Conceptual complexity and apparent contradictions in mathematics language

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Mathematics is like a language, although technically it is not a natural or informal human language, but a formal, that is, artificially constructed language. Importantly, we use our natural everyday language to teach the formal language of mathematics. Sometimes we encounter problems when the technical words we use, as formal parts of mathematics, conflict with an everyday understanding or use of the same word, or related words. This article discusses this problem, including some examples, and offers some suggestions for handling the difficulties.

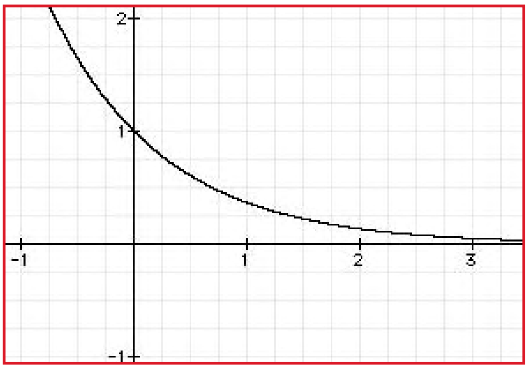

Figure 1. $y = e^{-x}$.

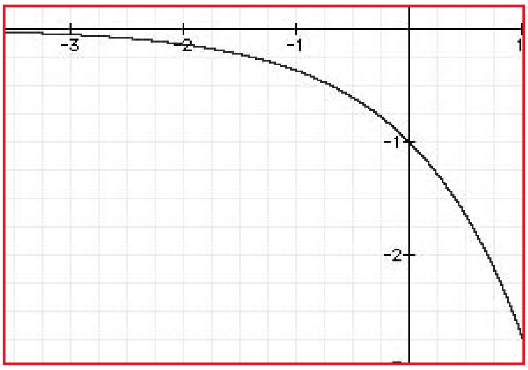

Figure 2. $y = -e^{x}$.

The first example arises in discussion of changes of gradient, and rates of change of gradient, of continuous functions. The AAMT list community (aamt-l@edna.edu.au) recently posted a message asking about particular functions that decrease at an increasing rate, or decrease at a decreasing rate, or are combinations of increasing and decreasing.

For example: "Would $y = e^{-x}$ be classified as decreasing at an increasing rate or decreasing at a decreasing rate?" (see Figure 1).

A reply was posted, saying that $y = e^{-x}$ can be described as decreasing at a decreasing rate, because the curve is definitely decreasing, and its rate of decrease is also decreasing (i.e., it is going down, but the rate at which it goes down becomes slower and slower as values of the independent variable x increase — it is the classic "exponential decay" function).

It was also noted that the function $y = -e^{x}$ is an example of a curve which is decreasing at an increasing rate (see Figure 2).

By contrast, $y = e^x$ increases at an increasing rate, (this is the classic "exponential growth" function; see Figure 3), and $y = -e^{-x}$ increases at a decreasing rate (see Figure 4).

Having got this far, the respondent ended, remarking: "Now I just need to go and lie down to stop my head spinning." Indeed.

The unusual combinations and close juxtapositions of words for up and down or increasing and decreasing are conceptually equivalent to feeling sea-sick. We read or hear the words "increase" or "decrease" and cannot prevent ourselves feeling some sensory version of the meaning of the words.

Technically, in calculus, the issue is one of comparing overall decrease or increase of the y-value of the function, and overall decrease or increase in the first derivative; is the second derivative:

- positive?
- negative? or
- variable (as in the gradient of a cubic)?

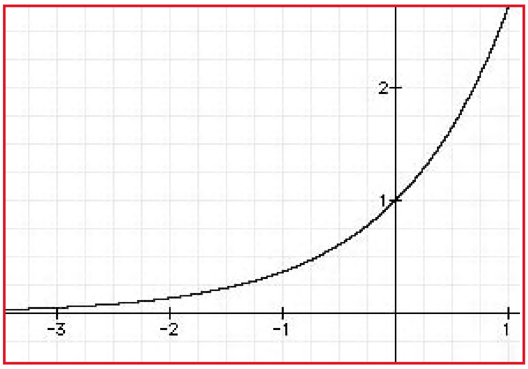

Figure 3. $y = e^x$.

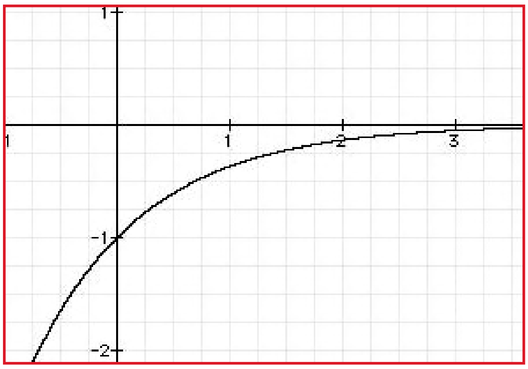

Figure 4. $y = -e^{-x}$.

What becomes conceptually tricky is the conceptual strength or metaphorical power of the words being used. We hear these dynamic metaphorical words and find it difficult to stop our imaginations interpreting them instantly — but there are conflicting imaginative pulls in words that are conceptual opposites, that appear so close to one another in the flow of discussion. (Also see Kristina Juter's (2004, p. 230) discussion of limits in functions: "everyday language can have a slightly different meaning compared to the language used in mathematics... [such as] convergence, arbitrarily close, tend to, and limit".)

Similar conceptual conflicts arise when we consider "least upper-bounds" and "greatest lower-bounds" in discussing sequences and series, convergence, and limits. I recently came across a similar conceptual word confusion in a television news Bureau of Meteorology weather report: "Today's maximum temperature was up to 5 degrees below the average minimum..." What struck me as odd, in this calculus-free everyday example, was the mixed conceptual implications of "up to" conflicting with "below".

Another oddity I have encountered occurs sometimes in accounts of sailing ships in the South Atlantic and South Pacific, for example in Geoffrey Blainey's *The Tyranny of Distance* (1966, p. 7), and Alan Villier's *Captain Cook: The Seamen's Seaman* (1967, p. 209): the further south the sailing ships go into the Roaring Forties, and further towards Antarctica, the stronger the prevailing winds blowing from west to east around the globe, across oceans unbroken by continental land. The odd expression is that of the ships sailing into "higher latitudes," where it is taken for granted that the context is the southern hemisphere. In my mind, at least, there is a conceptual-lexical clash between "higher" (numerically higher, as ships pass into latitude 40 degrees south, and move further towards 50 degrees south), conflicting with my concept of "high" relative to the Earth as a physical globe, orientated, visually and spatially (and arbitrarily but conventionally) with the North Pole at the "top". In my spatially-challenged mind the North Pole is "high", but the South Pole is "low". So the further south the ships sail, the "lower" (spatially, on the surface of my mentally imagined globe) they are going. Similarly Arctic explorers, in my thinking, trek up to the North Pole, and Antarctic explorers trek down to the South Pole. (This is further confused, in this latter case, because I also know that the South Pole is on a high central ice-covered plateau, so the trek is also "up" in terms of altitude above sea-level — another spatial concept!)

I am not alone in this way of thinking about the Earth, spatially. For example, discussing global warming, Fen Montaigne (2004) refers to a "sub-Antarctic system" (p. 39), meaning oceanic islands such as Heard, South Georgia and the Falklands, which are in geographic proximity to the Antarctic continent. Yet oddly, here, "sub-Antarctic" literally combines Latin roots meaning under, opposite and Arctic; or more simply, "below" the "Antarctic". But what is "below" or "under" the Antarctic? The atmosphere vertically over the South Pole — "lower" than the Earth's globe, within the plane of the Solar System? The Earth's crust or magma beneath the tectonic continental plate? Note, too, that Montaigne also speaks of the "high Arctic" (p. 48), when speaking of polar bears in Canada's Hudson Bay, where the bay-ice is diminishing, and the bears may need to change their hunting and breeding range if they are to survive loss of habitat. In this case, Montaigne imagines the globe with the North Pole at the "top", but with the ocean fringing Antarctica being "sub-Antarctic" — literally "below" the Antarctic (whatever that denotes). Implicitly, the meaning of "below" makes sense only in terms of numerical magnitude of latitude, not in spatial or global terms, in the way 30 is (numerically) "below" 40.

A similar conceptual difficulty arises when we read, for example, of someone saying, "I'd like to go up the Nile, wouldn't you?" (Christie, 1937, Chapter One: Part 8), and then mentally imagining the map of North Africa, with the mouth of the Nile at the Mediterranean coast, the northern top of the continent, and the mysterious origins of the river lost below, underneath, in the southwards African hinterland near the Mountains of the Moon. Where is "up" on the Nile?

This is perhaps more easily untangled, conceptually, because we also know that water flows downwards, downhill, and sea-level is a kind of "zero" towards which most rivers flow. Hence the convention is to speak of moving, physically, upwards, uphill, or "upstream" (towards the higher altitudes where rivers have their source) or "downstream" towards the sea. Hence we also speak of going "up the airy mountain" (and "down the rushy glen"), and going "down to the beach" (where "sea-level" is usually a notional "zero"). Here, in the case of the Nile, the implied meaning of "up" is relative to the flow of the water (and, implicitly, the lie of the land), rather than to compass bearings, latitude, or the globe.

In all four cases (graphed functions, temperature variations, southbound sailing ships, and flowing rivers), on examination, we can find nothing literally wrong with the words, or with the concepts. The concepts are correct, and the words are being correctly used. Despite this, my brain usually teeters or registers a kind of mental jolt when I encounter these verbal-conceptual clashes.

These examples would be trivial, especially for secondary teachers, except for two factors. First, it is through the secondary years that mathematics teachers formalise trigonometric and spherical calculations based on latitude and longitude. Second, when students, even at advanced levels of curriculum, are confronted by cognitive conflict, they are likely to revert to much earlier ideas and conceptual-experiential metaphors (e.g., Davis 1984; Lakoff & Nunez, 2000).

Consider the less trivial, more problematic examples, such as what happens when students encounter fractions and negative numbers. We learn that fractions, such as one-quarter, one-fifth, one-sixth, and beyond, get (numerically) smaller and smaller (as the denominator gets larger and larger — and later we also encounter the idea of sequences and limits). We also grasp the idea that zero is a kind of full-stop to this process. However, we learn that, before reaching zero, the number-line contains an infinite or unending succession of numerically (and spatially) smaller and smaller (positive) fractions.

Incidentally, on the topic of fractions and misleading word use, John Allen Paulos (1988) remarks wickedly that when he hears that something is "selling for a fraction of its normal cost" he (mentally) comments "that the fraction is probably 4/3" (p. 122). Our initial, and unhelpfully prolonged exposure to fractions as not-quite-numbers and as parts-of-a-whole, and hence as, preponderantly, less than 1, does not help us later when we need to think far more flexibly about "fractions" as numbers of a particular kind, and possibly of any numerical size. Fractions re-expressed as wholes (i.e., percentages) compound this.

Then we encounter negative numbers! For example:

> Which is "smaller": –1000 or 0.000001?

We know 0.000001 is very small, compared with years of experience of positive whole numbers, some of which are very large. We also learn to see negative numbers becoming "larger" (if only in absolute magnitude), in the left-hand-side of the number-line, extending left past the zero-full-stop of diminishing smallness for positive numbers. The mental shift across the boundary of zero, from positive to negative (or from AD dates to BC dates) can remain occasionally difficult for adults.

Zero causes other difficulties. Consider this example:

> A fence needs a fence post every 10 metres. We have a fence that is 1000 metres long: how many fence posts are needed?

Having grown up counting from 1, and having learned multiplication and division facts and processes that tend to neglect the zero-times multiplication table, we are more likely just to divide 1000 by 10 and reach the wrong answer, neglecting the first fence-post that stands at the zero-starting point of the fence.

Relationships, interactions and possible conflicts between language and mathematics have been extensively discussed (e.g., Durkin & Shire, 1991;

MacNeal, 1994). More recently, the discussion continues under the synonymous terms “literacy” and “numeracy” (although so called “literacy” is also often taken as including, oddly, “oracy” — I prefer to distinguish “spoken” from “written” language skills).

In particular, regarding the conceptual difficulties arising over “zero”, MacNeal (1994, pp. 83–84) suggests that we:

- put a 0 at the left end of every ruler;
- start counting with zero, in print and orally, from the beginning of counting experiences (Sesame Street scriptwriters, take note!);
- put “zero is a number” into school policy;
- display in every classroom a large dummy thermometer with a prominent zero and a moveable degrees-pointer; and
- speak of children younger than 1 as zero-year olds.

R. C. Ablewhite (1969, p. 32) gives an interesting way to regularise counting and place-value:

> One, two, three… eight, nine, one-ty,
> one-ty one, one-ty two… one-ty eight, one-ty nine, two-ty,
> two-ty one, two-ty two…

We might also consider starting oral counting with, “none-ty, none-ty-one, none-ty-two…” if only as a helpful step in remedial intervention with students who are struggling with early place-value concepts.

As noted, mathematics is not a natural human language, but artificial, supported by special alphanumeric characters and usages, non-alphanumeric symbols, special written formats within a single line, the clever use of two or more lines at a time, and set-theoretic logical connectives. Also, in important non-verbal ways, this “language” is supported crucially by spatial-textual formatting devices and non-verbal images (see Barling’s (2005) challenging discussion of specialised mathematical text and symbolic formatting in our computer-keyboard and CAS era).

Importantly, as a deliberately constructed language where it does not invent new terms (this is a rare event), mathematics borrows words that already exist, with everyday meanings, and reshapes or redefines the intended, specialist technical meaning. The result is that in classrooms we speak to our students using our everyday language as the medium of instruction, while trying to teach them how to speak and think in terms of new, often different, technical meanings, using words that overlap with lay-talk. It is valuable to discuss this overlap, and explore any possible confusions arising from the tensions between language-of-instruction versus subject-language.

Consider the potential conflicts between everyday and mathematical meanings of common mathematical words such as: identity, axis (and axes), volume, root, segment, power, exponent, cycle, etc.

Familiarity, as we know, breeds proverbial content: but this can be a danger for teachers. Once we have learned the specialist technical meanings, we are likely to forget that we might have ever ourselves have been uncertain what the new meanings were about. It is valuable to develop a heightened sensitivity to the vocabulary, the tools of our trade, regarding our students as both practitioners and novices, thus helping them move towards our own familiar multilingual expertise.

How do our non-English speaking students cope with technical terms that exist in the language of instruction, but which do not have effective equivalents in their mother-tongue? When non-English mother-tongue-speaking students talk about mathematics, for example, do they use their mother tongue to do this? To some extent, this may occur with arithmetic and numerical ideas, where the mother tongue's almost-everyday words for counting and numbers (and days of the week, etc.) can be used. Apart from this mother-tongue translation of arithmetic, and some almost-everyday measurement situations such as in shopping, other mathematical words (such as hypotenuse, diagonal, rectangle, hexagon, triangle, circle, sine, logarithm, area, surface, volume, length, mass and weight) may not have a mother-tongue equivalent that is useful. You will only know if you ask your students.

We should encourage our students to talk with each other, and with us, about what we are trying to teach and they are trying to learn.

There is more at stake in learning mathematics as an abstract set of concepts and technical processes than clarifying clashes between alternative meanings of words. Forming a conception, as we have seen, is itself problematic.

Edward MacNeal (1994) explores some of Piaget's ideas about young children's thinking, combined with some of Alfred Korzybski's theory of semantics, the science of meanings in language. Korzybski's key argument is that our language, through different stages of our learning development from child to autonomous adult, shapes the way we think in subtle ways usually beyond our everyday awareness. Even the very structure of the language we use influences how we think and how we use our thinking to learn.

Consider this statement: "Here is the number, 4." According to Korzybski, this is not actually the number 4; it is not even the numeral 4; it is simply an example of something we call the numeral 4. Four strokes | | | | can be represented as Roman IV, in Base-ten Hindu-Arabic as 4 or 4 or 4, in French as "quatre", in German "vier", in Base-two as 100, in Base-three as 11, using Dienes MAB blocks as four "minis", on a hand as four extended fingers and a closed thumb, and so on. Yet beneath the symbols, the intended concept-object (a mental construct abstracted out of real distinct individual events) remains the same, and unique.

Despite possible semantic confusions, we hope to communicate shared meaning with our students, building ideas we think we understand, on what seem to us to be ideas our students already possess, using deceptive examples and slippery words whose ambiguity and incorrectness we are unaware of, failing to recognise the fundamental difficulties that may arise, and, instead, talking about our students' "learning difficulties," cognitive confusion, attention deficits, (and in the face of a sentence such as this, how is your attention?) and so on.

Related issues arise when we are tempted to confuse a measurement (which is a Korzybskian symbolic statement) with the thing being measured (which Korzybski refers to as an "event"), compounded by the need to accept the approximation (due to "experimental error") unavoidably entailed in any attempt to measure, and the need to be clear about what unit is being used (which Korzybski calls an "object," a mental construct). These issues are forced on us when we try to teach students to round figures, to work realistically with numbers, to approximate and then report the approximation sensibly (pp. 140–143).

Similar issues also arise when we confront the idea of a variable, and its value. The letter C may represent some varying number of cheesecakes, in a pre-algebra context. We may have 3 cheesecakes — in which case C = 3. Students will often accept an initial letter interpretation of the C as an abbreviation of the word “cheesecake”, and hence write 3C to represent “three cheesecakes” — a valid approach in vectors, but flawed in simple algebra (I have discussed this, and other algebra “traps” in Gough, 2004).

MacNeal gives a valuable self-diagnostic test of maths-semantic competence, including: round 0.098 to the nearest whole number. Many people have difficulty rounding to zero. Why? Confusing “nothing” with “zero”, they feel that zero is not a number, because a number is “something”. As an example of MacNeal’s wit and constructive analysis, consider one of his many summary points: “Nothing is a maths-semantic problem” (p 84). Nice pun! Consider this: “How many polka-dotted zebras are in the staffroom?” Or consider these possible replies to a typical mathematics question

Q: Does the equation have a number of solutions?
A: No, only one; or,
A: No, there is no solution.

These examples of language usage (there is one solution, or the number of solutions is zero) suggest that 1 and 0 are not “numbers”. The point is Korzybskian: language is slippery — so are the concepts intended by the language.

Recommendations

1. Be alert for possible confusion in word meanings and usage.

This is one of the major problems. By the nature of learning, once we have learned something we tend to forget what it was like not to know what has now been learned. Hence, as we become familiar with technical terminology and specialist concepts, we lose sight of earlier, vaguer alternatives. Those teachers who can remember themselves struggling, as students, or recall helpful advice from their own teachers, are well placed to be sensitive to the potential struggles of their own students. Otherwise, we should listen to what our students are saying, and respond constructively to things that are wrong, only partly right, or confused.

2. Use student talking to negotiate and construct correct understanding.

It is essential that students become familiar with technical terminology and specialist concepts; they need to learn to “speak” and “do” and “think” mathematics in the way trained mathematicians do. This partly depends on students working through earlier stages of being able to put their new ideas into their own words. Sometimes this will be faulty. Sometimes the words chosen will be imprecise or unhelpful slang. In middle primary school, students often get into the wicked habit of talking about multiplying as “times-ing.” One extremely clever undergraduate friend of mine used to talk about “hitting” something with a function, when he meant substituting a value. His mathematical thinking was always correct but his explanations to less able friends were not always clear or helpful. What is needed is a progressive shaping or refining of initially rough approximations to correct usage.

3. *Examine new terms, symbols, techniques, diagrams, and technical "apparatus".*

- Is the new item clearly defined? Strong mathematical thinking and learning depends crucially on clear, consecutive definitions, supported by vivid experiences of what is defined, as well as learning what the definition does not mean.
- Is it accompanied by simple, sensible examples, alternatives, and counter-examples? This can be problematic. It is hard to introduce (or review) fractions, and present convincing examples of numbers that are not fractions, namely, irrationals. At successive stages through the developing curriculum we need to keep the curriculum as rich and honest as our students can stand.
- Does the new item depend on possibly weakly grasped sub-concepts or skills? If so, review and clarify these in direct association with the new material.
- Can simple sketch diagrams be used to show the idea(s)? If so, draw and discuss them. Verbal and symbolic learning of mathematics is greatly strengthened by visual imagery, and sometimes by concrete three-dimensional manipulatives.
- Does the new item have potentially confusing non-technical alternative meanings? For example, the mathematical distinction between "sequence" and "series" is not observed in these everyday synonyms: emphasise crucial differences.
- Are there potentially confusing similar but different concepts? For example, "volume" and "capacity." If so, examine and clarify these.
- Can the new item be directly related to existing concepts or skills? For example, is there a numerical counterpart to an algebraic expression or process? Does a three-dimensional situation have a two-dimensional counterpart? Does a verbal or algebraic concept have a diagrammatic representation?
- Does the new item involve special notation, syntax, and/or text-formatting? If so, this needs to be clearly and repeatedly explained; e.g., Greek deltas (for increments, differences) and sigmas (sums, evolving later into long-S integrals).

Consider the importance of distinguishing (even for advanced students) a handwritten multiplication symbol from a lower-case x, used as a pronumeral. Similarly, emphasise the need for careful handwriting when using exponents, superscripts, and subscripts — smaller characters, and deliberate raising or lowering, spatially, relative to the main (invisible?) baseline for writing.

It sounds trivial and/or silly to suggest that good book-keeping habits, along with good handwriting, can be important in learning and doing advanced mathematics; but it is true that slipshod penmanship, careless use of columns and rows, and poor attention to "managing and showing all (or enough) working," can make life harder than it needs to be for conscientious students.