Expansion of binomials and factorisation of quadratic expressions: Exploring a Vedic method

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Many students have traditionally found the processes of algebraic manipulation, especially factorisation, difficult to learn. This study investigated the value of introducing students to a Vedic method of multiplication of numbers that is very visual in its application. We wondered whether applying the method to quadratic expressions would improve student understanding, not only of the processes but also the concepts of expansion and factorisation. We found that there was some evidence that this was the case, and that some students also preferred to use the new method.

Introduction

Research has documented the difficulties students face in algebra and how these can often be traced to their limited understanding of numbers and their operations (Stacey & MacGregor, 1997; Warren, 2001). Of growing concern is the artificial separation of algebra and arithmetic, since knowledge of mathematical structure seems essential for a successful transition. In particular, this mathematical structure is concerned with (i) relationships between quantities, (ii) group properties of operations, (iii) relationships between the operations and (iv) relationships across the quantities (Warren, 2003). Thus it has been suggested by Stacey and MacGregor (1997) that the best preparation for learning algebra is a good understanding of how the arithmetic system works. An understanding of the general properties of numbers and the relationships between them may be crucial, and students need to have thought about the general effects of operations on numbers (MacGregor & Stacey, 1999). This study sought to test the hypothesis that arithmetic knowledge can improve algebraic ability by applying a Vedic method of multiplying arithmetic numbers to algebra, based on the similarity of structural presentation.

Vedic mathematics

Vedic mathematics has its origins in the ancient Indian texts, the Vedas, an integrated and holistic system of knowledge composed in Sanskrit and transmitted orally from one generation to the next. The first versions of these texts were possibly recorded around 2000 BC, and the works contain the genesis of the modern science of mathematics (number, geometry and algebra) and astronomy in India (Datta & Singh, 2001; Joseph, 2000). Sri Tirthaji (1965) has expounded 16 *sutras* or word formulas and 13 sub-sutras that he claims have been reconstructed from the Vedas. The sutras, or rules as aphorisms, are condensed statements of a very precise nature, written in a poetic style and dealing with different concepts (Joseph, 2000; Shan & Bailey, 1991). A sutra, which literally means thread, expresses fundamental principles and may contain a rule, an idea, a mnemonic or a method of working based on fundamental principles that run like threads through diverse mathematical topics, unifying them. As Williams (2002) describes them:

> We use our mind in certain specific ways: we might extend an idea or reverse it or compare or combine it with another. Each of these types of mental activity is described by one of the Vedic sutras. They describe the ways in which the mind can work and so they tell the student how to go about solving a problem. (Williams, 2002, p. 2).

Examples of the sutras are the "Vertically and Crosswise" sutra, which embodies a method of multiplication with applications to determinants, simultaneous equations, and trigonometric functions, etc. (this is the sutra used in the research reported here — see Figure 3), and the "All from nine and the last from ten" sutra that may be used in subtraction, vincula, multiplication and division.

Barnard and Tall (1997, p. 41) have introduced the idea of a *cognitive unit*, "A piece of cognitive structure that can be held in the focus of attention all at one time," and may include other ideas that can be immediately linked to it. This enables compression of ideas, so that a collection of ideas or symbols that is too big for the focus of attention can be compressed into a single unit. It seems as if the sutras nicely fit this description, with the mnemonic or other memory device being used as a peg to hang the collection of ideas on. Thus the theoretical advantage of using the sutras is that they allow encapsulation of a process into a manageable chunk, or cognitive unit, that can then be processed more easily, sometimes using a visual reminder, such as in the *Vertically and Crosswise* sutra. Here the essential procedure is signified holistically by the symbol $|5|$, unlike the symbol FOIL that signifies in turn four separate procedures. It might be possible for a symbol such as to be used in much the same way for FOIL, but this may appear more visually complex, and it is not usually separated from the accompanying binomials like this. In this way sutras often make use of the power of visualisation, which has been shown to be effective in learning in various areas of mathematics (Booth & Thomas, 2000; Presmeg, 1986; van Hiele, 2002). Such visualisation accesses

the brain's holistic activity (Tall & Thomas, 1991) and intuition, and this assists in providing an overview of the mathematical structure. The sutras also aid intuitive thinking (Williams, 2002) and being based on patterns and mnemonics they make recall much easier, reducing the cognitive load on the individual (Morrow, 1998; Sweller, 1994). The sutras were originally envisaged as applying both to arithmetic and algebra, and Joseph (2000) and Bhatanagar (1976) have explained that since polynomials may be perceived as simply arithmetic sequences, the principles apply equally well to them. This research considered a possible role of the *Vertically and Crosswise* sutra for improving facility with, and understanding of, the expansion of algebraic binomials and the factorisation of quadratic expressions.

Method

The research employed a case study methodology, using a single class of Year 10 (age 15 years) students. The school used is a co-educational state secondary school in Auckland, New Zealand and the class contained 19 students, 11 boys and 8 girls. The students, who included 9 recent immigrants, were drawn from several cultural backgrounds, and accordingly have been exposed to different approaches and teaching environments with respect to learning mathematics. This also meant that nine of the students have a first language other than English and these language difficulties tend to hinder their learning (for example, three of the students are on a literacy program at the school).

Two anonymous questionnaires (see Figure 1 for some questions from the second) were constructed using concepts we identified as important in developing a structural understanding of binomial expansion and factorisation, such as testing the concept of a factor and the ability to apply a procedure in reverse. Questions included: multiplication of numbers; multiplication of binomial expressions; factorisation of quadratic expressions; word problems on addition and subtraction of like terms; and expansion of expressions in a practical context. Some questions also involved description of procedures and meanings attached to words. In particular, the second questionnaire contained items on the use of the Vedic method applied to binomial expansion and factorisation.

The lessons were taught by the first-named author in 2003 in a supportive classroom environment that encouraged student-to-student and teacher-student interactions. Students were assured that the teacher was genuinely interested in their mathematical thinking and respected their attempts, that it was fine to make mistakes and that understanding how the mistake occurred was a learning opportunity for everyone concerned. Students were encouraged to explain and check the validity of their answers, and positive contributions were praised.

The first teaching session comprised work on multiplication of numbers and revision of work on algebra that the students had learned in Year 9 (age 14 years). Substitution, collection of like terms and multiplication of a bino-

2. Multiply the following two expressions together showing all your working and simplify the answer.
   a) $x + 6$ and $x + 3$
   b) $2x - 3$ and $x + 4$
   c) In part 2(a), describe what you are doing at each step.
3. a) Expand the following by the "vertically and crosswise" method.

$$\begin{array}{r} x+3 \\ x+4 \\ \hline \end{array} \qquad \begin{array}{r} x-6 \\ x-3 \\ \hline \end{array} \qquad \begin{array}{r} 2x+5 \\ x+7 \\ \hline \end{array} \qquad \begin{array}{r} 2x-3 \\ 5x+2 \\ \hline \end{array}$$

   b) For the first question in 3a, describe in your own words what you are doing at each step.
4. Fill in the blank boxes in the following:

$$\begin{array}{r} 2x+\square \\ \square+2 \\ \hline 2x^2 \;/\; 7x \;/\; +6 \end{array} \qquad \begin{array}{r} x+\square \\ x+3 \\ \hline x^2+\square+21 \end{array} \qquad \begin{array}{r} \square-3 \\ x-\square \\ \hline 2x^2-13x+\square \end{array}$$

$$\begin{array}{r} x+\square \\ x+\square \\ \hline x^2+12x+32 \end{array} \qquad \begin{array}{r} \square-\square \\ \square-\square \\ \hline x2-9x+14 \end{array}$$

5. Factorise the following expressions:
   a) $x^2 - 6x + 8$ b) $x^2 - 11x + 18$ c) $3x^2 + 17x + 10$

Figure 1. Some of the questions taken from the second questionnaire.

mial expression by a single value were revised, using, for example, expressions such as $5(x - 4)$, $(p + 2)4$, and $k(4 + k)$. Students were also reminded of the meanings of words such as term, expression, factor, expansion, coefficient and simplify. Diagrammatic representations of $3(5 + 6)$ and $k(4 + k)$ using rectangles were drawn and discussed, and then students drew similar rectangle diagrams representing multiplications such as $(3 + 5)(2 + 5)$ and $(k + 2)(k + 4)$ (see Figure 2). Following a review of factorisation of expressions such as $15p + 10$, the FOIL (First, Outside, Inside, Last) method of expanding binomials was taught, where the First terms in each bracket are multiplied together, then the Outside terms, the Inside terms and then the Last term in each bracket, to give four products. Finally factorisation of quadratic expressions, followed by a "guess and check" method for factorising quadratic expressions was covered. The students did not find these topics easy, especially factorising of quadratic expressions. This took a total of four hours, after which, questionnaire one was administered.

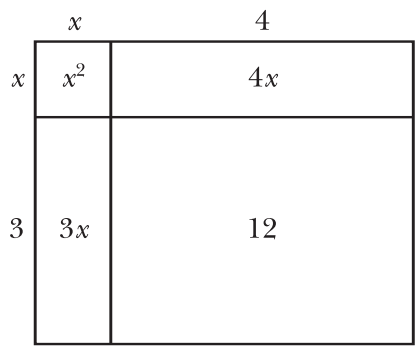


Figure 2. A rectangle diagram showing $(x + 3)(x + 4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$.

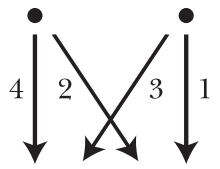
Students were then exposed for one hour to the Vedic *vertically and crosswise* method, where initially they practised multiplying two- and three-digit numbers with this approach. Subsequently, the next three hours were spent expanding binomials and factorising quadratic expressions with the *vertically and crosswise* method. This method (see Figure 3) involves a sequence of four multiplications, the answers to each of which are placed into a single answer line. The middle two terms are added together mentally to supply the final answer.

The binomials are arranged as below

$$\begin{array}{ccc} 2x & + & 3 \\ 4x & + & 5 \\ \hline \end{array}$$

$8x^2 \,/\, 10x + 12x / 15$

which is $8x^2 + 22x + 15$



The order of multiplications is as shown left (2 & 3 interchangeable)

Most working is done mentally and the answer is written down in one line.

Figure 3. The vertically and crosswise method of multiplication of binomials.

Results

The first question (1a) in each questionnaire was a two-digit multiplication. In the first, it was 37×58, and the second 23×47, and in this second case the question asked that this be done by the *vertically and crosswise* method. The aim was to check students' facility with arithmetic multiplication and to see if the *vertically and crosswise* sutra was of assistance in this area. In the event 11 of the 18 (61%) students who completed both questionnaires, correctly answered the question in the first test, and 13 (72%) in the second, with only one student not using the sutra in that test. There was no statistical difference between these proportions ($\chi^2 = 0.125$). When asked to explain what they had done using the Vedic approach, students who were able to write down the final answer were often able to write something like student 4's explanation for 1c), 32×69:

2 times 9 is 18
3 times 9 + 2 times 6 is 39 + carried 1 = 40
3 times 6 + carried 4 = 22

Expansion of binomials

A summary of the results in the first of the algebra questions (Q2 — see Figure 1 for format), requiring students to multiply together two basic binomials, is given in Table 1. Question 2 was presented in each case with the binomials in the same line.

Table 1. Results of Questions 2 on each questionnaire.

Questionnaire	***Item***	***Correct***	***Wrong***
1	2a) Multiply x+6 and x+3 (FOIL method)	8	10
2	2a) Multiply x+7 and x+2 (Vertically & Crosswise)	10	8

Since the first questionnaire was administered immediately after the FOIL method was taught, most students employed this method, and only 8 (44.4%) correctly performed the expansion in the simplest example, question 2a). In the second questionnaire, where students were asked to use the Vertically and Crosswise method, 10 (55.6%) were correct, but there was no statistically significant difference ($\chi^2 = 0.11$, $df = 1$, ns). However, when the presentation format was changed from a single line to the grid format of question 3 (see Figure 1), the facility on 3a) (Multiply $x + 3$ and $x + 4$) improved to 14 correct answers (77.8%). Thus there was weak evidence ($\chi^2 = 2.92$, $df = 1$, $p = 0.1$) of an improvement in students' performance on this basic expansion of binomials in the second questionnaire, using the Vedic method with a grid presentation. However, the improvement was not sustained for question 3b) which had negative signs (see Table 2: $\chi^2 = 0.45$), or for 2b), which involved $2x$ and negative signs. It seems that the arithmetic complexity caused problems, with only 3 students answering 2b) correctly on each test.

The other results of question 3 on the second questionnaire also support the idea of improved understanding; Table 2 gives the results of the multiplications in this question. In this case question 3c) was the question of a standard corresponding to 2b) in the first test ($2x - 3$ times $x + 4$), and yet the students did significantly better (50% correct) than they did on that question ($\chi^2 = 3.13$, $df = 1$, $p < 0.05$).

Table 2. Results of Question 3 of Questionnaire 2: Grid Presentation, Vedic Method.

Question	***Item***	***Correct***	***Wrong***
3a	Multiply x+3 and x+4	14	4
3b	Multiply x–6 and x–3	11	7
3c	Multiply 2x+5 and x+7	9	9
3d	Multiply 2x–3 and 5x+2	5	13

It is also of interest that on question 2 in the first test all eight students who successfully applied the FOIL method wrote out all four terms and then added the two middle terms. No student wrote the final answer without the intermediate step of giving both middle terms. In contrast, when using the vertically and crosswise approach, all 14 students who were successful were able to write the answers straight down, as seen in the example in Figure 4.

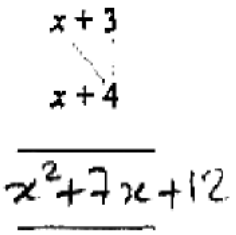


Figure 4. Student 11 multiplies two binomials by the Vedic method, writing down the final answer.

Furthermore, 5 students (12, 15, 16, 17, and 18) who were unable to multiply binomials using the standard method achieved success using the Vedic method. Figure 5 shows the corresponding working of student 18 on these questions. In test 1, following the FOIL method she was unable to multiply any terms together, and even seemed to confuse the question with factorisation. However, with the Vedic method she makes a good attempt, correctly answering two parts.

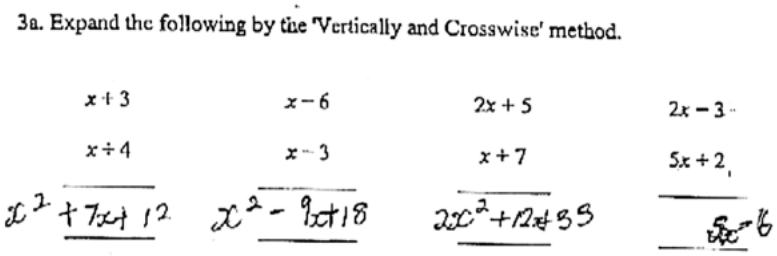


Figure 5. Student 18's work on multiplying binomials by the Vedic method.

It might be argued that the improvement above was due to students spending more time learning how to perform such expansions, or that the previous FOIL learning took time to assimilate. However, it should be noted that the questions where the improvement occurred specifically involved use of the Vedic method, which, while it can be related to FOIL by an experienced mathematician, would no doubt look quite different to these students, since it is set out in a grid format rather than being performed in a single line. Furthermore, it was noted during the teaching episode that there was some resistance from the students to learning a second method when they already knew the FOIL method, and this could be expected to have a detrimental effect on performance.

Factorisation of quadratic expressions

Table 3 contains a summary of the results of question 3 from the first questionnaire and question 5 in the second, these being corresponding single line, traditional format factorisation questions (see Figure 1). Individually the results of these questions did not show any statistical difference between the performance before and after the Vedic method was introduced. For example, between questions 3a) and 5a), $\chi^2 = 0.9$, which is not significant. However, if the question parts are grouped together and the number of students correct on question 3 compared with those correct on question 5

then we find that there is a significant improvement (Q3 v Q5 $\chi^2 = 6.65$, $p < 0.01$) on the second questionnaire.

Table 3. Results of Questions 3 and 5 — Factorisation with a linear presentation

Questionnaire	***Item***	***Correct***	***Wrong***
1	3a) Factorise $x^2 + 7x + 10$	6	12
1	3b) Factorise $x^2 - 12x + 35$	2	16
1	3c) Factorise $2x^2 + 10x + 12$	1	17
2	5a) Factorise $x^2 + 6x + 8$	10	8
2	5b) Factorise $x^2 - 11x + 18$	7	11
2	5c) Factorise $3x^2 + 17x + 10$	4	14

For question 4 in each questionnaire the students were asked to supply the missing terms in two binomials that were multiplied together. Boxes were provided for the missing terms, which were in the binomials, the quadratic, or both (see Figure 1). While these are not "standard" straight factorisation questions they do require students to work backwards from the answer and thus display some conceptual knowledge of how to "undo" multiplication of binomials. There was no difference in facility on any of these matching questions (see Table 4), or on a comparison of the total number of correct scores between the two questionnaires ($\chi^2 = 0.39$, ns). The students appeared to be able to do them equally well using either guess and check and decomposition, or the Vedic approach.

Table 4. Question 4 results — Factorisation with a grid presentation.

Questionnaire	***Item***	***Correct***	***Wrong***
1	4a) $(x + 3)(? + 2) = x^2 + 5x + 6$	16	2
1	4b) $(x + ?)(x + 4) = x^2 + ? + 24$	16	2
1	4c) $(? - 5)(x - ?) = 2x^2 - 17x + ?$	3	15
1	4d) $(x + ?)(x + ?) = x^2 + 10x + 21$	10	8
1	4e) $(? - ?)(? - ?) = x^2 - 11x + 18$	11	7
2	4a) $(2x + ?)(? + 2) = 2x^2 + 7x + 6$	14	4
2	4b) $(x + ?)(x + 2) = x^2 + ? + 21$	14	4
2	4c) $(? - 3)(x - ?) = 2x^2 - 13x + ?$	8	10
2	4d) $(x + ?)(x + ?) = x^2 + 12x + 32$	12	6
2	4e) $(? - ?)(? - ?) = x^2 - 9x + 14$	12	6

Note: The format of these questions was not that above, but was a grid presentation as shown in Figure 1.

While the discussion above shows that the evidence for a better performance on individual questions following the teaching of the Vedic method was

rarely present, a consideration of the students' overall scores on the expansion and factorisation algebra questions did show a significantly better performance on the second test, ($\bar{x}_1 = 41.4\%$, $\bar{x}_2 = 51.5\%$, $t = 2.66$, $p < 0.05$). Thus it appears that, overall, the Vedic approach may have contributed to student understanding of the methods, either by cementing in place the previous methods, or by complementing them.

It is worth noting too that some students preferred to use the *Vertically and Crosswise* method even when not directed to do so. For example, in question 2 of the second questionnaire students were simply asked to multiply two binomials together, with no method specified. In the event 4 students (4, 5, 16, and 18) chose to use the *Vertically and Crosswise* approach, setting out their work in a grid. In addition, 3 of these students (15, 16 and 18) used it for the factorisation in question 5. An example of their work is shown in Figure 6. While they used the method with varying levels of success, it seems to have benefited both student 5, who answered no algebra questions correctly in test 1 (and 2 in test 2), and student 16 who used it for factorisation and went from 3 correct to 8 correct (see Figure 6). While student 18 preferred the Vedic method to expand and to factorise expressions, she used the traditional method to multiply numbers even when asked to multiply by the *Vertically and Crosswise* method. This seems to suggest that she was comfortable using different methods in algebra from those employed in arithmetic.

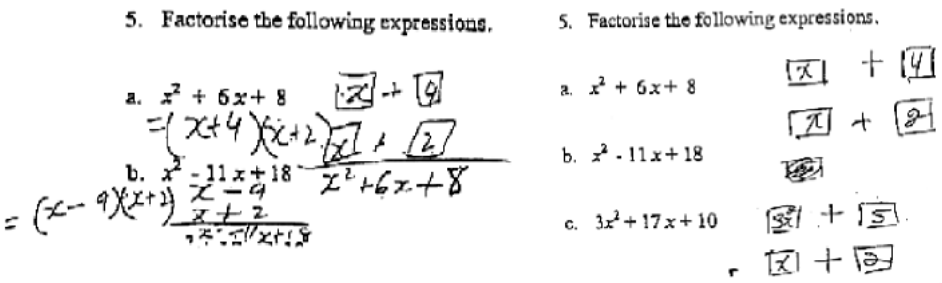


Figure 6. Students 5 and 16 choose to use the Vedic method.

Conclusion

In this study students were taught an appropriate Vedic sutra following teaching of the traditional FOIL method of multiplication of binomials, and the decomposition method for factorisation. We found that afterwards the students performed significantly better overall on these types of algebra questions, and specifically on the factorisations, and there was weak evidence of better results on expansion using a grid format. The reasons for the improvement are not easy to pinpoint since they appear in some areas and not in others. This seems to indicate that the value of the method may lie in what it adds to the students' overall algebraic conceptions and knowledge of mathematical structure. Thus we have found no evidence that it should be seen as a replacement for the former approaches, but our results suggest it could rather be recommended as a useful adjunct, a complementary method. While

this may take longer in terms of teaching time, the results indicate that possession of a range of strategies may have value above and beyond their individual benefit.

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