as the origin, then a single number or co-ordinate is enough to tell us where we are in relationship to that origin (this co-ordinate tells us how far to the left or right of the origin we are). Although we can now move, we still cannot go up and down, nor backwards and forwards.

If, instead of allowing the dot to move an infinite distance, I allow the original dot to move only 3 cm to the right, as suggested in Figure 2 (imagining that the line has no thickness), then I end up with a one-dimensional line segment that is 3 cm long. This segment is just a part of the whole one-dimensional universe, but it is a very important shape in its own right.

Now, let us take our one-dimensional line and imagine that we can move it at right angles to itself. Since we have used horizontally left and right across the page as one direction, we need to go vertically up and down the page. The result is a two-dimensional plane, extending infinitely left-right and up-down. By moving various amounts left and right, and up and down, I can get anywhere in this two-dimensional universe, but I am stuck in the plane and cannot move backwards and forwards (or inwards and outwards). In two-dimensional space, once I pick a point as the origin, it takes two co-ordinates to describe the position of another point: how far to the left/right of the origin it is, and how far up/down.

If we take only the 3 cm one-dimensional line segment and move it at right angles up the page for just 3 cm, then we get a square. This is illustrated in Figure 2. The square is two-dimensional object—a subset of all of two-dimensional space—and includes both the interior and the boundary. In order to illustrate better the rest of our discussion, however, we are going to highlight the edges of the square, and make the interior transparent as shown in the first part of Figure 3.

Figure 2. From zero to two dimensions in two moves.

Things now start getting a little tricky, especially since this is a written article and so I can only show you pictures on two-dimensional paper. To make a cube, we take our square and move it at right angles to itself by a distance equal to the length of the sides of the square (in this case 3 cm). So, where is this new right angle? We have moved left–right, we have moved up–down, and so the only direction remaining is to go backwards–forwards, into and out of the page. If you imagine holding the square in front of your face and then moving it away from you for a distance of 3 cm, then the space that is traced out is a cube. Here, then, is a three-dimensional object. If you had taken the whole two-dimensional plane, and moved it backwards and forwards an infinite amount, you would have created a whole three-dimensional universe. This would, of course, fill all Euclidean space. Now there is freedom to move up–down, left–right, and backwards–forwards, and the position of every point can be described in relation to an origin by three co-ordinates.

Let us return to our cube, and the way I got you to imagine its construction, moving the square away from your face. This is all well and good for you in three-dimensional space, where you can actually move the square away from you. For me writing this article, however, I have a problem in trying to show this. I have to represent that third dimension—the one that goes into and out of the page—by a sloping line. So, I move my square upwards and to the right on the page, in order to try and show a third direction that is mutually perpendicular to the other two. This third right angle ends up not being a right angle at all, but it is the best we can do on a flat piece of paper. This approach gives us the quasi-perspective representation shown in Figure 3—the standard drawing of a cube—with sloping lines and some of the faces looking like parallelograms. In fact, it should be noted that the way that people often draw such a cube is to draw two squares, offset from one another, and then join the corners with sloping lines.

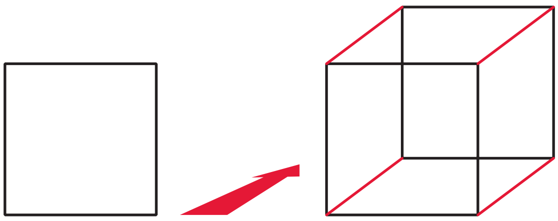

Figure 3. From two to three dimensions, with a problematic right angle.

Before we go any further, I want you to compare your “real” cube (the one you pictured by moving that square away from your nose) with my drawing in Figure 3. There is much that is missing from my drawing. For starters there is an inside space within the real cube, where you can look out through six faces by looking up, down, left, right, forwards and backwards. My drawing has no inside space. The faces that you see from inside the real cube are all square; my drawing has only two faces that actually look like squares whereas the rest look like parallelograms. At every corner of the real cube there are three edges that meet as a set of three mutually perpendicular lines (i.e., each one is at right angles to each of the others). On my diagram there are three lines meeting, but very few of them are shown to meet as actual right angles. Above all, my diagram has no depth: that third dimension is missing.

Thinking outside the square/cube

Let us review what we have done so far, and think about it more generally. At each stage we have taken an n-dimensional object and turned it into an

(n+1)-dimensional object by moving it in the direction of a new right angle. This allowed us to go from a point to a line segment to a square and finally to a cube. In the case of going from two to three dimensions, the third right angle can be shown correctly provided you are working in three-dimensional space; but, if working on paper, it is difficult to show that third direction and capture all the features that characterise a three-dimensional cube.

We are going to extend these ideas to see if we can at least start to visualise four dimensions.

I want you to imagine a cube, a real three-dimensional cube sitting in space. Now take this cube and mentally move it at right angles to all the other right angles that you have, and move it a distance equal to the lengths of the edges of the cube. (Just like we did with the square, to make it a cube.)

"But wait," I hear you say. "There's no such right angle." Okay, you are right; there is only room for three right angles in three-dimensional space. So let us cheat—the way I had to cheat when I drew a three-dimensional cube on a two-dimensional piece of paper—by drawing the extra right angle at something other than 90 degrees. Consequently, imagine the cube, in space, and move it along a sloping line for a distance equal to the cube's side length. It might help if I give you a recipe for building what I am trying to describe: instructions for making a (three-dimensional representation of a) tesseract or hypercube, or the four-dimensional equivalent of a cube. "Hypercube" is the more general term and applies in dimensions higher than three; the word tesseract is usually reserved for the specific four-dimensional case.

A recipe for the fourth dimension

Ingredients

32 plastic drinking straws (preferably of the same bright colour)
A reel of cotton
8 medium-sized blobs of Blu-tack
Needle
Magnet
Easily accessible ceiling or a box (half-straws in a photocopy paper box work quite well)

Method

The model of a tesseract can be constructed by threading the cotton through the straws. It is best to use 1–2 metre lengths of cotton, and when one length is nearly used up tie on a new length and continue threading. Furthermore, it is easier to build the tesseract in its final location, as hypercubes are not easily transportable (particularly in three dimensions!). Consequently, make sure you have a comfortable platform from which to work.

Attach the cotton firmly to one end of a straw, and then thread the cotton through this straw and three others, before passing the cotton through the first straw to make a square. Keep the square taut (or adjust it as you go)

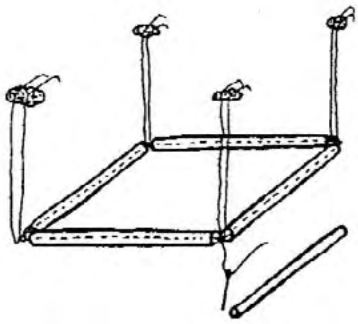

Figure 4. Stage 1 of the tesseract: a two-dimensional square, suspended in three-dimensional space.

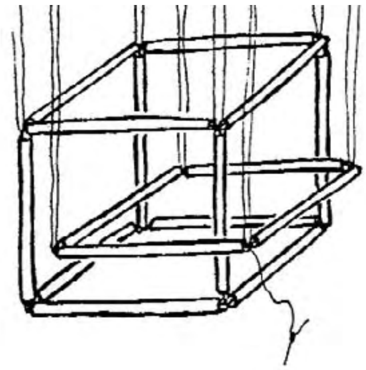

Figure 5. Stage 2 of the tesseract: a three-dimensional cube, with the start of an interlocking second cube.

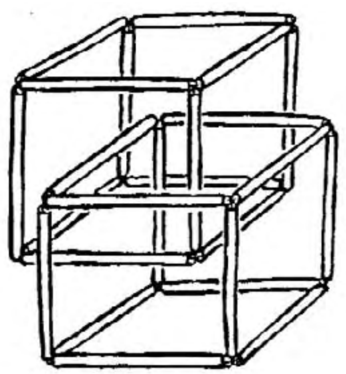

Figure 6. Stage 3 of the tesseract: two interlocking cubes.

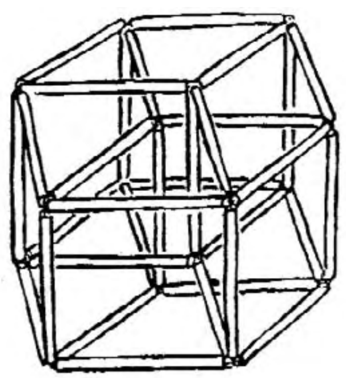

Figure 7. Stage 4 of the tesseract: the finished product.

and use four lengths of cotton and the Blu-tack to suspend the square from the ceiling, or from the inside top of a large box, as shown in Figure 4.

Use this square as the top face of a cube which you now build using another eight straws. The cube hangs down from the top face, and is constructed by threading the cotton through both the new straws and those already in position, with the magnet being used to draw the needle and thread through the straws. It is necessary to keep pulling the cotton taut—if you do this you will not have to tie any knots, although you may like to for added security. While threading the cube you may like to contemplate if there is a minimum length of thread to use! However, use as much as is needed to make everything tight. When you have finished building the cube, tie off the end of the cotton so that the tension does not ease. You now have a three-dimensional cube suspended in three-dimensional space, where in this case we added the third dimension to the original square by coming downwards.

From here the job becomes a little more delicate—and a lot more mind-blowing. Thread four straws onto a new piece of cotton, but before joining these into a square locate them as shown in Figure 5, so that the new square is interlocked with the first cube. Suspend this square from the ceiling with four pieces of cotton and Blu-tack, so that its innermost corner is about at the centre of the cube. Carefully build a second cube as before (suspended from the new square); this cube is interlocked with the first cube but not connected to it, as in Figure 6.

Before we go any further, note that what we are doing is the three/four-dimensional analogue of what most people do when drawing a cube in two dimensions by first drawing two interlocking squares. Also observe that, at the moment, everywhere that edges meet they do so at right angles because in three-dimensional space we have all three right angles available and so far we have only built cubes.

Our next step is the three/four-dimensional analogue of drawing sloping lines to connect the corners of the squares in the two/three-dimensional cube. Connect a straw from the corner of one cube to the corresponding corner of the second, for each of the eight corners. In the process of threading through new and already positioned straws you will be able to tighten any imperfect connections. Finally, adjust the length and location of the hangers (the lengths of cotton attached to the ceiling from which the tesseract is suspended) so that both of the original cubes are nice and square, and the diagonal struts are as parallel to each other as possible. This allows us to visualise what I tried to describe earlier: take a cube and move it along a sloping line that represents the extra right angle. The final eight straws that we have just placed allow us to fake the fourth right angle that we need for four-dimensions, in much the same way as we faked the third right angle when drawing a cube on two-dimensional paper. Where four lines meet at the vertices, we cannot have four mutually perpendicular lines in our three-dimensional world, and so some of them have to cheat by sloping off at an angle other than 90°.

You should now have the finished model of a tesseract as in Figure 7 (see also Figure 8, which is a photo of the one that hangs outside my office door). Of course, what we have built is not really a tesseract, as they only exist in four-dimensional space, but it is a three-dimensional representation of one. In fact, talking about Figure 7 is quite hard: it is actually a two-dimensional representation (a drawing on paper) of a three-dimensional representation (the straw construction hanging in space from the ceiling) of a four-dimensional object (the real tesseract in four-dimensional space with four mutually perpendicular directions apparent at the vertices). Pretty scary stuff!

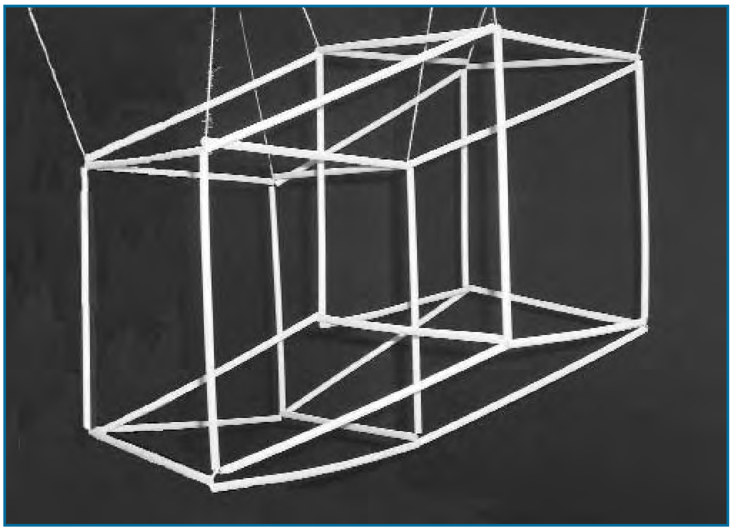

Figure 8. Photo of one of the author's tesseract models (all edges have the same length).

If you build such a model—or if you have a vivid imagination and can work with the diagrams—you should be able to pick out eight cubes: the two original right-angled cubes, and an additional six cubes which have square and parallelogram faces (and which therefore appear as parallelopipeds). The two-dimensional diagrams and photo do not do the object justice, however. By looking at the three-dimensional model you will get more of a sense of the structure, especially if you move around it and take advantage of the fact that the model will flex, so you can temporarily make the parallelopipeds appear more cube-like.

In fact, the three-dimensional model does not do justice to a real tesseract either. Just as a two-dimensional drawing of a cube has serious shortcomings in depicting all the properties of a real cube, so does the

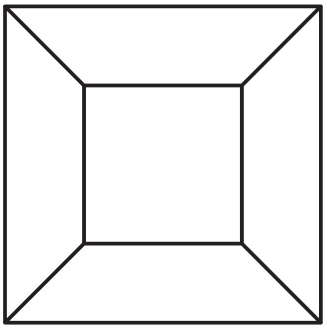

Figure 9. Alternative two-dimensional representation of a cube.

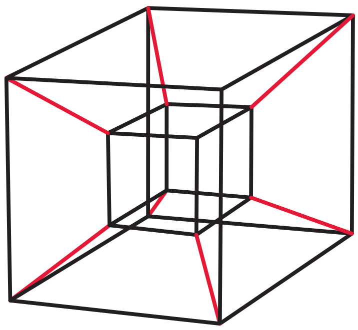

Figure 10. An alternative two/three-dimensional representation of a tesseract.

three-dimensional model of a tesseract fail to capture all the characteristics of a real four-dimensional hypercube. Most significant of these is that one dimension is missing, and if we appreciate the huge difference between "flat" and "solid"—or between "plane" and "space"—we can start to appreciate the huge difference between the three-dimensional model and the object that exists in four-dimensional hyperspace. As another example, the two-dimensional drawing of a cube does not readily reveal that each square face has four other faces as its neighbours; similarly the three-dimensional model of the tesseract cannot easily show that each cube has six other cubes as its neighbours and that you can pass through the walls of one into the next.

One final note, again limited by my capacity to show things only in two dimensions. There is an alternative way of drawing a cube that is shown in Figure 9. Instead of having two squares the same size we show one "nearer" (larger) and one "further away" (smaller), but then join corresponding corners as before. (For those of you who know your graph theory this is a planar graph representation of a cube.)

We could use a similar approach to building a model of a tesseract. Take two cubes, one inside the other, and then connect corresponding vertices. A colour-coded depiction of this is shown in Figure 10; this is an alternative two/three-dimensional representation of a tesseract. You might like to think about whether this representation provides us with any additional understanding of the fourth dimension.

I hope this has helped you to visualise the fourth dimension, at least to some extent. Can you imagine what the fifth dimension is like or explain how you might show a five-dimensional hypercube? It is amazing stuff. If you are interested in learning more, try some of the following.

Abbot, Edwin A. (1952). *Flatland: A Romance of Many Dimensions. (6th rev. ed.).* New York: Dover. (This is a story about a two-dimensional being's experiences in a three-dimensional world, and helps us appreciate the difficulty of comprehending four-dimensions.)

Heinlein, Robert A. *And He Built a Crooked House.* Available online at http://www.scifi.com/scifiction/classics/classics_archive/heinlein/heinlein1.html (This is a science fiction short story about a tesseract house.)

Stewart, Ian (2001). *Flatterland: Like Flatland, Only More So.* London: Macmillan. (Presents a wide variety of different geometries.)

Author's note

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