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A Study of Kindergarten Children's Spatial Representation in a Mapping Project

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This phenomenological study examined kindergarten children's development of spatial representation in a year long mapping project. Findings and discussion relative to how children conceptualised and represented physical space are presented in light of theoretical notions advanced by Piaget, van Hiele, and cognitive science researchers Battista and Clements. Analyses of the processes the children used and their finished products indicate that children can negotiate meaning for complex systems of geometric concepts when given opportunities to debate, negotiate, reflect, evaluate and seek meaning for representing space. The complexity and "holistic" nature of spatial representation of young children emerged in this study.

Theoretical Background

Theoretical Perspectives on the Development of Geometric Thought

The National Council of Teachers of Mathematics (NCTM), in their *Principles and Standards for School Mathematics* (NCTM, 2000a), state that "geometry has long been thought of as the place in the school mathematics curriculum where students learn to reason and to see the axiomatic structure of mathematics" (p. 41). For young children (i.e., pre-kindergarten – grade 2), the NCTM *Principles and Standards for School Mathematics* Geometry Standard specifies that:

> The geometric and spatial knowledge children bring to school should be expanded by explorations, investigations and discussions of shapes and structures in the classroom… [and that] …students should use their notions of geometric ideas to become more proficient in describing, representing, and navigating their environment. (NCTM, 2000a, p. 97)

The NCTM also suggest that young children use varied representations to build new understandings and communicate mathematical ideas, and that teachers should analyse children's representations and listen carefully to their discourse to gain insight into the development of their mathematical thinking (NCTM, 2000a).

Jean Piaget and Barbel Inhelder, in their book *The Child's Conception of Space* (Piaget & Inhelder, 1948), advanced the theory that the spontaneous intuitive structures young children build correspond with structures mathematicians have developed. It is in this book that Piaget and Inhelder report their research on the development of a child's representational space.

In this extensive work, they contend that topological representation develops first in the child. Among the topological relationships described by Piaget and Inhelder are proximity, separation, order and continuity. Topological space refers to a child's ability to represent separate objects in isolation, instead of as a part of a comprehensive system within a holistic, organised layout or spatial map. Martin (1976) gives examples of topological properties as "…the interior of a set, exterior of a set, boundary of a set, connectedness…and openness and closedness of curves" (p. 27). Piagetian notions of topological relations refer to the qualitative correspondences involving proximity and separation, order or spatial succession, relative position, enclosure and continuity. Projective relationships involve perspective, and Euclidean relationships involve proportion and distance. More specifically, Euclidean ideas include coordinates, scale, relative distance, and relative size, involving comparison and seriation (Piaget & Inhelder, 1948).

Projective and Euclidean concepts derive from these topological concepts. Here, children are required to locate objects and their structures relative to one another, in accordance with perspective and projective systems. This coordination is more complex than topological representation and Piaget hypothesised that projective and Euclidian structures are usually observed later in a child's development. Piaget advanced the theory that projective and Euclidean notions are slow to appear in the realm of representation, in contrast to their development in perception. This thesis is known as the Topological Primacy Thesis.

Piaget and Inhelder have developed a hierarchical theory explaining the stages of spatial representation in which topological, projective, and Euclidean relationships are analysed. They suggest that a drawing is a representation, and that a representation implies the construction of an image, which may be different from perception. Sometimes, there is a gap between internal image and drawing as outlined in the stages of synthetic incapacity, intellectual realism, and visual realism. Table 1 presents Piaget and Inhelder's (1948) Stages of Spatial Representation.

The long-standing hypothesis of Piaget and Inhelder's understanding of children's spatial representation has been studied systematically over the years (e.g.,. Drake, 1982; Everett, 2000; Geeslin & Shar, 1979; Kapadia, 1974; Lovell, 1959; Martin 1976a, 1976b; Moyer, 1974; Peel, 1959). Clements and Battista (1992) provide a thorough analysis of this theory and related research and conclude that "…it is still possible that children show a bias toward topological versus Euclidean characteristics" (p. 424), and that:

> Overall, while not totally disproven, the topological primacy theory is not supported. It may be that children do not construct first topological and later projective and Euclidean ideas. Rather, it may be that ideas of all types develop over time, becoming increasingly integrated and synthesized. These ideas are originally intuitions, grounded in action… (Clements & Battista, 1992, pp. 425–426).

Table 1
Stages of Spatial Representation (Piaget & Inhelder, 1967)

Synthetic Incapacity	**Intellectual Realism**	**Visual Realism**
Characteristic of 3 and 4 year olds. Drawings fail to correspond with perception. Also, it is a representation of space, which neglects Euclidean relationships (proportions and distance) and projective relationships (perspectives). An example would be a young child's drawing of a person as a head with four strokes representing arms and legs, with eyes, nose and mouth, in some configuration. Drawings are representations that imply the construction of an image, which is something different from the perception itself. A child may see arms and legs attached to a person's trunk, but when the child tries to re-present these images, he/she cannot.	Characteristic of 5, 6 and 7 year olds. This refers to spatial representation in which Euclidean and projective relationships are just beginning to emerge. The interconnectedness of these relationships is not always accurate. In this stage, proximities are correct or at least attempted; separations are made among the elements of the representation; order and direction are found; the relationship of surrounding or enclosure is very important; and continuity is well defined. Elementary topological relations are demonstrated. At this level, there are the beginnings of the accurate copying of Euclidean shapes and a start made in constructing projective relationships but no co-ordination of perspective in the drawings as a whole, no understanding of proportions and especially a lack of co-ordinate systems applicable to a complex layout.	Characteristic of children approaching 8 or 9 years old. This stage refers to a child's drawings in which the beginnings of the use of perspective, proportions and distance are evident in a coordinated and simultaneous fashion. Such evidence is predicated on the existence of topological relationships and the increasingly accurate demonstration of the inter-relationships among perceptual and represented space. At this stage, what the child perceives is beginning to be represented more accurately and realistically. Objects are represented with attention to scaling/proportion, perspective/uniform points of view, and relative distance between and among them.

This research study analysed a small group of kindergarten children's mapping projects in light of Piaget's Topological Primacy Thesis with consideration of Clements and Battista's critical analysis of Piaget's thesis. Notions across the three types of geometric frameworks were considered in analysing the children's spatial representational work. While the researchers (the two authors) studied individual children's work, the focus of the study was on the changes that occurred in the group work as a result of group processes.

Other theoretical perspectives that can be simultaneously considered with Piaget's perspective come from studies related to children's drawing. Similar to Piaget's hierarchical theory of spatial representation, as illustrated in Table 1, according to Kellogg (1969, 1979), Lowenfeld and Brittain (1947), and Isenberg and Jalongo (2001), children's drawing can be categorized as *nonrepresentational*, meaning that it does not look like the item being represented, or *representational*, meaning that it does resemble the item being depicted. Isenberg and Jalongo (2001) indicate that these stages correspond roughly to two stages in writing: *pre-alphabetic* (e.g., scribbles, shapes, mock letters) and *alphabetic* (e.g., writing that looks like letters of the alphabet). Table 2 presents the developmental relationships among children's drawing, writing, and Piaget's theory of young children's development of spatial representation.

Table 2
Developmental relationships among children's drawing, writing, and Piaget's theory of spatial representation.

Nonrepresentational		**Representational**		
Pre-alphabetic			**Alphabetic**	
Scribbling stage 1	Scribbling stage 2	Scribbling stage 3	Emerging Representationa, Stage 4	Representational Stage 5
Approximate chronological age: 2–3 years Explores media through senses. Can make random marks on paper. Can beings scribbling.	Approximate chronological age: 3 years Explores and manipulates objects/materials /tools. Can make scribbles on top of each other. Scribbles are more controlled with some sense of directional movement. May cover paper with layers of color. Process (not product) is important/ meaningful to the child.	Approximate chronological age: 3–4 years Perceives shapes (maybe with names) in work. Attempts to make shapes. Often names one's own scribbles.	Approximate chronological age: 4–5 years Can combine two shapes with an intention or an agenda (e.g, a circle and a cross to make mandalas. Can draw "suns". Drawing represents humans as a circle with stick arms and legs, a "tadpole" person. Figures appear to float on the page (2–D space). Can represent feelings and ideas through art.	Approximate chronological age: 6–8 years and older Child's art work clearly resembles whatever it represents. Baselines beings to appear in drawings. Exhibits more preplanning and inclusion of details. Strives to master various art skills and begins to evaluate own work. Work tends to be more realistic in terms of proportion and arrangement.
		Synthetic incapacity	**Intellectual realism**	**Visual realism**

As illustrated in Table 2, theoretically, young children's spatial representation may start to emerge during the early stage of representational drawing along with the emerging stage of alphabetic knowledge.

Following Piaget, another significant contribution to understanding the development of geometric thinking was advanced by Pierre and Dina van Hiele (van Hiele, 1986, 1999). They suggested that children progress through levels of thinking when engaged in geometric inquiry, and that for young children, geometry begins with play (van Hiele, 1999). The van Hiele theory of geometric thought includes five levels. Table 3 presents Van Hiele's levels of geometric thinking.

Table 3
Van Hiele's descriptions of children's levels of geometric thinking.

Visual	This level begins with nonverbal thinking. At the visual level, figures are judged by their appearance. "It is a square because it looks like one." Attributes or properties of shapes are not thought about. Determination of a shape is made in a global, holistic way.
Descriptive/Analytic	At this level, figures possess their properties. Children recognise and categorise shapes by their properties. Properties are established experimentally by observing, measuring, drawing and model making. Students however, do not see and cannot explain the relationships between classes of shapes. At this level, language is important for describing shapes. However, properties are not yet logically ordered, so, to a child at this level, a triangle with equal sides is not necessarily a triangle with equal angles.
Informal deduction	At this level, children can form abstract definitions, classify hierarchically and pose informal arguments to justify classifications. Children can deduce properties of figures and express interrelationships both within and between figures. Understanding inclusion relationships among classes of shapes is demonstrated. However, the role of axioms, definitions, theorems and their converses are not yet understood.
Formal deduction	At this level, students are capable of creating formal deductive proofs. They can conceptualise about relationships between properties of shapes and are able to construct formal proofs. Such skills are necessary for successful work in a high school geometry course.
Rigor	At this level individuals are able to compare different axiomatic systems. This level is usually attained by mathematics majors and mathematicians.

According to Van Hiele's description of children's levels of geometric thinking, most children in grades kindergarten through to grade 6 [ages 5–12] are at the visualisation or analysis level; some middle school children [ages 12–14] are at the informal deduction level; and successful completion of a typical US high school geometry course requires an understanding at the formal deduction stage. However, in studies of American children's geometric knowledge, the National Assessment of Educational Progress (see NCTM, 2000b) report noted that most students across the grade levels tested [i.e., grade 4 (9–10 year olds), grade 8 (13–14 year olds) and grade 12 (17–18 year olds)] appeared to be performing at the "holistic" level of visualisation. The children in this study, as kindergarten children, are within the *visual* level of the van Hiele model. Clements and Battista (2001) challenge van Hiele's theoretical perspective related to geometry in stating that "conceptualizing geometric growth as being strictly visual, then strictly descriptive/analytic, then logical, and so on, is neither accurate nor optimal for educational theory or practice…" (p.129–130).

Other researchers have studied, analysed, and validated the levels of geometric thought proposed by Pierre and Dina van Hiele (e.g., Burger& Shaughnessy, 1986; Gutierrez & Fortuny, 1991; Mayberry, 1983; Wilson, 1990). However, it is suggested that within this hierarchical structure of levels, not all people use a single level of reasoning at one time. Sometimes, several levels are evident at the same time. The notion was reported that people do not behave in a simple linear manner, which the van Hiele model of geometric thinking suggests.

Clements and Battista (1992) suggest that there is another level of geometric thinking that develops before the van Hiele level of visualisation. At this level, children perceive geometric shapes, but attend to only one aspect of a shape's visual characteristics. For example, children may differentiate between figures that are curved and those that are angular, but not among figures in the same class. They postulate a *pre-recognition* level where children perceive geometric shapes by attending to only a part of the shape's characteristics and state that:

> …Children perceive geometric shapes, but perhaps because of a deficiency in perceptual activity, may attend to only a subset of a shape's characteristics. They are unable to identify many common shapes. They may distinguish between figures that are curvilinear and those that are rectilinear but not among figures in the same class; that is, they may differentiate between a square and a circle, but not between a square and a triangle. Thus, students… may be unable to identify common shapes because they lack the ability to form requisite visual images. (Clements & Battista, 1992, p. 429).

In summation, according to Clements (2000) and Clements and Battista (1992, 2001), the existing theories—Piaget's Topological Primacy Thesis and Van Hiele's description of children's level of geometric thinking—about children's stage-oriented and linear developmental characteristics of spatial and geometrical thinking, and representational skills have not been fully favourable in understanding the complexity and holistic spatial representational skills that young children develop and are able to use in their two and three dimensional representational work such as mapping. Yet, it is clear that both Piaget and van Hiele, in their developmental approaches to cognition, emphasise the active construction of knowledge and the importance of children building relationships among geometric concepts and processes. It is the construction and abstraction of these relationships that is important in the personal meaning making for children in their mathematical thinking.

Most studies that have dealt with young children's spatial and representational skills were designed to measure children's conceptualisation of space by giving tests, conducting clinical interviews, or engaging children in haptic perception or visual tasks (Darke, 1982; Gutierrez & Fortuny, 1991; Kapadia, 1974; Martin, 1976a, 1976b; Mitchelmore, 1980; Moyer, 1974; Wilson, 1990). These studies have not revealed children's deep conceptual understanding of geometric ideas and relationships, nor have they captured in depth the naturalistic phenomena young children engage in and create as their emergent interests lead to complex levels of constructing spatial representations. As Clements (2000) discusses, geometric thinking can develop in a variety of naturalistic learning experiences, although cognitive and physical development also play a role. Clements and Battista (1992) state that "research is needed to identify instruction facilitating the construction and selection of increasingly sophisticated reference systems for organizing spatial information…" and that "…more articulated research from a constructivist position is needed" (p.426).

In this study, we attempted to explore a group of kindergarten children's emerging spatial representational understandings through a year long mapping project built into their kindergarten curriculum. The study was designed to capture the naturalistic phenomena of how a group of young children collaboratively sought meaning and used basic geometric intuitions to solve problems related to constructing maps of familiar space.

Research scope and research question

In this study a group of kindergarten children's development of spatial representation was explored as they were engaged in conceptualising and constructing a wide variety of maps in their classroom and in a computer-based, technology-rich environment. The study was particularly aimed at examining the evidence of spatial representation as the children collaboratively used materials and computers as tools for developing concepts and theories about mapping. Rich opportunities emerged for them to engage in conversation, negotiation, conceptualisation, and construction of two and three-dimensional maps.

Specific interest was given to investigating how the children conceptualised, debated, sought meaning for, and represented physical space through their two-dimensional constructions, drawings, and pictorial space, and their three-dimensional constructions. Based on their tactile or visual perception, the children constructed a variety of maps depicting their immediate physical space (classrooms, corridors, school building, swimming pool, gymnasium, track, planetarium, technology-rich classroom, etc.) and part of the campus of the university setting (route from the school to the technology-rich classroom and a map of the front part of campus, called The Loop). Children were presented with these tasks of map building after many conversations about where and how they have seen maps used in their daily lives.

Notions of spatial representation analysed in the children's map constructions were: topological relations involving proximity and separation, spatial order, enclosure and continuity; projective relationships involving perspective; and Euclidian relationships including proportion, distance, and relative size. Although Piaget and Inhelder have considered these topological, projective, and Euclidean relations separately, they are inter-related and become progressively integrated and synthesized concepts. In analysing and interpreting the data, the researchers considered Clements' (2000) and Clements and Battista's (1992, 2001) critical perspectives on the complexity and holistic aspect of the development of children's spatial representation. It is important to understand that projective and Euclidean space are very much related to, and are derived directly from, topological relationships. However, children construct projective and Euclidean representations of space as they incorporate their perceptions within a system of relationships and viewpoints. Thus, the main focus and research question in this investigation was: How do children construct their own understanding and representation of space? Specifically, how do children set about making a plan or layout of a familiar spatial environment? How do children, individually and collaboratively, conceptualise and represent proximity, perspective, projective relationships, and Euclidean relationships of proportion and distance in constructing representations of real space?

This was a social phenomenological study attempting to explore and examine the development of spatial representation in young children as they worked with a variety of media to conceptualise and construct representations of physical space. The research methodology is similar to the approach used in Gerhardt's (1973) work about young children's spatial orientation. In phenomenological studies, researchers attempt to understand participants' perceptions, perspectives, and understandings of particular social realities (Leedy & Ormrod, 2001; Schutz, 1970). Shank (2002) explains that the phenomenological process begins with a person's awareness of the world. As researchers in a phenomenological study, we sought to uncover the meanings about geometric concepts and ideas that were constructed by a group of young children. As the rich contexts of the children's thinking, processes, and constructions are analysed, new knowledge about how children conceptualise and construct understanding of their physical world can be gained.

Research Methodology

Participants

A group of 18 culturally and linguistically diverse and inclusive kindergarten children (nine boys and nine girls), aged five or six, were participants in the study. The children were attending a child development centre at a northeast Ohio university in the United States. The research team participants included university academic staff, centre teachers, and the centre director. Kindergarten marks the year before grade one. In the US, children typically begin kindergarten at the age of five, or at least turn five years of age within the first two to three months of attendance. Three of the children in this study did not speak English as their first language and one child had a documented learning disability. All children were similar with respect to socio-economic status; they were all from middle class families.

Classroom settings

As a part of a year long mapping project, two different classroom environments were used. One classroom setting was the kindergarten children's regular classroom; the other was a technology-rich classroom environment at the same university. The kindergarten curriculum was designed to meet the criteria of a developmentally appropriate program set forth by the National Association for the Education of Young Children (1996). The kindergarten children had one lead teacher, one assistant teacher, and one art teacher. In the kindergarten, the children had a wide variety of media to explore and use in their construction, including: blocks, paint, clay, paper, markers, yarn, construction materials, and a small number of computers equipped with word processing, language and paint programs, and Internet access.

The technology-rich University Classroom of the Future, called Ameritech Classroom, is designed as a place for K-12th grade students and their teachers to work on class-defined projects utilising the latest technology. This setting includes twelve networked computers with Internet access, a scanner, colour printers, video conferencing cameras connected to several computers, digital still frame cameras, camcorders, word processing tools, a variety of reference books, art supplies, and two large tables for group work. At the front of the room is a central instruction area with a computer, large projection screen, a VCR, and an Elmo video document camera.

In the Ameritech Classroom the children worked in pairs at the twelve computer stations. The software available included: Microsoft Word, AlphaSmart, Golden Books Encyclopedia for Kids, Community Construction Kit, KidPix, Neighborhood MapMachine, Microsoft PowerPoint, Intel Video Still Camera Movie Maker, Diorama Designer, and Paint Shop.

The Ameritech Classroom is designed for research. An observation booth is adjacent to the classroom and is equipped with a panel of one-way mirrors and video recording devices to capture images of the children working. The cameras that are mounted in the Ameritech Classroom are situated in four strategic places so that researchers can observe, record, and study, with state of the art technology, how students and teachers work within the context of their topics of exploration.

Role of the teachers and the researchers

The role of the researchers in this project included teaching and observing. With respect to teaching, the kindergarten teacher and the art teacher worked with the kindergarten children from the beginning of the school year and introduced them to a vast range of varied media for creative art expression and construction. The children were given ample opportunities to explore different media and techniques to create their own works of art. The teacher often provided provocations and prompts to guide the children's individual and collaborative investigations and creations. For example, the art teacher would give the children opportunities to "map" objects and space in the children's immediate environment based on their emergent interests, such as a bookcase, a door, or a wall. After experience with mapping some focused and immediate objects, the teachers provided the kindergarten children with a challenge to make a map of their classroom and then of their building. The role of the teachers was to first introduce the tasks of map making, then to provide the media and encourage problem solving, analysis, and approaches to the mapping investigations and constructions.

With respect to observation, the teachers and the researchers observed and interacted with the children to gain information about their thinking. In order to engage in the experiences with the children and to capture the process of children's sense making about mapping, the researchers participated with the children weekly in the first semester and daily in the second semester. There, they worked alongside the kindergarten children and the teachers in the classroom in an attempt to observe, talk with, interview, assist, and learn during the full academic year of 2001–2002.

Data Collection

From the two different classroom settings, the researchers collected documentation of the children's drawings and sketches, photographs of block construction representing the kindergarten children's classroom, photographs of a 3-D tabletop map construction representing the front part of the campus called The Loop, classroom and small group conversations, individual children's journal writings, parent meeting notes, video and audio tapes of children's work in both classrooms, researchers' field notes, and computer graphics.

For the first five months (September, 2001–January, 2002) the children were engaged in a mapping project in their regular classroom using multiple forms of representational tools such as blocks and two-dimensional drawing tools (paper, pencils, markers, crayons, paint brushes, etc.). This project work was typically done daily for approximately one hour. During the two months of February and March 2002, the children were in the technology-rich classroom from 12:30–2:30 p.m daily. In this technology-rich Ameritech classroom the children were exposed to a variety of mapping software and hardware as tools to support their mapping project. In the mornings for approximately one hour, they were in their regular classroom continuing the expansions of their mapping projects using various software and hardware as tools, in addition to the same multiple forms of representational tools that they had been using since the beginning of the school year. From March 25 to May 10, 2002, after the Ameritech classroom experiences, follow-up observations were made, and documentation of the children's learning experiences in their regular classroom setting were collected.

Data Analysis

Data were collected and analysed qualitatively. The researchers looked for evidence of the children's levels of sophistication in their many representations of space. The data were coded according to the types of spatial representation used by the children, including topological, projective, and Euclidean relationships (see Table 4). The main data analysis techniques used were open, inductive, axial, selective coding (Strauss & Cobin, 1990), and data reduction (Miles & Huberman, 1994, Shank, 2002). The focus of the data analysis was to find emerging patterns of the kindergarten children's meaning making about spatial representation, in terms of the related research questions: How do children set about making a plan or layout of a familiar spatial environment? How do children conceptualise and represent proximity, perspective, projective relationships, and Euclidean relationships of proportion and distance in constructing representations of real space?

Open coding was used to analyse and categorise the data. This coding system was used as the first procedure for the data analysis. As part of early categorisation in the open coding, Piaget's different spatial representational qualities (see Table 1) were referenced to identify how children's spatial representation changed during the year long mapping experiences. In this process, the children's work was counted to produce descriptive data. These descriptive data (e.g., frequency) were used to search for patterns from the children's work in the next stage of data analysis. Axial coding was a set of procedures whereby data were put back together based on the research questions to see what made up the children's representation of space. Selective coding was the process of selecting the core categories of spatial representation, systematically relating them to other categories, and filling in categories that needed further refinement and development for theory building. As a result of the selective coding, we were able to classify the children's individual and collaborative work in terms of Piaget's stages of spatial representation and van Hiele's levels of geometric thinking. Table 4 presents the results of the selective coding. In the section on findings, the characteristics of the children's geometric thinking and mapmaking that were uncovered are discussed.

This qualitative study incorporated several strategies that Merriam (1988) outlined to achieve internal validity and reliability. The researchers used triangulation with multiple sources of data (e.g., researchers' field notes, teachers' reflective notes, class daily reflection notes, parents' meeting notes, children's mapping work in drawings, photographed three dimensional table top map, and block construction, etc.) and engaged in theoretical triangulation by examining multiple perspectives of the children's geometric thinking (Denzin, 1970; Denzin & Lincoln, 1994; Merriam, 1988). The theoretical triangulation was based on the theories discussed in the literature review above. The process of peer examination regarding the analysis of the children's work was conducted during the entire study.

Results

The children's work was analysed according to the descriptive geometric contexts of Piaget's stages of spatial representation – synthetic incapacity, intellectual realism, and visual realism; the van Hiele levels of geometric thinking; and the theoretical ideas advanced by Clements and Battista. The data, representing a year long process, included maps of familiar, immediate spaces such as the classroom, classroom door, swimming pool, planetarium, track, gymnasium, skating rink, etc., and maps of expansive spaces not able to be viewed all at once, for example, a map of the corridors within their building, the route from their school building to the Ameritech Classroom, the map of the front of campus. It is important to note that this year long mapping project emphasised social-constructive group learning processes. This study did not directly aim at analysing individual children's developmental changes; rather it focused on changes that occurred in their group work as a result of group processes. Table 4 presents a summary of the data. See Figures 1a-1d, 2a, and 4a for representative samples of the children's work.

Table 4
Descriptive data of the frequency of types of spatial representation across the school year:

Spatial Representation Contexts	**Fall Semester: September – December**	**Spring Semester: January – May**
Topological Structures of proximity, separation, order, relative position, enclosure and continuity. Each figural object is considered in isolation.	<u>2-D representational work presented in independent sketches and drawings:</u> Two-thirds of the children's drawings and sketches contained objects and people drawn with little coordination of spatial relationships or inter-relatedness of the contexts of the subject they were trying to draw.	<u>2-D representational work presented in independent sketches and drawings:</u> One-fourth of the children's drawings and sketches continued to contain objects and people drawn with little coordination as well as displaying no inter-relatedness of the subject they were trying to draw.

		However, more attempts were made to represent these relationships.
Projective Structures: General perspective where objects/patterns are not viewed in isolation but are considered in relation to a "point of view" of the person or of an object; there is an inter-coordination of objects separated in space.	2-D representational work presented in independent sketches and drawings: One third of the children's drawings and sketches captured the inter-relatedness and inter-coordination of objects and people in the space. Some of the computer generated pictures and maps clearly demonstrated this coordination.	2-D representational work presented in independent sketches and drawings: Although not always accurate, close to three fourths of the children's drawings and sketches captured the inter-relatedness and inter-coordination of objects and people in the space. Many attempts were made to use enclosures and to represent the relationships of objects in the correct order. Pictures evidenced attempts of accurate copying of shapes, direction and connectedness.
	3-D representational work presented in a collaborative project of block construction of their classroom: Successful attempts to coordinate each structure and facet of the classroom were exhibited. All aspects of the constructed classroom reflected an interconnectedness and fluency of structure and directional placement and orientation.	2 and 3-D representational work presented in a collaborative project as a map of the front campus loop: Successful completion of a three dimensional representation of the front campus was exhibited. All buildings were placed in the correct order and directional orientation. The landscaping was constructed to represent actual trees, vehicles, traffic signs, and bushes. There was an intricate inter-coordination of objects within the space.
Euclidian Structures: proportion/relative size, distance/relative distance, measurement, coordinates, and scale.	3-D representational work presented in a collaborative project: of block construction of their classroom. Children attempted to represent relative size and shape of the classroom with respect to proportion, relative size, relative distance, and measurement; primitive attempts to represent scale.	2 and 3-D representational work presented in a collaborative project as a map of the front campus loop: Scaling the trees, cars and bushes according to the large sizes of the buildings was attempted. Children has some rudimentary notions about proportion but had difficulty capturing the proportional relationships. Difficulty was experienced in seriating the sizes of the buildings. Comparisons were made in pairs of buildings, bigger and smaller; but coordination and interrelationships among three buildings could not be done.

Piaget and Inhelder's model of spatial representation describes the development of two-dimensional constructions. The contexts of their model, which include ideas of topological, projective and Euclidean space, are related to three-dimensional constructions as well. As indicated in Table 4, at the beginning of the school year when the children worked independently, they produced spatial representational work that indicated the end stage of Piaget's synthetic incapacity and some emergent evidence of intellectual realism (refer to Tables 1 & 2). However, during the same time period, their collaborative work in the construction of a representation of their classroom with blocks revealed complex coordination and integration in their geometric thinking. This work reflected the strong intellectual qualities of Piaget's intellectual realism, such as projective and Euclidian spatial representations. This particular quality of intellectual realism also supports van Hiele's level of descriptive and analytic geometric thinking (refer to Table 3). At this van Hiele level, children construct geometric concepts and ideas by making models, observing, measuring, and drawing. He also suggested that, at this level, language is important for describing shapes. However, his theory contends that children do not see and cannot explain the relationships between classes of shapes. Yet, in this research, it was found that through a rich contextualised group process, and through the use of multiple representations, the children established meaning for conceptualising and constructing complex geometric shapes and for representing elementary Euclidian and projective space. It was the continuation of the group process and dialogue that improved the children's geometric thinking and spatial representation. The children's resulting constructions exhibited evidence of visual realism (see Table 1). Examples of the phenomena of a group process appear in the findings related to growth in spatial representation in sketches and drawings, and growth in representation of three-dimensional space.

Findings and Discussion

Finding 1. Growth in spatial representation in sketches and drawings

The children's sketches and drawings, done independently, demonstrated a clear progression in geometric complexity across the year. During the first semester of the school year, the ratio of work exhibiting purely topological structures to work beginning to exhibit projective and Euclidean structures was approximately 3:1. In the following semester this ratio changed to approximately 1:4. Children's sketches and drawings gradually took on characteristics of projective and Euclidean relationships, that is, exhibiting Piaget's intellectual realism. This type of spatial representation is typically characteristic of five, six, and seven year olds (Piaget & Inhelder, 1948). The children began with their perceptions and observations of many places and spaces, such as a bookcase, their classroom, school hallways, a swimming pool, a running track, a bus route, and a small part of the college campus. They were guided by their teachers to use tools such as observing, touching, walking, sketching, and describing the space. They were also given tools such as digital cameras, video cameras, software, blocks, and writing implements to support their map making. Teachers helped the children learn how to use the tools so that they could work independently in their map planning and construction. The children were continuously encouraged to reflect and debate on their constructions as they developed a conceptual framework for their work.

Figures 1a, 1b, 1c and 1d provide representative samples of sketches from the beginning of the school year. These sketches can be also classified as Kellogg's (1969,1979) emerging representational stage where children can combine shape with an intention to represent what they see and what they are trying to draw. The drawings of people, for example Figures 1a, 1b and 1d, show people with stick arms and legs who appear to float on the paper, depicting more of a two dimensional representation than a three dimensional one in which perspective is captured. The portrait representations, Figures 1a and 1b, characteristic of Piaget's synthetic incapacity stage, do not match an accurate perception of people; bodies drawn out of heads and, in general, incomplete bodies can be seen. The map representation, Figure 1c, the sketch of room 6, is also an emerging representational stage. Although it lacks elements of projective and Euclidean space, there is some order in its representation, and boundary lines are drawn depicting enclosure. Figure 1d, the representational sketch of a child in a swimming pool, neglects Euclidean relationships such as proportion and relative distance. In this sketch, the child can be seen to be taking up the whole area of the pool. This sketch also neglects projective relationships involving perspective. Such representations can be classified into Piaget's synthetic incapacity stage, where children's drawings do not correspond to actual perceptions. These sketches were completed independently within the context of small group activities.

Figure 1a

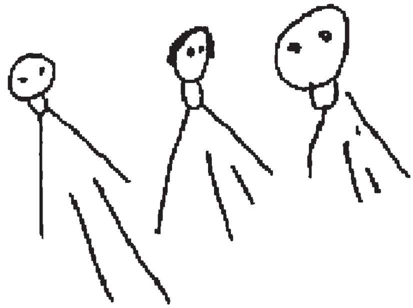


Figure 1b

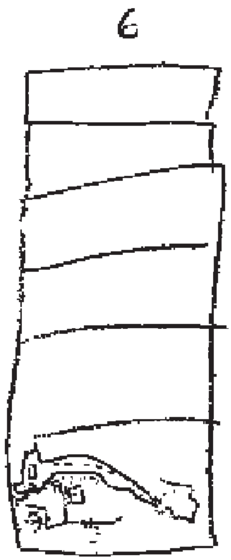


Figure 1c

Figure 1d

Figures 1a, 1b, 1c, and 1d. Samples of kindergarten children's drawings at the beginning of the school year.

The classroom teacher reflected on the power of group work along with how drawing two-dimensional representations informed the three-dimensional block construction map of the classroom. As the children worked together to construct the various maps, they debated with each other, listened to each other's ideas, and tried to construct maps that represented the perceived space. The teacher allowed the children to work through and build meaning for the block construction process without any direct instruction or transmission of information.

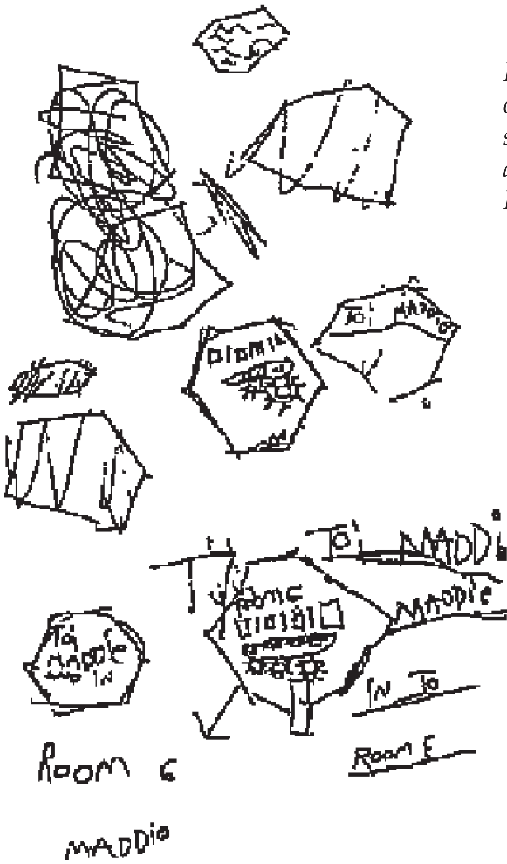
From the following teacher reflection notes, it is clear that the approaches the teachers took facilitated the children's natural thought processes for organising the spatial information. As the children examined the boundaries of the classroom, seeing the observation booth jutting out in a trapezoidal shape challenged them. Walls that were straight in the classroom were easily translated into straight blocks; but the trapezoidal protrusion of the observation booth caused much discussion. To the teacher's credit, she did not direct the children to look at the corners and replicate the shape with blocks; instead, she guided them to observe the booth from many perspectives and then collectively they decided how to figure out how best to represent it with appropriate blocks of their own choosing.

> Teacher's reflections about: How to support the process that the children initiated. She helped the children to decide what approaches would be helpful in their task.
>
> November 1, 2001:
> *Sketch/map out [the] room to help with concept and children's thinking.*
> *Blocks—use platform—how do you make room 6 out of blocks? How high? Ceilings?*
> *[The children] count steps in hallway/classroom to use as unit of measure.*
> *[one child] built wall out of blocks.*
> *Which materials afford map making [and] spatial relationships?*
>
> Teacher's reflection notes, November 5, 2001:
> *One group drew pictures of our classroom walls and doors and windows. They are getting ready to build on the platform.*

At this point in the process, the children indicated that they were ready to represent the perimeter of their classroom, feeling and seeing the straight walls. However, the observation booth posed its own challenge:

> Teacher's reflection notes, November 8, 2001:
> *The children were [ready] to build a model of our kindergarten room using blocks. [Straight walls were represented with rectangular blocks]. They had a hard time figuring out the shape of the observation booth, so we thought that by drawing a picture of the booth they would be able to figure out how to build it using blocks. When the children were inside the booth, they couldn't figure out the shape until one child said it was a special kind of circle. After that they were able to look at the shape differently and draw it.* [The special kind of circle was constructed with curved blocks. Upon more analysis, the children agreed that it did not "look like" the booth.]

The children were perturbed with the incongruence between the actual observation booth and the curved structure they built. Although topologically similar, the children knew that curves did not represent the actual contours of the booth. Figure 2a is a drawing of the observation booth in the kindergarten classroom. In this sketch, many attempts to replicate the trapezoid shaped protrusion of the observation booth can be seen. Here is the beginning of accurate copying of a Euclidean shape, representative of Piaget's intellectual realism stage. Projective and Euclidean relationships are just beginning to emerge as a kindergarten child attempts to place the people around the table in the booth facing the windows. Although this work did not demonstrate accurate proportional relationships, the placement of objects and people was correct. The door was also in the correct orientation (indicated by the arrow).



It's this shape – special kind of circle. Special circle – the shape was hard to draw…look at all the shapes I made till I got it right.

Figure 2a. A child's iterations of the drawing of the observation booth in the kindergarten classroom (November 2002).

The following conversations present how the children approached and negotiated meaning for constructing the three-dimensional representation of the trapezoid-shaped observation booth in their classroom:

> As one child observed the trapezoid-shaped observation booth, she said,
> *[It is] a special kind of circle in walls.* [Research notes 11/6/01]
>
> *It's kind of like square, but it's round kind of circle.*
> *The shape was hard to draw…look at all the shapes I made till I got it right* [Research notes 11/8/01]
>
> The children shared this statement in writing about their work [Daily Reflection note November 8, 2001]: *The building and drawing group went into the observation booth to look at the shape. Everyone discovered a different way to draw the shape.*

Children responded to a description of the trapezoid-shaped observation booth:
…straight here and then a curve
…part to make it curved
…the curvy wall
…big blocks for wall and circles for [observation booth]…
…like a circle
…it's curvy
…the shape that it is and the curves in it
… [it's] hard to make observation booth because hard to make the curve part …

When the child stated that the shape of the observation booth was a "special kind of circle", the children selected curved blocks to construct the observation booth.

This representation did not capture what they "knew" the observation booth looked like. They agreed that this representation of the booth was not right. Throughout their conversations, debate, and attempts to find meaning for representing the shape of the observation booth, the teacher continuously, and indirectly, helped the children to discover the characteristics of the trapezoid shape. She suggested that they re-visit the observation booth, study it, and then describe what they saw in comparison to the curved structure that they built. This type of constructively-guided instruction is essential if children's natural development of spatial representation is to be supported. An environment that promotes multiple representations and dialogue with interesting provocations encourages children to seek their own meaning. In this research, the continuation of group processes helped the children solve the complex geometric problem. This finding speaks directly to Clements and Battista's (1992) charge, mentioned earlier, that "research is needed to identify instruction facilitating the construction and selection of increasingly sophisticated reference systems for organizing spatial information…More articulated research from a constructivist position is needed. (p. 426)

It is interesting to note that in constructing the trapezoid shape shown in Figure 2a, the child drew its reflection and constructed a hexagon. This discovery occurred during their regular group work in map making. Thus, her discovery became a part of the group's knowledge in the attempt to figure out how to represent the shape of the observation booth. In addition, as Isenberg and Jalongo (2001) indicated, children who are at the representational stage also exhibit emergent writing skills, as part of their drawing, which is infused into the representational work. The child's writing served as a tool for documentation of the drawing record—what it was and directions. Figure 2a also represents that particular characteristic.

On the same day as this breakthrough sketch occurred, another kindergarten child helped to conceptualise the special trapezoid-shaped protrusion. The child traced and measured the observation booth by touching across each surface in a linear motion. He said, in front of the group members, "Straight and up". As the child discussed the characteristics of

[It is] a special kind of circle in walls.

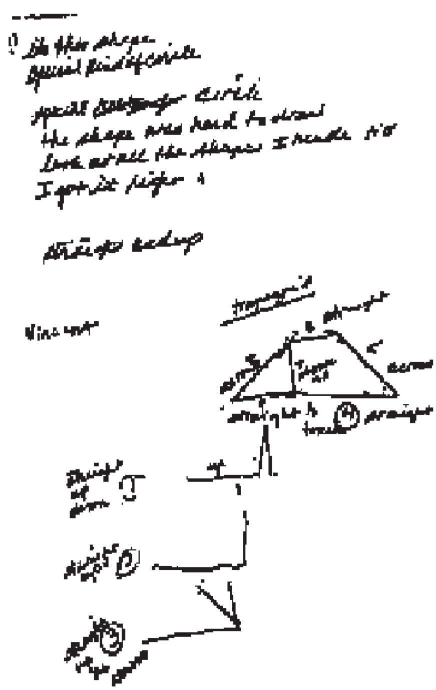


Figure 2b. Teacher's documentation of a child's exploration of the observation booth in the kindergarten classroom (November 2001).

the trapezoid-shaped observation booth, he was able to find and share his own words to describe his new geometric discovery. Then, he followed the shape and tried to replicate it on paper. Figure 2b presents the teacher's documentation of his work.

This evolution of representing the trapezoid shape with curves, to representing it with straight sides was a long, deliberate, collaborative process. Certainly, after seeing the children's struggle with this construction, it would have been easier for the teacher to show the children how to construct this shape. However, she knew that deeper understanding would be gained by the children solving the problem themselves. After two weeks of exploration, reflection, and negotiation, the children were able to correctly build an accurate representation of the observation booth – see Figure 2c (Kindergarten Room 6 – a block construction of the observation booth).

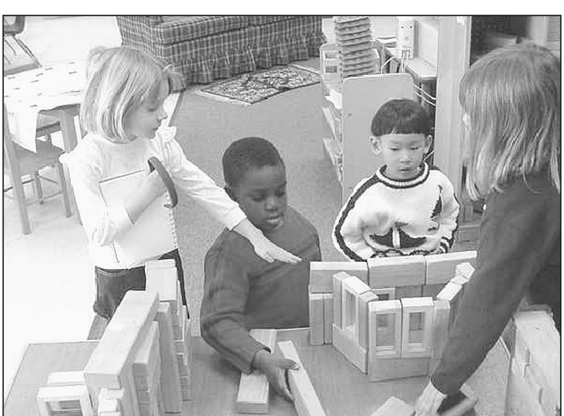


Figure 2c. Kindergarten Room 6 – a block construction of the observation booth.

Finding 2: Growth in three-dimensional spatial representation

The children's collaborative block construction of the classroom replica (see Figure 3) and The Loop of front campus (see Figures 4a and 4b) also represent advanced developments in geometric thinking. In the block construction of the classroom, the map of the room was first studied and planned with sketches of each part of the room, that is, the wall, the door, the window section, the observation booth, and the teacher's office. In addition, the children demonstrated in their representations of space and in their constructions, that they were representing increasingly complex geometric concepts involving projective and Euclidean space. By walking, touching, describing and looking at the space from different perspectives and angles, the children were able to construct it with blocks. Clements and Battista (2001) maintained that:

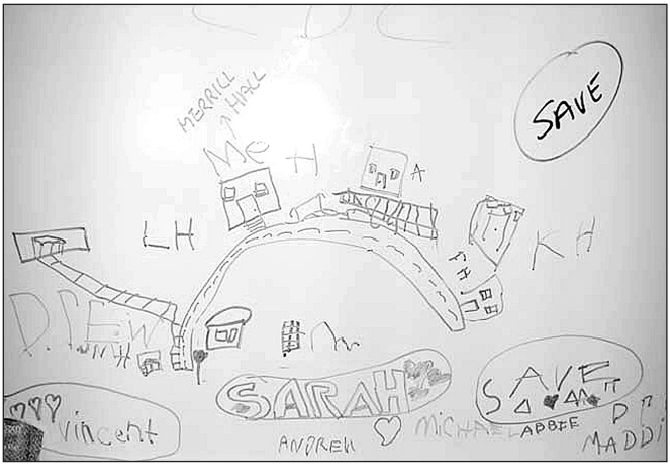
> A variety of geometric ideas appear to develop over time, becoming increasingly integrated and synthesized. Certainly, some Euclidian notions are present at an early age. Originally these ideas are intuitions grounded in such actions as building, drawing and perceiving. Children might develop actions that produce curvilinear shapes before they develop those actions that produce rectilinear shapes. Even young children have basic geometric intuitions that might be productively built on in the elementary school classroom (p. 3).

This study is validated by this notion of early competence in children's basic geometric intuitions which were cultivated by the social experiences of map making. It may also shed light on the reasons the children first represented the trapezoidal observation booth with curved blocks.

Figure 3. Children's collaborative block construction

Frequently, the children were leading pedagogical decision-making by probing, giving or suggesting ideas of what to do or how to do it, in order for them to pursue their objective of making maps that were meaningful and understandable to them. When the children were making the three-dimensional version of The Loop of the front campus (see Figures 4a and 4b), they went out to The Loop five to six times to walk it and capture the real images of it with cameras before and during the map-making project. This process is the same process that cartographers (professional map makers) use to construct maps. Using digital cameras, video imaging cameras (e.g., Intel Video Still Camera Movie Maker), and sketchbook drawings, they captured the real images of the area and brought them back to the classroom. They downloaded the images into the computer and then began to conceptualise the space. The children worked in teams to analyse the photographs and computer generated images of buildings, vehicles, people, and landscaping. The computer generated printouts were made with map-making software (e.g., Community Construction Kit), or map-drawing software (e.g., Neighborhood MapMachine). In that process, they faced several conflicts in problem solving which prompted them to explore and suggest new approaches and ideas to the teacher as well as among themselves (field notes, March 4 and March 8, 2002):

Sasha:	Why are there two buildings?
Vic:	It (the pictures) didn't turn out clearly.
Daniel:	Take the picture again.
Vic:	Why do pictures always get mixed up? It (a building's location on the 3-D map) was up hill. But, It's supposed to be down hill.
Koonwah:	Why do we have two places for Moulton Hall?
Taro:	It has to be in The Loop.
Jen:	Up there! Turn around, turn around on Mae's side.
Jane:	We could look [at it again] when we go out.
Pilmook:	It needs more signs.
Nakil:	Maybe we can have more buildings and signs. It will help us.
Sasha:	Maybe we could look at the map [meaning the Campus Map for reference] or go outside and take a picture and put them on the map. And look at the map and see where it goes.
Vic:	The Loop is like this. I want to draw it on here. (He decided to draw the map of The Loop on the white board, See Figure 4a)
Daniel:	I think we should start all over [It] again.
Jane:	We made the building bigger because the car was bigger than the building. We had to make the building bigger than what you made at first. [Correcting the scale of the objects that will represent the real scale of the buildings and the relationship with other objects around the building]



The Loop is like this. I want to draw it on here [white board].

Figure 4a. White board drawing of The Loop: two-dimensional

We made the building bigger because the car was bigger than the building. We had to make the building bigger than what you made at first.

Figure 4b. The Loop of front campus: three-dimensional

In their class meeting, following these deliberations, the children discussed the relative size, scale, and proximity of objects within the environment that they were representing. This evidence is indicative of Piaget's intellectual realism stage and the emerging conceptualisation of visual realism (see Table 1). They used words such as "tallest" and "next to" as they were looking at the objects in the three dimensional model. As the teacher was listening to the children's discussion, she used the word "proportion" to help the children to anchor their developing concept of proportional relationships: "[you] may want to re-think the size [of the buildings and vehicles] to be more in proportion" (March 12, 2002).

The children exhibited their capabilities of comparing their work (maps) with their original plans (map designs) or comparing them with the purposes or intentions as detailed in the child's brief thinking processes (Solomon & Hall, 1999; Fleer, 1999). As Fleer (1999) indicated, this is clear evidence of the kindergarten children using their meta-cognitive thinking, which requires higher-order thinking skills. As Anning (1997) mentioned, the recursive cycle from images generated in the mind's eye to their explication in speech, gesture, writing, models, or drawings is the essence of [map] designing. The kindergarten children's work of map-making took place in a recursive mode: talking about making a map; going outside to observe/explore the actual physical images of the area with talking, and drawing/sketching the map; emerging need for reviewing examples of what the map may look like; assessing their own map drawing by talking among themselves; going back to the outside to re-check the reality; drawing again/finding missing parts; adding or relocating parts in the map; and drawing the map on the computer as well as constructing the three dimensional model of the map using blocks and a table top model. In learning to draw or make a map the kindergarten children learnt to look. They taught themselves to see and to feel what they see. In that process, emergent literacy (reading the word, e.g., stop sign, speed limit sign, name of the each building) experiences occurred naturally. Simultaneously, this representational map drawing seemed to serve the children's collective cognition of reading the world in careful observation of real objects, phenomena, events, or people, and of enhancing their hand/eye co-ordination in translating a perceived image into a graphic outcome. In that process the children clearly seemed to realise that computer software can be a troublesome tool due to the fact that certain programs did not fully support their fluent map drawing ideas. They preferred drawing the map on paper, or even on the large whiteboard in the classroom. As Anning (1997) articulated, drawing is a vehicle for rehearsing or clarifying ideas for the production of two- or three-dimensional work, or to explore ideas and feelings. It is the power of the graphicacy used as a tool for learning and for recording thinking that helped the children's expansion of geometric thinking. Sketching and drawing are the basic components of communication (Anning, 1997; Henderson, 1990) that served the young children's geometric discovery and new knowledge construction.

Conclusion and Educational Implication

This research study supports a social constructivist approach to teaching and learning, and suggests that children can negotiate meaning for complex systems of geometric concepts in a culture that values curiosity, wonder, exploration, reflection, provocation, and conversation. Teachers can provide opportunities for young children to perceive structure, form, and space through firsthand observation and action (Davis, 1996). This research study demonstrates that teachers can help children make sense of complex mathematical concepts and ideas by providing rich environments, encouraging conversation, and presenting interesting provocations and materials to embrace and explore. Collaborative activity in which children question each other and themselves, debate their ideas, and try to understand different points of view were helpful as the children negotiated meaning for complex geometric representations to construct their maps. The approaches used in this study support the charge of the National Council of Teachers of Mathematics, Principles and Standards for School Mathematics Pre-K–grade 2 Geometry Standard that spatial relationships and structures should be explored and investigated, and that young children should "use their notions of geometric ideas to become more proficient in describing, representing, and navigating their environment" (NCTM, 2000a, p. 97). Moreover, this research supports NCTM's position on the role of representation in young children's thinking that varied representations should be used to help them build new understandings and express mathematical ideas.

This research study acknowledged the stage-based developmental aspects of Piaget's and van Hiele's theories of geometric thinking. Van Hiele (1986) stated that "the transition from one level to the following is not a natural process; it takes place under influence of a teaching-learning program" (p. 50). With respect to the hierarchical levels of geometric thinking proposed by Piaget and van Hiele, this study found that children can simultaneously construct geometric meaning for topological, projective, and Euclidean space in an integrative and synthetic way when given constructivist opportunities for collaborative engagement, discourse, and reflection. This finding agrees with Clements and Battista (1992) who claimed that "it may be that children do not construct first topological and later projective and Euclidean ideas. Rather, it may be that ideas of all types develop over time, becoming increasingly integrated and synthesized" (p. 425-426). In this sense, the data in this study suggested that children's thinking occurred across levels. Clements' (2000) and Clements and Battista's (1992, 2001) critical perspectives on the complexity and holistic nature of the spatial representations of young children clearly emerged in this study.

The kindergarten children demonstrated interesting and productive problem-solving approaches as they negotiated their way to representing familiar space. Their work exhibited topological relationships and the beginnings of projective and Euclidean space. It was when the children were able to problem solve together and engage in dialogue to help them construct maps that the most advanced expressions of spatial representation were realised. Children need time to think and reflect upon their work and be involved with others as they think "out loud" and try out new approaches. Children's intuitive, tacit sense of geometric ideas can be nurtured with stimulating environments and interesting and meaningful provocations. As the children worked together to construct maps, they challenged themselves to capture the space that they observed.

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