Easing students' transition to algebra

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Traditionally, students learn arithmetic throughout their primary schooling, and this is seen as the ideal preparation for the learning of algebra in the junior secondary school. The four operations are taught and rehearsed in the early years and from this, it is assumed, "children will induce the fundamental structure of arithmetic" (Warren & Pierce, 2004, p. 294). Recent research has shown that the emphasis on computation can actually lead to many misconceptions in the students' minds, which in turn will make the learning of algebra more difficult.

This article will focus on two categories of student misconceptions, the first concerns difficulties with the notion of equivalence and the second concerns difficulties with the application of the four operations. The last section of the article presents suggestions on easing the transition to algebra through problem-solving.

Student misconceptions of equivalence

Falkner, Levi and Carpenter (1999) asked 145 American grade 6 students to solve the following problem:

$$8 + 4 = \square + 5$$

All the students thought that either 12 or 17 should go into the box. Referring to the same study, Blair (2005) reports: "It became clear through subsequent class discussions that to these students, the equal sign meant "carry out the operation". They had not learned that the equal sign expresses a relationship between the numbers on each side of the equal sign." This is usually attributed to the fact that in the students' experience, the equal sign always "comes at the end of an equation and only one number comes after it" (Falkner et al., 1999, p. 3). One expert has suggested to me that another possible origin of this misconception is the "=" button on many calculators, which always returns an answer.

Figures 1 and 2 show two typical responses from my Year 8 class to the question: "Explain the meaning of the '=' sign" (Toth, Weedon & Stephens, 2004). One third of the class (nine out of 27 students) gave an operational definition despite the fact that we had previously discussed the meaning of the equal sign in that class.

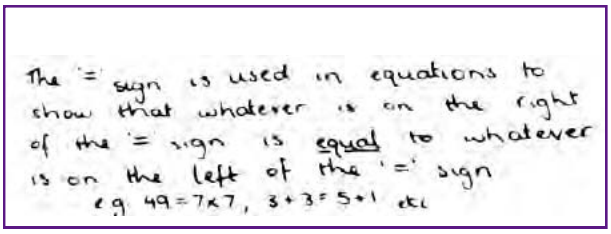

The '=' sign is used in equations to show that whatever is on the right of the '=' sign is equal to whatever is on the left of the '=' sign
e.g. 49 = 7×7, 3+3 = 5+1 etc

Figure 1. Relational and operational understanding of the equal sign by a Year 8 student.

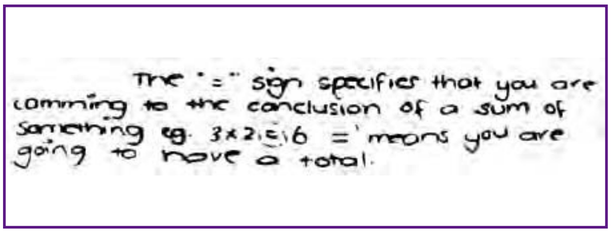

The "=" sign specifies that you are comming to the conclusion of a sum of something eg. 3×2=6 ='means you are going to have a total.

Figure 2. Relational and operational understanding of the equal sign by a Year 8 student.

While all of my Year 8 students are capable of solving problems such as the one above, those with an operational understanding of the equal sign perform the sum on the left hand side (8 + 4 = 12) and then resort to different strategies to find the missing number on the right. Having found that number, they then perform the operation on the right hand side in order to verify their answer. In contrast, those with a relational understanding of the equal sign recognise that the missing number must be one less than 8, since it is being added to a number that is one more than 4. Figures 3 and 4 contrast two students' justifications of the truth of the equation: 46 + 33 = 45 + 34. Clearly, those with an operational understanding can establish the truth of the statement, but their understanding proves to be a hindrance when learning algebra.

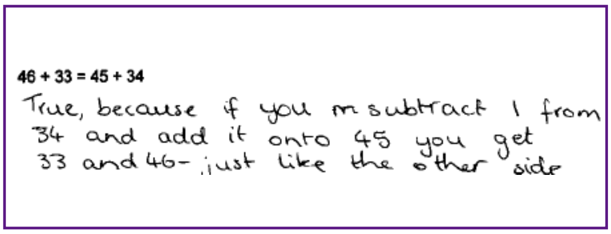
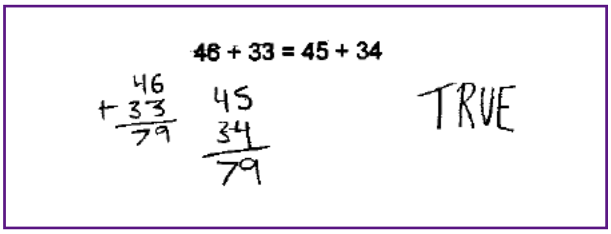

46 + 33 = 45 + 34
True, because if you subtract 1 from 34 and add it onto 45 you get 33 and 46 – just like the other side

Figure 3. Application of the relational and operational understandings of the equal sign.


46 + 33 = 45 + 34

46 + 33 = 79

45 + 34 = 79

TRUE

Figure 4. Application of the relational and operational understandings of the equal sign.

When teaching students to solve equations, we teach them the necessity of doing the same thing to both sides. This is "particularly important as students encounter and learn to solve algebraic equations with operations on both sides of the symbol (e.g., $3x - 5 = 2x + 1$)" (Knuth, Alibali, McNeil & Weinberg, 2005, p. 69). Unless a student understands that this rule exists to preserve the equality of both sides, then that student will have little chance of experiencing success. Teachers often pass over these difficulties by teaching their students to take terms to the other side of the equal sign and change their sign. I recently asked my Year 11 students to explain to me how they understood this method, and their answers are best summed up by the following statement from one of them: "It gets zapped by the equal sign!" Clearly, while the justification of that method may have been taught, the practice of taking terms over to the other side does nothing to address students' misunderstandings of equivalence.

Equivalence and teacher discourse

Booth (1986) suggests that teachers should emphasise the equivalence of an equation in the way they read number sentences. For instance, when working with the sentence "2 + 3 = 5", teachers should sometimes read the left hand side as "the number that is 3 more than 2", and avoid reading the equal sign as "makes" as this reinforces the operational meaning of the sign (p. 4). This use of language is not lost on curriculum writers. The following performance indicator comes from level 1 (prep.) of the Curriculum Standards Framework II, used in Victorian schools at the time of writing: "Use materials and models to develop and verbalise part–whole relationships (e.g., 6 is 5 and 1 more, two more than 4, one less than 7, double 3)" (2000, p. 31).

Building generalisations in arithmetic

Recent research is suggesting that students need to be helped, from an early age, to construct valid generalisations of the arithmetic operations. Fuji and Stephens (2001) introduce a concept built on elements of the Japanese program which they call a quasi-variable. They define this term as "a number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used" (p. 260). For instance, before students are able to understand an equation expressed as $a - b + b = a$, they can be introduced to equations such as 78 – 49 + 49 = 78. The truth of this relationship is independent of the number 78 “provided the same number is taken away and then added back” (p. 260). If such use of numbers is followed by class discussions, students can arrive at mathematical generalisations.

Falkner et al. found that, with guidance, students at first and second grade levels were able to make such generalisations as: “Zero added to another number equals that other number”, and “any number minus the same number equals zero”. In the same study, fourth and fifth grade students were able to explain the commutative law by generalising that “when you multiply two numbers, you can change the order of the numbers” (p. 2).

Student misconceptions of mathematical structure

Warren (2003) administered a written test to 672 students in grades 7 and 8 in two Brisbane schools. Neither group had as yet received formal training in algebra. She found that subtraction was regarded by 16% of the students as commutative, as was division by 18% of them. This means that they considered the following statements to be true:

$$2 - 3 = 3 - 2$$
$$2 \div 3 = 3 \div 2$$

Knowledge of the commutative law necessitates exposure to subtractions with a negative result and divisions with a result that is less than one. In interviews I conducted with six Year 7 students, I found that they recognised that division was not commutative. The reasons four of these students gave were concerned with the fact that dividing a number by a larger one “does not make sense” or “gives a really small number, like a negative number!” It is possible that these students' experiences with division were limited to whole number quotients. So, in addition to the excessive emphasis on computation, it may be that a limited experience with computation is another source of misconceptions.

In Warren's study, more students had trouble with the associative law, 17% answering false to a sentence involving addition, and 21% answering true to one involving subtraction. The New South Wales curriculum (Board of Studies NSW, 2002, pp. 74–75) includes the following objectives at stages 1 and 2:

- building addition facts to at least 20 by recognising patterns or applying the commutative property, e.g. $4 + 5 = 5 + 4$
- relating addition and subtraction facts for numbers to at least 20, e.g. $5 + 3 = 8$; so $8 - 3 = 5$ and $8 - 5 = 3$
- applying the associative property of addition and multiplication to aid mental computation, e.g. $2 + 3 + 8 = 2 + 8 + 3$, $2 \times 3 \times 5 = 2 \times 5 \times 3$

Facilitating a relational understanding of the operations

Gunningham (2004) describes buying lunch from a former student of hers. The student struggled to work out the correct change as she endeavoured to apply the subtraction algorithm she had learned at school. The author suggests that a better way of teaching number skills is to present students with authentic tasks where the numbers mean something. An enduring memory of my initial teacher training is a video we were shown in our very first mathematics method lecture. It was a video recording of interviews with primary students who were given “take-away sums” to perform. The success of these students was very limited. However, when asked more complex subtraction problems in the context of buying bars of chocolate, they were able to solve them mentally and describe their strategies in clear terms.

Before teaching my Year 8 class to find the percentage of a quantity, I asked them how they would work out the discount on a shirt that normally costs $40, if the store had a 15% sale. One student, Jackie, described her strategy of working out 10% of $40, which she found by dividing 40 by 10, and then adding half of that amount. Hence, 15% of 40 is the same as

$$(40 \div 10) + \frac{1}{2}(40 \div 10) \quad \text{or} \quad 4 + 2$$

My regret is leaving the strategy at that and launching into teaching the more standard algorithm:

$$15\% \text{ of } \$40 = \frac{15}{100} \times 40$$

A student who experienced a lot of success with this algorithm came back from the mid-year break and asked me to spend some time teaching her Jackie's method as she had found the standard algorithm too difficult to apply when shopping!

The idea of allowing students to discuss their intuitive strategies and helping them build on those is gaining currency in recent research. Carpenter, Levi, Franke and Zeringue (2005) report on interviews with two primary students, one of whom is described as "a fairly typical student in the class [who] was actually struggling more than most of the students, but she was learning with understanding" (p. 55). This student, referred to as Kelly, struggled with her multiplication tables, but used the distributive law to combine smaller multiplications with which she was more comfortable. For instance, she broke 4×6 into $2 \times 6 + 2 \times 6$ and got 32! When encouraged to check her answer, she said: "4 times 6 is 24, because 10 and 10 is 20, and 2 and 2 is 4, put those together and its (sic) 24" (p. 56).

Redefining algebra: Problem modelling and solving

Some researchers widen the definition of algebra and regard it as being more than a system of symbols. Instead, they concentrate on algebraic thinking, which "can be interpreted as an approach to quantitative situations that emphasises the general relational aspects with tools that are not necessarily letter symbolic, but which can ultimately be used as cognitive support for introducing and for sustaining the more traditional discourses of school algebra" (Kieran, 1996, cited in Johanning, 2004, p. 372). The idea behind this approach is to discover students' informal strategies in solving problems. These strategies are then used as a foundation for teaching the students to model the problem and their solution using symbolic notation. This may be the intent of the writers of the *Victorian Essential Learning Standards*, which will be implemented as of 2006, when they state: "Students identify variables and related variables in everyday situations, and explain the ideas of change, dependency and allowable values in relationships between pairs of variables" (Victorian Curriculum and Assessment Authority, 2005, Mathematics Level 4, Standards, Structure, paragraph 2).

Johanning gave two worded problems to 31 students in grades 6 to 8 who had not yet learned algebra and whose exposure to problem solving was limited. One of these was the "candy bar" problem:

> There were three kinds of candy bars being sold at the concession stand during the Friday dance. There were 22 more Snickers bars sold than Kit Kat bars and there were 32 more Reese's Peanut Butter Cups sold than Kit Kat bars. There were 306 candy bars sold in all. How many of each kind of candy bar was sold?(p. 374)

Nine out of the twelve students who elected to solve this problem did so successfully, and seven of those used a systematic guess and check. The difference in the solutions was mainly in the quantity which the students guessed first. Three approaches were observed:

(a) Most started with the number of Kit Kat bars, as this was the smallest quantity. To that, "they added 22 to find the number of Snickers bars. Next, they took the number of Kit Kat bars and added 32 to find the number of Reese's Peanut Butter Cups. Finally, the amounts of the three candy bars were totalled to see if the sum was 306" (p. 378).

(b) One student began by guessing the number of Reese Cups and used subtraction to obtain the other two quantities.

(c) One student adjusted the first approach in that he guessed the number of Kit Kat bars, multiplied it by three, and then added 22 and 32.

While not explored in the study, Johanning proposes that such problems could form the basis of teaching algebra, with symbolic notation being introduced to model the students' solutions. For instance, the solution in (a) can be modelled as $n + (n + 32) + (n + 22) = 306$, while (c) can be modelled as
$3n + 22 + 32 = 306$.

Using intermediate representations

In this section, I present examples from two research projects that have taught students algebra through realistic tasks. Both have used the students' intuitive solutions and encouraged them to develop their own symbolic representations. The researchers then built on those intermediate representations in teaching them formal algebra. I hope that these examples will show that students do have abilities which, if allowed to come to the fore, can form a sound basis for our teaching.

In Montreal, Bednarz worked with 24 students in secondary 2 (13 and 14 years old) who had been identified as having "low-ability". She aimed to build on these students' experiences with problem-solving in arithmetic. She argues that continuity with arithmetic can be achieved through a gradual "complexification of reasoning procedures" (2001, p. 71) and the use of problems that show the advantage of using a symbolic language, thus endowing the symbols with meaning. Towards the end of the intervention, the following problem was posed with the aim of forcing the students to concentrate on the relationships between three unknown quantities:

> A son (hired by his father to do an inventory) left him the following message: three types of articles were counted. There are 2 times more rackets than balls, and 3 times more hockey sticks than rackets. (p. 74)

The students had to model the problem using multiple representations. An example is given of a student who drew three columns of symbols. In the first column, a symbol is drawn to stand for the total number of balls. In the second, two symbols of the same type are presented to stand for the number of rackets (which is twice that of balls). The third column contains six such symbols to represent the number of hockey sticks. The same student then proceeded to define variables to model the problem using the letters *b*, *r* and *h* which he used in two letter-symbolic equations. A description in words was also presented, followed by a restatement of the standard equations using pictures of the balls, rackets and hockey sticks.

Brizuela and Schliemann (2003) used a similar approach of posing problems and inviting students to develop multiple representations. They write of the success of some grade 4 students, with whom they had worked from the time they were in grade 2, at solving equations. The following problem was posed:

> Two students have the same amount of candies. Briana has one box, two tubes, and 7 loose candies. Susan has one box, one tube, and 20 loose candies. If each box has the same amount and each tube has the same amount, can you figure out how much each tube holds? Each box? (p. 3).

Samples are presented by the author of students representing the problem with pictures and solving for the unknowns. Of particular interest is one student's sophisticated solution which starts by modelling the problem as: $20 + 1N + J = 7 + 2N + J$. The student then proceeded to eliminate common terms from both sides, to obtain $N = 13$. He explained: "I broke 20 into 7 and 13. Then matched 7 and 7. Then broke $2N$ into N and N and matched them. Then $13 = N$" (p. 3).

As these examples show, students can surprise us if we give them meaningful problems and the necessary support, trust and time to discuss and solve them.

Conclusion

This article has discussed some of the misconceptions that many students have at the end of their primary years of schooling. It has argued for the promotion of a relational understanding of equivalence and the four operations in the primary school. The article has also attempted to present some suggestions for building a bridge between arithmetic and algebra through problem-solving. A common theme in all the cited research is the necessity to expose students to arithmetic in a variety of contexts and to allow them to discuss their intuitive solutions.

The transition to algebra can be eased if we acknowledge that our students need explicit assistance from us to see its relationship with arithmetic, and if we show them that algebra is a tool for them to model realistic situations.

Current approaches confuse students, presenting algebra as the art of manipulating symbols. This art is made up of arbitrary rules to be remembered, it is hoped, until the unit of work is finished and the test is passed. Year after year, we teach the students the same skills and lament their lack of proficiency. We need to break this cycle and make room for meaningful tasks that build a good understanding in the middle years.

Finally, I must acknowledge the many limitations of this article. Among those is the absence of a discussion of the misconceptions related to the concept of a variable and the uses of technology and manipulatives.

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