# Carl Friedrich Gauss

KATHRYN RICE & PAUL SCOTT

## Education

Johann Carl Friedrich Gauss was born on 30 April 1777 in the German city of Braunschweig (Brunswick). He was the only child of Gebhard Dietrich Gauss and Dorothea Benze. Neither of Gauss's parents had much education, his father could read and write, but earned his living doing menial jobs such as gardening and as a street butcher; his mother could read but not write.

In 1784 Gauss entered elementary school, where he was put in a class of fifty students. Gauss knew how to read and write before entering school, skills he appears to have learnt without any help from his parents. Gauss was lucky to have a competent and concerned teacher, Buttner, and assistant teacher, Bartels, who took a personal interest in Gauss, trying to help and encourage him. His teachers were amazed that when asked to sum the numbers 1 to 100, Gauss added (1 + 100) + (2 + 99) + (3 + 98)… all the way to (50 + 51), reducing the whole problem to (50 × 101) = 5050.

Carl Friedrich left his parents home in 1788 when Buttner helped him be admitted to secondary school. The lectures were orderly and regular with smaller class sizes and with students who all had the same level of knowledge. The curriculum was old-fashioned and unbalanced, with an over-emphasis on Latin and High German. While still a teenager, Gauss propounded the theory of least squares, demonstrated a solution to the age-old problem of dividing a circle into 17 parts, and made important mathematical discoveries which he was too shy to publish, and entrusted only to his diary.

In 1791 Gauss was introduced to his prince, Duke of Brunswick-Wolfenbuttel, who undertook to finance his education and, in the process, became his lifelong patron and friend. Gauss attended the University of Göttingen and in 1799 obtained a Doctorate in Mathematics from the University of Helmstedt. About this time he turned his attention to astronomy, making brilliant computations of orbits of asteroids.

Gauss married Johanna Ostoff on 9 October 1805. He now had a happy personal life for the first time, although his benefactor, the Duke of Brunswick, was killed fighting for the Prussian army. In 1807, Gauss became director of the observatory in Göttingen, a post he held until his death in 1855. In 1808 his father died, and a year later Gauss's wife Johanna died after giving birth to their second son, who was to die soon after her. Gauss was married for a second time the next year, to Minna, the best friend of Johanna, and although they had three children, this marriage seemed to be one of convenience for Gauss.

The period 1817–1832 was a particularly distressing time for Gauss. He took in his sick mother in 1817, who stayed until her death in 1839, while he was arguing with his wife and her family about whether they should go to Berlin. He had been offered a position at Berlin University and Minna and her family were keen to move there. Gauss, however, never liked change and decided to stay in Göttingen.

Most of Gauss's contributions to mathematics and science were written in about 155 papers. He published anything only after the most thorough investigation and after he was sure it met his motto: *Ut nihil amplius desiderandum relictum sit* — that nothing further remains to be done. His mind was so full of numbers that when he was interrupted in solving a problem, and told that his wife was dying, he reportedly replied: "Tell her to wait until I am done."

In 1831 Gauss's second wife died after a long illness. Gauss died in his sleep early in the morning of 23 February, 1855.

## The fundamental theorem of algebra

The fundamental theorem of algebra was first stated by d'Alembert in 1746 but was only partially proved. Gauss, at the age of 21, was the first to give a rigorous proof of the theorem. In fact, Gauss gave three alternative proofs to the theorem. All three proofs can be found in the third volume of his *Works*.

The fundamental theorem of algebra states that every equation of the $n$th degree has $n$ roots. We can alternatively express this theorem as:

> the polynomial
> $f(z) = z^n + c_1 z^{n-1} + \ldots + c_{n-1} z + c_n$
> can always be factored into $n$ linear factors of the form $z - a_i$.

The proof of this theorem is done in two steps, first showing that an equation of the $n$th degree has at least one root, and then showing that the equation has $n$ roots and no more.

Note that it is possible for several of the $n$ roots $a_1$, $a_2$, … $a_n$ to be the same; e.g., $a_1 = a_2 = a_3$. In this case $a_1$ is a *multiple root* and for the example given is a *root of multiplicity three*.

## Number theoretical work

*Disquistiones Arithmeticae* was published in Leipzig in 1801 and centred on Gauss's work in number theory. *Disquistiones Arithmeticae* is divided into seven chapters, called sections. The first three are introductory, sections IV–VI form the central part of the work and section VII is a short monograph devoted to a separate but related subject. The work is dedicated to the Duke of Brunswick.

The first section, only five pages long, deals with elementary results and concepts, such as the derivation of the divisibility rules for 3, 9 and 11. As the most basic concept in the work, congruences for rational integers modulo (a natural number) are defined and their elementary properties are proved, among them the division algorithm.

In section II (24 pages) Gauss proves the uniqueness of the factorisation of integers into primes and defines the concepts of greatest common divisor and least common multiple. After defining the expression $a \equiv b \bmod c$, Gauss turns to the equation $ax + k \equiv c$. He derives an algorithm for its solution and mentions the possibility of using continued fractions instead of the Euclidean algorithm.

Section III (35 pages) contains an investigation of the residues of a power of a given number modulo (odd) primes. The basis of the investigation is Fermat's 'little' theorem:

$$a^{p-1} \equiv 1 \pmod p$$

where $p$ is any prime which does not divide $a$.

Gauss gives two proofs, one by 'exhaustion' which goes back to Euler or possibly Leibniz. The other uses the 'binomial theorem'

$$(a + b + c + \ldots)^p \equiv a^p + b^p + c^p + \ldots \pmod p$$

In section IV the main topic is the law of quadratic reciprocity. The law derives its name from a formalism invented by Legendre and is defined as follows. If $p$ and $q$ are positive, odd primes, we define

$$\left(\frac{q}{p}\right)=\begin{cases}+1 & \text{if } x^2 \equiv q (\text{mod } p) \text{ is solvable} \\ & \quad\text{in whole numbers} \\ -1 & \text{otherwise}\end{cases}$$

The *law of quadratic reciprocity* is the identity:

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right)=(-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$$

This result had been formulated by Euler and discussed at length by Legendre but had not been proved correctly. Try some simple examples!

Section V is the main section of *Disquistiones Arithmeticae*. It deals with the theory of binary quadratic forms; i.e., algebraic expressions of the type:

$$f(x,y) = ax^2 + 2bxy + cy^2$$

for given integers $a$, $b$ and $c$. A substantial amount of the fifth section is not original but merely summarises the work of Legendre. Gauss clearly indicates where his original work begins and gives credit to others were required.

In section VI Gauss presents several important applications of the concepts included in Section V. The principal topics are partial fractions, periodic decimals and the resolution of congruences. Another interesting topic is the derivation of criteria for distinguishing between prime and composite numbers. The remaining section of *Disquistiones Arithmeticae* deals with ruler and compass constructions and will be discussed shortly.

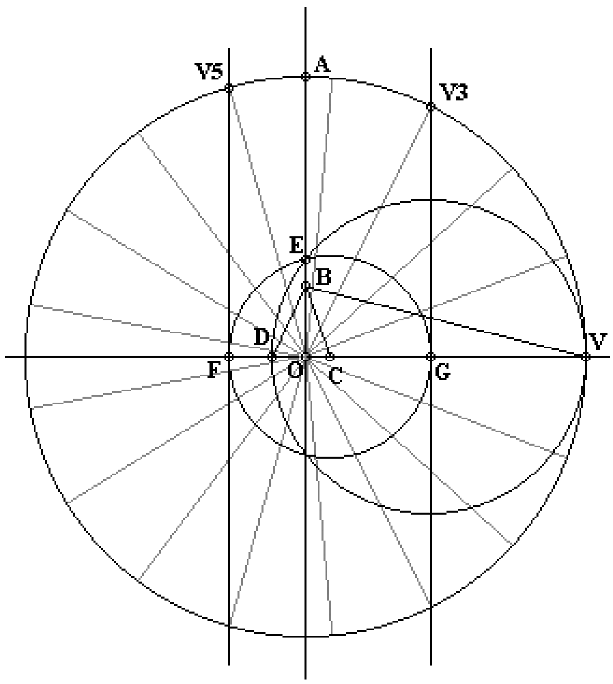
## Ruler and compass constructions

The division of the circle using only a ruler and a compass is an old and classical problem. For many years it had been an open and often discussed problem as to whether the regular 17-gon can be constructed with the use of these two tools only. The ancient Greeks knew how to construct an equilateral triangle (3-gon), a square (4-gon), and a regular pentagon (5-gon), and of course they could double the number of sides of any polygon simply by bisecting the angles. They could also construct the 15-gon by combining a triangle and a pentagon. For over 2000 years no other constructible $n$-gons were known. Then, in 1796, the 19 year old Gauss discovered that it was possible to construct the regular heptadecagon (17-gon). Subsequently Gauss presented this result at the end of *Disquistiones Arithmeticae*, in which he proves the constructibility of the $n$-gon for any $n$ that is a prime of the form

$$2^{2^k} + 1,$$

also known as *Fermat primes*.

One of the nicest actual constructions of the 17-gon is Richmond's (1893), as reproduced in Stewart's *Galois Theory*. Draw a circle centred at $O$, and choose one vertex $V$ on the circle. Then locate the point $A$ on the circle such that $OA$ is perpendicular to $OV$, and locate point $B$ on $OA$ such that $OB$ is a quarter



of $OA$. Next locate the point $C$ on $OV$ such that angle $OBC$ is a quarter the angle $OBV$. Then find the point $D$ on $OV$ (extended) such that angle $DBC$ is half of a right angle. Let $E$ denote the point where the circle on $DV$ as diameter cuts $OA$. Now draw a circle centred at $C$ through the point $E$, and let $F$ and $G$ denote the two points where this circle cuts $OV$. Then, if perpendiculars to $OV$ are drawn at $F$ and $G$, they strike the main circle (the one centred at $O$ through $V$) at points $V_3$ and $V_5$, as shown in the figure.

The points $V$, $V_3$, and $V_5$ are the zero, third, and fifth vertices of a regular heptadecagon, from which the remaining vertices are easily found (i.e., bisect angle $V_3OV_5$ to locate $V_4$, etc.). Gauss was clearly fond of this discovery, and there is a story that he asked to have a heptadecagon carved on his tombstone, like the sphere inscribed in a cylinder on Archimedes' tombstone.

## Useful links and references


Eves, H. (1990). *An Introduction to the History of Mathematics, (6th ed.).* Saunders Publishing Co.

Stewart, I. (2004). *Galois Theory (3rd ed.).* New York: Chapman & Hall.

*The MacTutor History of Mathematics archive.* Available at: www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Gauss.html

*Gauss' biography.* Available at: www.geocities.com/RainForest/Vines/2977/gauss/g_bio.html

*Carl Friedrich Gauss.* Available at: www.genie-gauss.de (in German).


*Laplace was once asked who was the greatest mathematician in Germany, and Laplace replied, "Pfaff." "But what about Gauss?" he was asked. "Gauss," replied Laplace, "is the greatest mathematician in the world."*


**Kathryn Rice**

**Paul Scott**
Wattle Park, SA
mail@paulscott.info