Modelling and the transit of Venus

Dave Quinn
University of Queensland
<pdq@pobox.com>

Ron Berry
University of Queensland
<rondberry@optusnet.com.au>

Introduction

Senior secondary mathematics students could justifiably question the relevance of subject matter they are being required to understand. One response to this is to place the learning experience within a context that clearly demonstrates a non-trivial application of the material, and which thereby provides a definite purpose for the mathematical tools under consideration. This neatly complements a requirement of mathematics syllabi (for example, Queensland Board of Senior Secondary School Studies, 2001), which are placing increasing emphasis on the ability of students to apply mathematical thinking to the task of modelling real situations. Success in this endeavour requires that a process for developing a mathematical model be taught explicitly (Galbraith & Clatworthy, 1991), and that sufficient opportunities are provided to students to engage them in that process so that when they are confronted by an apparently complex situation they have the thinking and operational skills, as well as the disposition, to enable them to proceed.

The modelling process can be seen as an iterative sequence of stages (not necessarily distinctly delineated) that convert a physical situation into a mathematical formulation that allows relationships to be defined, variables to be manipulated, and results to be obtained, which can then be interpreted and verified as to their accuracy (Galbraith & Clatworthy, 1991; Mason & Davis, 1991). The process is iterative because often, at this point, limitations, inaccuracies and/or invalid assumptions are identified which necessitate refinement of the model, or perhaps even a reassessment of the question for which we are seeking an answer.

This article develops a model of a physical situation that can be explained through application of senior secondary mathematical concepts, and which has elements of all phases of the modelling process. A transit of Venus is a very rare astronomical event that occurred most recently on 8 June 2004. We were the first in several generations that was provided with an opportunity to witness it, 1882 having been the previous opportunity. In the 18th century, calculations stemming from extremely precise observations of a transit of Venus were central to the determination of the size of the Sun, the distance of the Sun from Earth, and thence the scale of the known universe, which were only guessed at prior to that time. There are web resources, such as Bueter's (2004) website and the Sun–Earth Connection (2004) document, where the techniques implemented are described. However, this article focuses on another aspect of the scenario, one for which we have been unable to uncover any mathematics-oriented web sites, and one for which the modelling process can be utilised: when will a transit occur?

This was an important question in Australia's recent history: James Cook's primary directive in 1769 was to observe a transit from Tahiti. Only after that did he read sealed orders that culminated in his landing on Australia's east coast (James Cook University's (2004) website has further details). Furthermore, when past transits are analysed, a mathematical curiosity emerges: it becomes apparent that transits occur in a regular cycle, but within that cycle, a seemingly unusual pattern appears. Transits are separated by periods of eight years (the next transit is scheduled to appear in 2012), followed by a gap of 105 years, then eight years, then 121 years. This pattern then repeats itself.

Specifying the problem

It is one thing to observe a pattern in past observations; another to replicate that pattern mathematically, and thereby create a tool than can be used outside the range of observed phenomena. When students are faced with the problem of determining when transits will occur, it is to be expected that many of them, if unfamiliar with modelling as a process, would be uncertain how to begin. They may be confident with algebraic manipulation to obtain an answer, but that is the relatively simple component of mathematical modelling. In the early stages of formulating a model, the essential skills required are those that involve identifying quantities and relationships that are deemed to be significant (Mason & Davis, 1991), and breaking a seemingly complex task into smaller subtasks with simpler parameters and goals. This will often require the making of simplifying assumptions.

A transit of Venus occurs when Venus passes directly in front of the Sun, as seen from Earth. The key values that deal with that physical situation are the speeds at which the planets orbit (or, their orbital periods). Because Venus' orbit lies closer to the Sun than Earth's, Venus' orbital period is less than Earth's. That is, it completes a revolution of the Sun in less time than Earth. So the first problem to solve is, how often does Venus, in its orbit around the Sun, pass the Earth? This phenomenon is called a planetary alignment.

However, an alignment does not imply a transit. The reason can be seen by looking at Figure 1, which shows the orbits of Venus and Earth.

Venus' orbital plane is inclined with respect to Earth's orbital plane (which is called "the plane of the ecliptic"). Therefore, the fact that Venus has caught up with the Earth does not automatically imply that Earth, Venus and the Sun

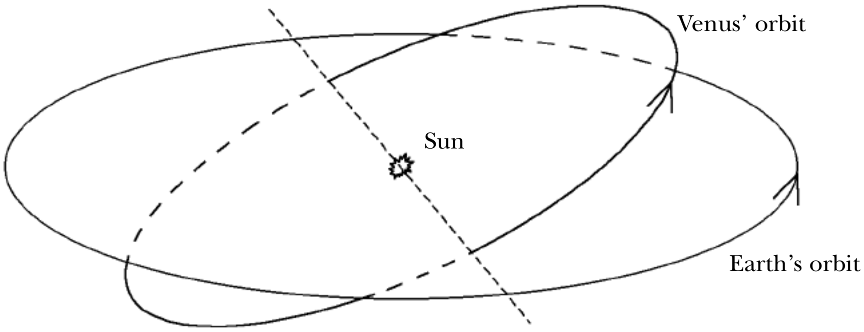

Figure 1

are collinear, which is the condition necessary for a transit. So the next problem is to work out how far Venus is above (or below) the ecliptic when an alignment occurs. If an alignment occurs when Venus is very close to the ecliptic, a transit results. To calculate this we need to know the angle of inclination of Venus' orbit, and the radius of Venus' orbit.

But just how close must Venus be to the ecliptic for a transit to result? Because the Sun is a relatively large disk in the sky, Venus need not be *exactly* crossing the ecliptic at alignment — there is some margin involved. As long as Venus traverses some part of the Sun, it is deemed a transit. To calculate the margin, we need to know how far Venus is from Earth at alignment, as well as the apparent size of the Sun.

Formulating the mathematical problem

Relevant information that has so far been identified includes:

- orbital periods of Venus and Earth;
- angle of inclination of Venus' orbit (with respect to the ecliptic);
- radius of Venus' orbit;
- angular size of the Sun (as seen from Earth).

Some of these values may be obtained by direct measurement, while others must be calculated. Direct observation provides us with the following data:

Earth's Period of Revolution	365.25636 days
Venus' Period of revolution	224.701 days
Inclination of Venus' orbit to ecliptic	3.3944°
Angular size of the sun, from Earth	0.534°

Students could be encouraged to engage in some research to locate this information for themselves, via a search of the World Wide Web, or their school library resources, for example. In collaboration with a school's physics department, an opportunity arises for students to explore the science that provides these observations and measurements, thereby extending utilisation of the scenario beyond the mathematics classroom.

We have also identified several subtasks, the solution to which will enable us to successfully proceed towards our original goal. These subtasks are as follows:

1. Calculate the frequency at which alignments occur.
2. Calculate the maximum distance above or below the ecliptic that Venus may be at alignment for a transit to occur.
3. Subtask 2 requires that we know the distance between Earth and Venus at alignment.
4. Determine how high Venus is whenever an alignment occurs.

The final step in the iterative process will be the implementation of the model using some technology — be it pencil and paper, graphing calculator, or a spreadsheet — to verify its accuracy and to use it to make predictions.

Solving the mathematical problem

Once we have reduced the problem to a few subtasks, generating results becomes far simpler, almost trivial in some cases. Simple application of ratios and trigonometry provide answers to most of the subtasks. Knowledge of periodic functions is required to model Subtask 4, Venus' distance from the ecliptic.

Alignment frequency

Planets closer to the Sun orbit the Sun in fewer days than planets further away. They therefore "catch up" and "overtake" the outer planets from time to time, depending upon their relative orbital periods. By simply calculating their orbital velocities, in terms of degrees travelled per day, we can determine how frequently an alignment occurs.

Venus, closer to the Sun than Earth, makes one complete orbit (360°) of the sun every 224.701 Earth days. Therefore, it travels 1.6021° per day. Similarly, Earth orbits every 365.25636 days, travelling an angular distance of 0.9856° per day (Figure 2).

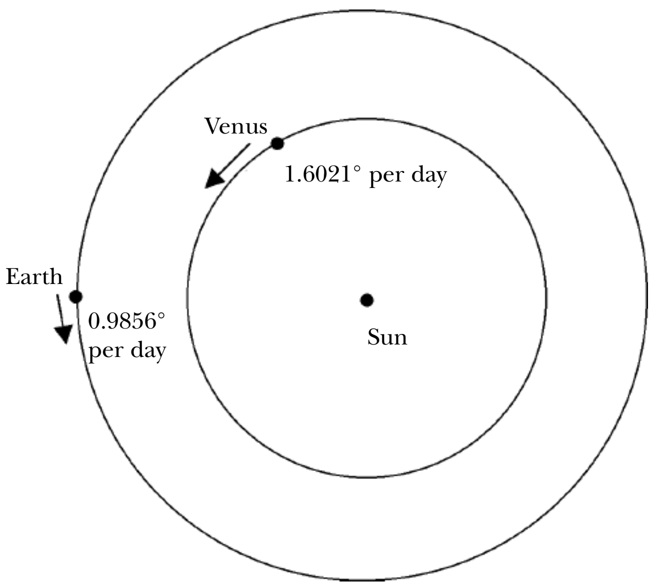

Figure 2

Hence, Venus is travelling faster than Earth by 0.6165° per day. The time it takes for Venus to "lap" Earth, from one planetary alignment to the next — that is, to travel 360° more than Earth — is approximately (360 ÷ 0.6165) ≈ 583.9 days. Or, more accurately,

$$\frac{360}{\left(\frac{360}{224.701}\right)-\left(\frac{360}{365.25636}\right)} = 583.9227 \text{ days}$$

Earth–Venus distance at alignment

If the distance from the Earth to the Sun is known, the distance from Venus to the Sun may be calculated by observing the maximum angular separation of Venus from the Sun, and applying trigonometry. So we have identified two additional quantities necessary to our model development.

The maximum angular separation between Venus and the Sun (as seen from Earth) is obtainable by direct observation at the relevant time of year:

Maximum angle of separation, Venus–Sun 46.324°

For the purpose of senior secondary mathematics students, it is appropriate to provide the Earth–Sun distance in terms of kilometres[1]:

Earth–Sun distance 149.6 million km

Given the angle and Earth–Sun distance, trigonometry can be applied to determine Venus' orbital radius, as shown in Figure 3.

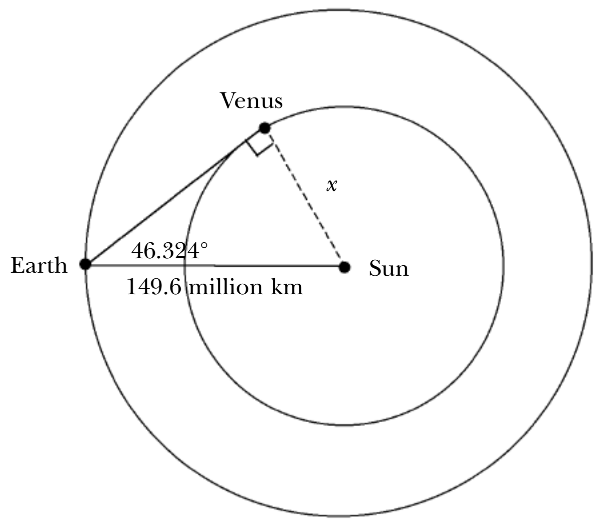

Figure 3

1. Note that in the 18th century, the Earth–Sun distance in kilometres was unknown. One of the primary motivations behind the study of the transit of Venus was to establish this value. Original calculations were made using Astronomical Units (AUs). One AU is defined as the distance from the Earth to the Sun. Calculation of the Earth–Sun distance, in kilometres, is beyond the scope of this article, but may be calculated with senior secondary mathematics. The procedure is detailed on several websites, including the online Sun–Earth Connection (2004) document.

$$x = 149.6 \sin(46.324) = 108.2 \text{ million km}$$

Venus' maximum distance from ecliptic for a transit

To calculate the maximum distance above or below the ecliptic that permits a transit to be possible we can use trigonometry, incorporating the angular size of the sun (as seen from Earth) and the distance of Venus from Earth, as shown in Figure 4.

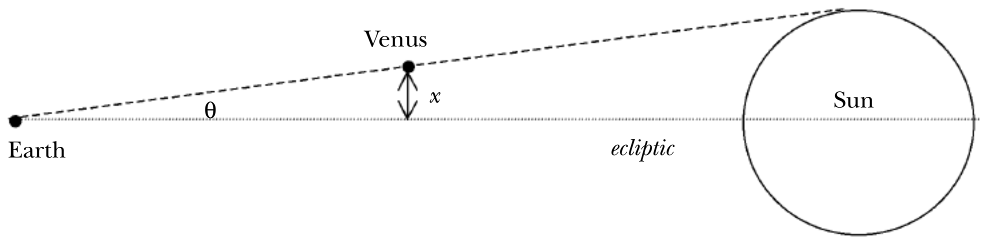

Figure 7

Sun's angular size from Earth $= 0.534°$, $\therefore \theta = 0.534° \div 2 = 0.267°$

Earth to Venus distance $= 149.6 - 108.2 = 41.4$ million km

$$x = 41.4 \tan(0.267) = 0.1929 \text{ million km}$$

Venus' distance from ecliptic

The underlying problem requires us to identify alignments, which we have determined to be 583.9 days apart and then, for each, decide whether it is a transit. To do this, we have to know how far above or below the ecliptic Venus is at that particular point in its orbit. If it is within 0.1929 million kilometres, then a transit will occur. So, how can we model Venus' position with respect to the ecliptic?

We are considering Venus to be travelling in a circular orbit at constant speed, inclined to the ecliptic. For half its orbit, it is above the ecliptic; for half, below. This regular oscillatory behaviour associated with circular motion can be modelled using a sine curve. The general form of a sine function is

$$y = A \sin(Bx + C) + D$$

where A is the amplitude, B is 2π divided by the period, and C is the horizontal displacement (or phase shift) of the curve. For now, we will consider C to be zero. We will return to this parameter later. D is the vertical displacement of the curve. Since Venus spends equal time above and below the ecliptic (see Figure 1), this parameter is zero. (Taking into account the latitude of an observer of Earth would affect this parameter slightly, but for simplicity of the model we are assuming all observations are taken from a point through which the ecliptic passes.) So the function simplifies to

$$y = A \sin Bx$$

where y is the height of Venus above the ecliptic and x is the number of days into Venus' orbit, measured from some nominated starting point, $x = 0$.

In this situation, the period is determined solely by the actual orbital period of Venus (224.701 days); that is, how long it takes for Venus to travel from its highest point above the ecliptic, to its lowest, and back to its highest — one complete cycle, which occurs in conjunction with its physical orbit. So B is simply

$$\frac{2\pi}{224.701}$$

A is the amplitude of the function; that is, the maximum distance of Venus above the ecliptic. This may be determined through, once again, application of simple trigonometry. All that is needed is Venus' distance from the Sun, which we calculated previously (108.2 million km) and its orbital inclination, obtainable through direct observation, 3.3944° (see Figure 5).

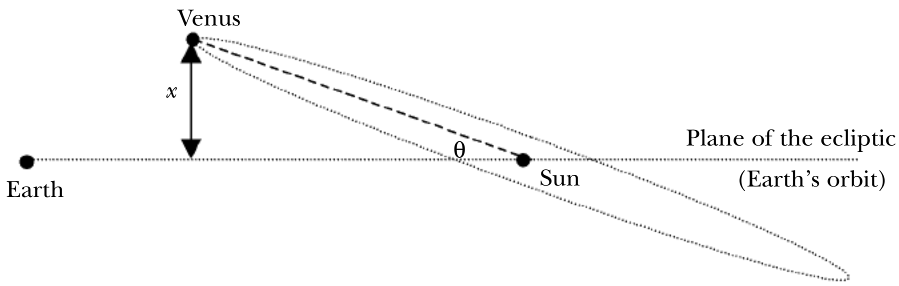

Figure 5

$$\begin{aligned} x &= 108.2 \sin(3.3944) \\ &= 6.4064 \text{ million km} \end{aligned}$$

We can now substitute parameter values into our periodic function:

$$y = 6.4064 \sin\left(\frac{2\pi}{224.701} x\right)$$

Remember, y represents the height of Venus above the ecliptic at a point x days into its orbit. This function appears as the graph in Figure 6.

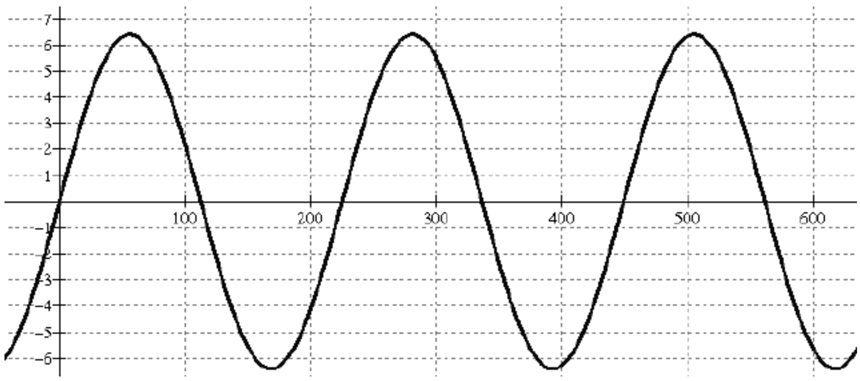

Figure 6

Acknowledging assumptions

Let us pause a moment to consider some of the assumptions that have been made in course of developing our model. It is essential that students recognise that assumptions are almost always present in a mathematical model. Models are only approximations, simplifications of actual phenomena. As long as the model answers a question to a desired level of detail or accuracy, it is satisfactory. But the underlying assumptions must be identified, understood and stated.

To simplify this model for high school students, we have made these assumptions:

1. That the orbits of Earth and Venus are circular. While planetary orbits are, in fact, elliptical, those of Earth and Venus have a very small eccentricity. That is, they are virtually circular. Modelling orbits with high eccentricities would require consideration of varying orbital speeds and distances from the Sun.
2. That Venus is a point. The Sun is a relatively large disk, as seen from Earth. In contrast, Venus' disk is very small, about 3% of the angular diameter of the Sun's disk (when viewed from Earth). By treating Venus as a point, we avoid complications that occur when Venus just "skims" the extremity of the Sun.
3. That Earth is a point. Another way of stating this is to assume that all observations are made from a point on Earth through which the ecliptic passes. Observations made from different latitudes generate different results as the observers have slightly different perspectives. The differences are very slight, but detectable. (In fact, it is these tiny differences that permitted transits to be used for calculating the Earth–Sun distance in the first place — refer to the Sun–Earth Connection (2004) document for details of the calculations.) For our purposes, the differences are negligible and are ignored.

Implementing and validating the model

We are now in a position to combine the elements we have calculated and generate a model of Venus' orbit that will allow us to identify when an alignment will occur, and whether that alignment will result in a transit.

The first step is to understand the origin of our function. When x is zero, y will also be zero (because $\sin 0 = 0$). What does this mean? At a point zero days into Venus' orbit, Venus is crossing the ecliptic. If we decide that an alignment is occurring on that day, it follows that a transit is occurring on that day (since $-0.1929 \leq y \leq 0.1929$). So it is most convenient to select the origin of our function as being a known transit. We have decided to choose the June 8, 2004 transit as our origin, but any other previously known transit is valid. If $x = 0$ represents 8 June 8 2004, then positive x values represent days after that date, and negative x values represents days further into the past.

We now have enough information to be able to identify transits. Recall that a transit can only occur during an alignment, and alignments occur every 583.9 days. Further, a transit can only occur when Venus is close to the ecliptic. So we need to examine the height of Venus (with respect to the ecliptic) at every alignment. This can be calculated manually for each multiple (both positive and negative) of 583.9 days, or by examining the graph of the periodic function (Figure 6) at every alignment value (i.e., each x multiple of 583.9) but is most conveniently implemented using a spreadsheet, or by setting up a table in a graphing calculator. For each alignment, we then need to examine the height of Venus with respect to the ecliptic. If it is within the critical range ($-0.1929 \le y \le 0.1929$) then we have a transit.

To set up a spreadsheet, we require two columns, representing x and y. The starting point is the origin, $x = 0$. We want to be able to extend our analysis further into the future by adding rows after the origin, but we also want to be able to look back into the past, so we need room to set up rows before (or above) our origin point. In the example in Figure 7, the origin has been positioned on spreadsheet row 171, which gives opportunity to look back to the 1700s. We need to set up subsequent rows where x is incremented by 583.9; similarly, prior rows are successively decreased by 583.9. The y value for each row can be calculated by the software simply by setting up each cell in the y column as our periodic function, using the value in the corresponding x column.

For convenience, it is also desirable to convert the x value (number of days) into a year. So $x = 0$, which is 8 June 2004, is 2004 $\frac{160}{366}$, or 2004.44. Then it will be easy to identify the year, and time of year, in which transits occur. As we add (or subtract) 583.9 to create successive x values, so too we add (or subtract) 583.9/365.25, or 1.599, years for each alignment. This can be represented as an additional column in our spreadsheet or table (see Figure 7).

It is now time to test this initial version of our model. For what values of x (or for what years) is a transit predicted to have occurred? This is determined by checking y values for each alignment to see if they fall within the critical range, ±0.1929. Investigating our spreadsheet into the past we see that, in addition to 2004, we also get transits in 1761, 1769 and 1882. The decimalised years are 1761.44, 1769.44 and 1882.94, corresponding with transits that occurred in June 1761 and 1769, and December 1882.

	C	D	E	F
11	**X value: Days (from origin)**	**Years (from origin)**	**Year**	**Y value: Ht above ecliptic**
19	-88756.25	-243.0	1761.44	**0.1147**
24	-85836.64	-235.0	1769.44	**-0.1539**
95	-44378.13	-121.5	1882.94	**-0.0573**
171	0.00	0.0	2004.44	**0.0000**

Figure 7

This seems fine until, upon checking historical records, we discover that there was another transit in December 1874, and the next transit is predicted (by far more sophisticated calculations than we have done here) to occur in June 2012. What has happened to these transits? Looking at the y values generated for each of these alignments (Figure 8), we can see that our model predicts near transits for these years — they are not far out of the critical range, but enough to cause us to consider the possibility that our model is not yet perfect!

	C	D	E	F
11	**X value: Days (from origin)**	**Years (from origin)**	**Year**	**Y value: Ht above ecliptic**
19	-88756.25	-243.0	1761.44	**0.1147**
24	-85836.64	-235.0	1769.44	**-0.1539**
90	-47297.74	-129.5	1874.95	-0.3258
95	-44378.13	-121.5	1882.94	**-0.0573**
171	0.00	0.0	2004.44	**0.0000**
176	2919.61	8.0	2012.43	-0.2685

Figure 15

Refining the model

We have generated a model, implemented it and attempted to verify it. It shows some promise, but it seems to be lacking in some respects. It is now time to return to some of the assumptions we made and see if we can improve the model.

Recall that one of the initial simplifying assumptions made was that $y = 0$ when $x = 0$. What does this mean? Simply, it means that during the June 2004 transit Venus was crossing the ecliptic ($y = 0$) at the precise time that the transit occurred. As viewed from Earth, Venus would be crossing directly in front of the centre of the Sun. We know that it is possible, however, for Venus to be slightly above or below the ecliptic at alignment and a transit will still occur. We come back to the critical range, $-0.1929 \le y \le 0.1929$. At $x = 0$, y may be anywhere in this range. Those who witnessed the 2004 transit may recall that Venus transitted quite some distance from the centre of the Sun.

How can we incorporate this factor into our model? This is where the C parameter in the periodic function comes into play. It represents the fact that y is non-zero at $x = 0$; or, alternatively, that y is zero at some value of x other than zero. Translating this into our scenario, it means that Venus crossed the ecliptic ($y = 0$) a short time before or after, rather than during, the transit (which is at $x = 0$).

The first step in introducing C into our model is to determine the possible range of values for C. This can be achieved by finding the value of C when x is zero and y is each of the minimum and maximum values allowable; that is, ± 0.1929. Substituting into our function, we have:

$$
\begin{aligned}
y &= A\sin(Bx+C) \\
\pm 0.1929 &= 6.4\sin\left(\left(\frac{2\pi}{224.7}\right)\times 0+C\right) \\
\frac{\pm 0.1929}{6.4} &= \sin C \\
C &= \sin^{-1}(\pm 0.0301) \\
C &= \pm 0.0301 \quad \text{(radians)}
\end{aligned}
$$

The next step will be done manually, although students familiar with manipulating spreadsheets may be able to come up with a technological solution. We want to find the value for C, within the range ± 0.0301, which will improve our model insofar as accurately identifying past known transits is concerned. So far we have tested a value of $C = 0$. We need to manually alter our function by iteratively incrementing C by amounts of, say, 0.001, and checking our transit predictions for the years 1761, 1769, 1874, 1882, 2004 and 2012. The best fit turns out to be a value for C of 0.012. So our function becomes

$$y = 6.4064\sin\left(\frac{2\pi}{224.701}x+0.012\right)$$

Using this formula, and checking values of y for all x multiples of 583.9, provides a quite accurate prediction tool. The model is still not absolutely perfect, but it is sufficient for most purposes, and that is as far as we will proceed in this analysis. It would be instructive at this point to return to the initial set of assumptions made, and investigate whether any of those may have had an undue effect on our result.

Implications for teaching

Certain familiar scenarios are often presented to students to demonstrate the application of trigonometric functions: tidal heights, and times between sunset and sunrise, for example. These are textbook examples where the information is handed to students and they convert it into a mathematical representation. What makes this task different is that it starts with a topical real world phenomenon for which students are required to "find the maths" in order to interpret it.

It is important to know, prior to embarking upon this (or a similar) exercise, what the purpose of the activity is, and to remain focused upon achieving that outcome. As far as demonstrating the applicability of mathematics to physical situations that are otherwise difficult to explain, describe or have predictions made about them, this example hopefully fulfils that aim. But when considered in the context of its being used as a learning experience of modelling and mathematical thinking, care must be taken to ensure the teaching and learning objectives are achieved.

Mathematical modelling takes place when students are able to interpret results, justify assumptions taken in the construction of their model, and use their model to explore and predict. But if a specific outcome is expected, and students are explicitly guided in such a way as to reach that outcome, then students, while experiencing a model and seeing its applicability, are not themselves modelling (Mason & Davis, 1991). The model that has been developed on the preceding pages can be presented as a detailed example of a mathematical model, but if it is required that the students develop the model themselves, clear emphasis must be placed on the modelling process, as distinct from the scenario being modelled: students must be able to identify relevant quantities, expose explicit or implicit assumptions, make refinements, validate the model, and interpret and communicate results.

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