Golden sections

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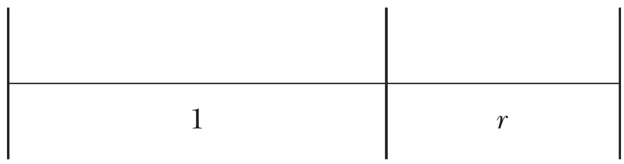
Introduction

Architects, musicians and other thoughtful people have, since the time of Pythagoras, been fascinated by various harmonious proportions. One, a visual harmony attributed to Euclid, is called "the golden section". We explore this concept in geometries of one, two and three dimensions.

One dimension

A finite line is divided by golden section when the smaller segment is to the larger as the larger is to the whole.



Write r for the length of the smaller segment in units of the larger; then,

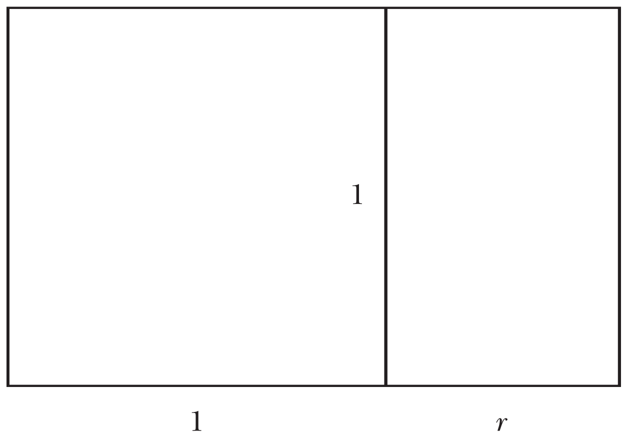
$$r : 1 :: 1 : 1 + r$$
$$r < 1,$$
$$r^2 + r = 1.$$

The positive root of the quadratic equation could be found by successive trial and error, using an electronic calculator, or via the usual formula:

$$r = \frac{1}{2}\left(\sqrt{5} - 1\right) = 0.61803$$

Two dimensions

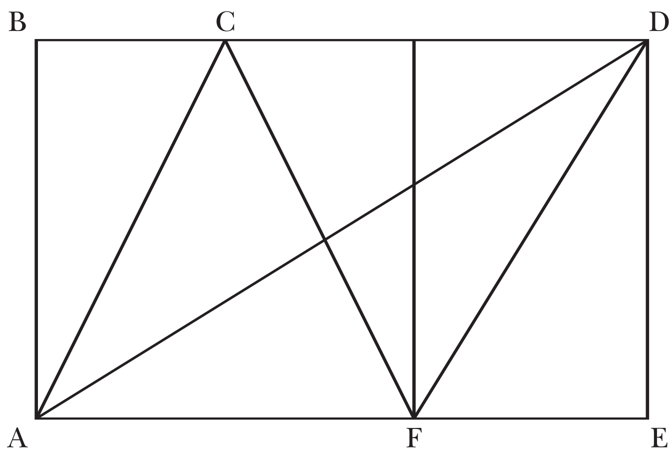
Imagine a square space, such as a courtyard, which is to be extended on one side by an area of particular significance. The symmetry of the square will be broken, but an element of harmony may be recovered if the extension and the combined space share the same proportions. Evidently the extension will be rectangular in plan. The shape in question is called the "golden rectangle".



The proportionality is the same as before and leads to the same quadratic equation, which may also be interpreted as an equation of areas — an exercise for the reader.

Geometric construction

Draw a triangle ABC with a right angle at B, AB having unit length and BC one-half. Continue the line BC to D so that CD = CA. Complete the rectangle about diagonal DA, with corner E, and mark the point F on the side EA, such that AF = 2BC and CF = CA.



The desired result follows arithmetically from Pythagoras' theorem.

$CD^2 = CA^2 = \frac{5}{4}$, $BD = CD + \frac{1}{2}$, $FE = CD - \frac{1}{2} = r$, so BD.FE = 1.

Therefore AB : BD :: FE : ED, QED.

Since the problem is one of similar shapes, there should also be a proof that appeals only to similar angles. Notice that AD bisects the angle CAF, since CAD is an isosceles triangle; and ∠CDF = ∠CFD, since CDF is another isosceles triangle. Now,

$$\begin{aligned}\angle\mathrm{CFD} &= \left(\frac{\pi}{2} - \angle\mathrm{CFA}\right) + \angle\mathrm{EDF} \\ &= \frac{\pi}{2} - 2\angle\mathrm{FAD} + \angle\mathrm{EDF} \\ \angle\mathrm{CDF} &= \frac{\pi}{2} - \angle\mathrm{EDF} \\ &\therefore \angle\mathrm{EDF} = \angle\mathrm{FAD} = \angle\mathrm{BDA}\end{aligned}$$

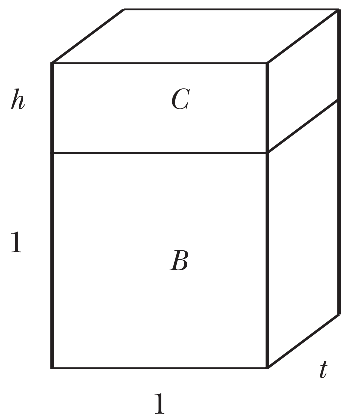
Consequently, the rectangles having diagonals DF and DA are similar, QED. Note that the lines DF and CA look parallel but are not quite.

(As a digression, this construction works even if the shape we start from is an arbitrary rectangle instead of a square. For example, the international standard series A of paper sizes is based on the ratio 1 : √2.)

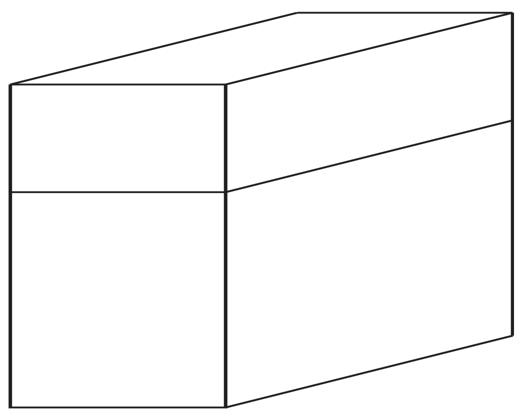
Three dimensions

When we ascend to three dimensions, we try to imagine what might be called a "golden column", keeping all angles right angles. Let us start with a box-like or solid base (B) having a square face with edges of unit length and some depth or thickness *t*. Can this shape be extended, we ask, by adding a capital (C) of height *h*, so that the combined column (B + C) has the same overall proportions as the capital (C) by itself? That is, the combined column is to be geometrically similar to the capital, though with a different orientation.

It turns out, yes, this can be done, and with two possible ways for the proportionality to come about.



column (i)



column (ii)

(i)
$$h : t : 1 :: t : 1 : 1 + h$$
$$\therefore h < t < 1$$
$$h = t^2$$
$$t^3 + t = 1$$

(ii)
$$h : 1 : t :: 1 : t : 1 + h$$
$$\therefore h < 1 < t$$
$$h = t^{-1} - 1$$
$$t^3 - t = 1$$

In both cases, the three dimensions of the similar shapes are in geometric progression, and the quantity t satisfies a neat cubic equation, i.e., an equation of volumes. To evaluate these numerically we could use an electronic calculator by successive trial and error, or we can appeal to Cardan's formula for the cubic. Writing

$$a \equiv \frac{4}{27}$$

we are led to solutions for t in surd form:

(i)
$$t_{\mathrm{i}} = \sqrt[3]{\frac{1}{2}\left(\sqrt{1+a}+1\right)} - \sqrt[3]{\frac{1}{2}\left(\sqrt{1+a}-1\right)} = 0.68233, \quad h_{\mathrm{i}} = 0.46557$$

(ii)
$$t_{\mathrm{ii}} = \sqrt[3]{\frac{1}{2}\left(1+\sqrt{1-a}\right)} + \sqrt[3]{\frac{1}{2}\left(1-\sqrt{1-a}\right)} = 1.32472, \quad h_{\mathrm{ii}} = 0.75488$$

These cubic equations are particular cases of the general form,

$$t^3 + pt = 1$$

distinguished by parameter $p = \pm 1$, the real solution of which can be written as a function of real p. When p is negative, this function contains the cube root of a negative number — discuss.

Further projects

The subject tempts curiosity at almost every point. Can anyone find in the neighbourhood some window frames or door panels that come close to golden proportions? A standard size for photographs is 20 cm × 12.5 cm: how do ATM cards, tennis courts and other things of regulated size compare? Keen students constructing the golden rectangle will notice how the construction may be iterated. Starting with the interior square on FE, smaller golden rectangles can be produced successively, and with the exterior square on AE, larger golden rectangles, all with corner D. If one starts with a rectangle of not quite golden proportions, say 5 × 8, adding a square on the long side and then repeating the growth process gives a sequence that converges toward the golden rectangle: numerically it is a Fibonacci sequence. In order fully to appreciate the three-dimensional proportions, model bases and capitals might be constructed from sheet-board and displayed. As to the aesthetic appeal of the figures, it is worth remembering the penetrating observation of John Keats, in "Ode on a Grecian Urn" (1819):

> Heard melodies are sweet, but those unheard
> Are sweeter.

Supplementary references

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