First one home the long way round

Brian Sherman
University of Adelaide
<brian.sherman@adelaide.edu.au>

The activity "First One Home" in the Shell Centre's *Problems with Patterns and Numbers* Blue Box gives rise to a number of interesting patterns for those looking beyond the basic solution given in the book.

The game is for two players, each taking turn and turn about to move a single counter on the grid down and/or to the left until one or the other player can put the counter on the FINISH square. Each move can cover as many squares as the mover likes, within the confines of the grid, but cannot change direction; and the direction must be either directly to the left, directly down, or diagonally down to the left. The starting square can be chosen by common consent (see Figure 1).

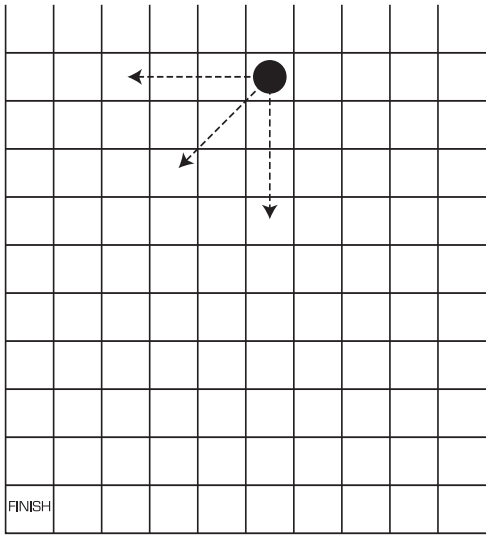

Figure 1

As indicated on the solution page, there are critical squares, shaded in the diagram, which determine the winning strategy. No critical square can be reached in one move from any other critical square, while from any non-shaded square there is at least one shaded square that can be reached in one move (see Figure 2).

The game can be won by the first player to put the counter on one of the

critical squares; after that, the next player must move to a non-shaded square, enabling the first player to once again move to a critical square. The FINISH square is just the last of the critical squares.

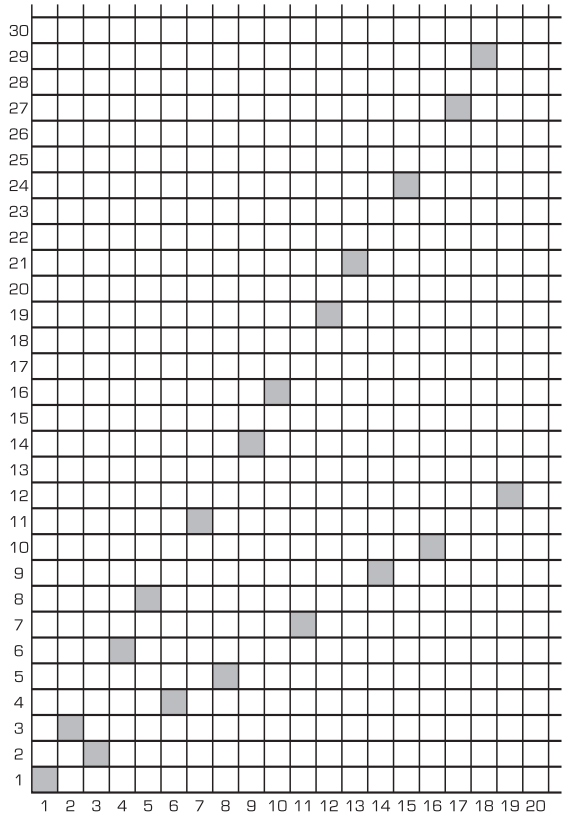

Figure 2

The solution page lists the coordinates of the early shaded squares in two columns, one for the squares in the upper arm, and the second for those in the lower arm; the lower arm coordinates are simply the reverse of the upper arm ones. However, the solution page offers no general rule for finding the shaded squares; the following traces a tortuous path through a number of recursive rules for doing this to eventually reach a specific rule, and a geometric construction for identifying the squares.

The first way is simply to start from the FINISH square (1,1), and rule out all squares from which this square can be reached; the next two squares to be shaded, (2,3) and (3,2), are the squares at the points of the two wedges remaining. The next two, (4,6) and (6,4), are at the points of the two wedges remaining after also ruling out all those squares from which (2,3) and (3,2) can be reached; and so on. Note that each square has the same name as its top right-hand corner point, if we use Cartesian coordinates for the points.

Each shaded square must be on exactly one south-east to north-west diagonal, and the squares on any one of these diagonals have a constant difference between y- and x-values. Thus the upper arm coordinates can be numbered consecutively by this difference in y- and x-values, so that Table 1 is a convenient way to exhibit them. Note that function notation $(x(n), y(n))$ rather than subscript notation (x_n, y_n) is being used, as there would be a need to use subscripts three deep, which is too unwieldy.

Table 1. The coordinates of the upper arm shaded squares.

$y - x = n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$x = x(n)$	1	2	4	5	7	9	10	12	13	15	17	18	20	22
$y = y(n)$	1	3	6	8	11	14	16	19	21	24	27	29	32	35

The first, obvious, pattern is that $y(n) = x(n) + n$; it arises from the way the table has been set up.

Next, each whole number appears once as an x-coordinate and once as a y-coordinate over all the shaded squares; for the upper arm squares, this means that, apart from 1, each whole number appears once, either as an x-value or as a y-value, as any which are a y-value for the upper arm are an x-value for the lower arm, and vice-versa.

Also, as the difference between any two x-values is at least 1, so the difference between any two y-values is at least 2:

$$\begin{aligned} & y(n+1) - y(n) \\ &= x(n+1) + n + 1 - x(n) - n \\ &= x(n+1) - x(n) + 1 \end{aligned}$$

Therefore, no two y-values are consecutive numbers, so that there cannot be more than one y-value between any two x-values. Thus the difference between $x(n)$ and $x(n+1)$ is either 1 or 2, and the difference between $y(n)$ and $y(n+1)$ is either 2 or 3.

These observations give the following procedure for generating the coordinates of the critical squares.

Assume that the table has been determined up to $n - 1$, and look for the next whole number above $x(n-1)$ which is not used as a y-value; then this number is $x(n)$. In addition, $y(n)$ is $x(n) + n$.

This procedure can be used to work out a number of patterns and relationships in the table.

Consider $y(n)$, which, of course, appears in column n in the y-row. As there are n of the y(i) before $y(n)$ (i.e., $y(0)$, $y(1)$, … $y(n-1)$), so the remaining $y(n) - n - 1 = x(n) - 1$ numbers less than $y(n)$, plus the extra 1 in column 0, must be in the x-row. These numbers are then $x(0)$, $x(1)$, … $x(x(n) - 1)$, and $y(n)$ is the number directly after them. Therefore $x(x(n) - 1)$ is the number just before $y(n)$, and $x(x(n))$ the number just after it.

So $x(x(n) - 1) = y(n) - 1$, and $x(x(n)) = y(n) + 1$. These are illustrated in Table 2, where the shading shows an example of the first, and the bold text shows one of the second:

Table 2. Displaying $x(x(n) - 1) = y(n) - 1$, and $x(x(n)) = y(n) + 1$.

$y - x = n$	0	1	2	3	4	5	6	7	8	9	10	11	**12**	13
$x = x(n)$	1	2	4	5	7	9	10	**12**	13	15	17	18	**20**	22
$y = y(n)$	1	3	6	8	11	14	16	**19**	21	24	27	29	32	35

From the second of these it follows that $x(y(n)+1) = x(x(x(n))) = y(x(n))+1$; $x(y(n))$ is either 1 or 2 less than $x(y(n)+1)$, but it cannot be $y(x(n))$, which is not an x-value. So, $x(y(n)) = y(x(n))-1 = x(x(n)) + x(n)-1 = y(n)+1 + x(n)-1 = y(n) + x(n)$.

Using this last relationship it can be seen that the Fibonacci sequence is generated as part of the table: start from the n-value and x-value of a column, and proceed to the column whose n-value is the y-value of that first column. The n-value $n' = y(n)$ and x-value $x(n') = x(y(n))$ of this new column are the next two terms in the sequence. It can be observed that each term is the sum of the two previous terms: either $n' = y(n) = n + x(n)$, or $x(n') = x(y(n)) = x(n) + y(n) = x(n) + n'$. So, starting with the first column, with $n = 0$ and $x = 1$, the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, … indicated by the dotted borders in Table 2, is generated.

Thus there is an improved procedure for determining the elements of the table:

If n is an x-value ($n = x(r)$), then $x(n) = x(x(r)) = y(r) + 1$;
and if n is a y-value ($n = y(r)$), then $x(n) = x(y(r)) = x(r) + y(r)$;
and, of course, in both cases $y(n) = x(n) + n$.

However, the procedure is still recursive — i.e., the later values of the sequences are determined in terms of the earlier values — so we need to find something outside the sequences to refer to; and for this we look to the golden ratio

$$\Gamma = \frac{1+\sqrt{5}}{2}$$

(cf. Coxeter, 1961). (It may look as though I have just pulled the golden ratio out of the air; it took some work with Microsoft Excel to see that the sequences

$$\frac{x(n)}{n} \text{ and } \frac{y(n)}{x(n)}$$

both approached Γ, though somewhat slowly. I should have known, though, that the golden ratio was involved because of its involvement with the Fibonacci sequence.)

Now Γ is the positive root of the quadratic $x^2 - x - 1 = 0$; for positive values of a, then, $a > \Gamma$ ($a < \Gamma$) if and only if $a^2 - a - 1 > 0$ ($a^2 - a - 1 < 0$). As

$$\frac{x(n)}{n} \text{ and } \frac{y(n)}{x(n)}$$

are both positive for all values of $n \geq 1$, it can be seen from the following that Γ is between

$$\frac{x(n)}{n} \text{ and } \frac{y(n)}{x(n)}$$

Suppose $$\frac{x(n)}{n} > \Gamma$$

then $$\left(\frac{x(n)}{n}\right)^2 - \frac{x(n)}{n} - 1 > 0$$

i.e., $$x(n)^2 - nx(n) - n^2 > 0$$

Therefore
$$\begin{aligned} & y(n)^2 - x(n)y(n) - x(n)^2 \\ &= (x(n) + n)^2 - x(n)(x(n) + n) - x(n)^2 \\ &= -x(n)^2 + nx(n) + n^2 < 0, \end{aligned}$$

and so $$\left(\frac{y(n)}{x(n)}\right)^2 - \frac{y(n)}{x(n)} - 1 < 0$$

and hence, finally, $$\frac{y(n)}{x(n)} < \Gamma$$

Equally, if $$\frac{x(n)}{n} < \Gamma$$

then $$\frac{y(n)}{x(n)} > \Gamma$$

either way, Γ is between $\dfrac{x(n)}{n}$ and $\dfrac{y(n)}{x(n)}$.

Observing the table, it would appear that
$$\frac{x(n)}{n} > \frac{y(n)}{x(n)}$$

and this can be proven by induction, as follows:

Define $A(n)$ to be $x(n)^2 - ny(n)$; then the inequality translates to $A(n) > 0$.

Firstly, $A(1) = x(1)^2 - y(1) = 4 - 3 > 0$.

Suppose that $A(r) = x(r)^2 - ry(r) > 0$ for all $r < n$.

If n is an x-value, i.e., if $n = x(i)$,

then $x(n) = x(x(i)) = y(i) + 1$

and $x(n-1) = x(x(i) - 1) = y(i) - 1$.

Therefore $x(n) = x(n-1) + 2$ and $y(n) = y(n-1) + 3$,

and so
$$\begin{aligned} A(n) &= x(n)^2 - ny(n) \\ &= (x(n-1) + 2)^2 - n(y(n-1) + 3) \\ &= x(n-1)^2 + 4x(n-1) + 4 - ny(n-1) - 3n \\ &= A(n-1) + 4x(n-1) + 4 - 3n \\ &> A(n-1) > 0, \end{aligned}$$
as $x(n-1) \geq n - 1$.

On the other hand, if n is a y-value, i.e., if $n = y(i)$, then
$$\begin{aligned} A(n) &= x(n)^2 - ny(n) \\ &= x(y(i))^2 - y(i)y(y(i)) \\ &= (y(i) + x(i))^2 - y(i)(x(y(i)) + y(i)) \\ &= y(i)^2 + 2x(i)y(i) + x(i)^2 - y(i)(x(i) + y(i)) - y(i)^2 \\ &= x(i)y(i) + x(i)^2 - y(i)^2 \\ &= x(i)^2 - (y(i) - x(i))y(i) \\ &= x(i)^2 - iy(i) \\ &= A(i) > 0. \end{aligned}$$

This last part indicates that, for those columns in the table which are part of the Fibonacci sequence, $A(n) = 1$; for, if n and $x(n)$ are in the Fibonacci sequence, then so are $n' = y(n)$ and $x(n')$, and the above argument shows that $A(n') = A(n)$. As $A(1) = 1$, so too is $A(n)$ for each column in the Fibonacci sequence.

Thus it is established that

$$\frac{x(n)}{n} > \Gamma > \frac{y(n)}{x(n)}$$

Hence $x(n) > n\Gamma$ and $y(n) < x(n)\Gamma$; also, $y(n) + 1 = x(x(n)) > x(n)\Gamma$, so that $y(n)$ must be the integer part of $x(n)\Gamma$.

A similar relationship can be established for $x(n)$ and n:
If $n = x(i)$, then $x(n) - 1 = x(x(i)) - 1 = y(i) < x(i)\Gamma = n\Gamma$.
If $n = y(i)$, then we must apply induction as follows:

$$x(1) - 1 < \Gamma.$$

Suppose $x(r) - 1 < r\Gamma$ for $r < n$; then

$$\begin{aligned} x(n) - 1 &= x(y(i)) - 1 \\ &= y(i) + x(i) - 1 < x(i)\Gamma + x(i) - 1 < x(i)\Gamma + i\Gamma \\ &= y(i)\Gamma \\ &= n\Gamma. \end{aligned}$$

As $x(n) > n\Gamma$, so $x(n)$ is the next integer above $n\Gamma$.
Thus there is a direct procedure for generating the table:
$x(n)$ is the next integer above $n\Gamma$, and $y(n) = x(n) + n$.

On the diagram of the solution of the First One Home game, if the lines $y = x\Gamma$ and $x = y\Gamma$ are drawn in, it will be seen that the shaded squares lie along the line $y = x\Gamma$ on the upper arm, and along the line $x = y\Gamma$ on the lower arm. More precisely, the shaded squares on the upper arm are those squares where the line $y = x\Gamma$ crosses the left edge, and those on the lower arm are the squares where the line $x = y\Gamma$ crosses the bottom edge.

To draw in the line $y = x\Gamma$, place the point of a compass on the point (2,1), and the pencil on (0,0), and draw the arc through (0,2) and (1,3) which will cross the line $x = 2$ at $(2, 1 + \sqrt{5})$. The line through this last point and (0,0) is the required line $y = x\Gamma$ for the upper arm, and the line $x = y\Gamma$ for the lower arm can be found similarly.

References

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