Playing outside: An introduction to the jazz metaphor in mathematics education

Jim Neyland
Victoria University, Wellington, New Zealand
jim.neyland@vuw.ac.nz

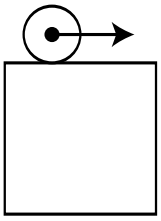
How exact is mathematics?

The trouble with mathematics is that it looks a little more logical and consistent than it is. Mathematics has a universally recognised exactitude. It also has an inexactitude that tends to remain concealed. Mathematics contains what we might call 'illogical truths'; that is, truths whose apprehension requires, not logical reasoning, but insight and imagination. These 'illogical truths' are experienced as surprises; as malformations. You may be a little perplexed by my use of this last term; it may appear non-mathematical. It is not. The history of mathematics contains many examples of these malformations. Imre Lakatos (1976) wrote a book, *Proofs and Refutations*, in which he highlighted their significance. And he used an even stronger term. What I have just called malformations, he called 'monsters'. So, what I am saying is this. The person who studies mathematics is more than a logician. The study of mathematics has an inescapable element of unpredictability. And, mathematics endlessly escapes being captured within the logical frames of any mathematical structures.

To put it differently, the trouble with mathematics is that it is neither structured, nor unstructured. It is nearly structured, but not fully. It is neither tamed nor wild. It is both nearly tamed and forever untameable. This fact is troublesome only for those who do not adequately recognise it. Of course, what we consider troublesome is often just a matter of perspective, and this is the case here. This troublesome inconvenience — that mathematics remains forever a little wild — differently considered, can be seen as the vital life force within mathematics that keeps it interesting, and ensures that it maintains an element of mystery and enchantment. All this has been argued by a number of philosophers of mathematics, and many of the key ideas have been nicely summarised for the general reader by Philip Davis and Reuben Hersh (1981) in *The Mathematical Experience*, particularly in Chapter 7.

An example for the purposes of illustration

I will use the following example to illustrate something of what I am saying, and I will return to it every now and then during this article. It concerns a circle rolling around the outside of a square. What is the length of the path formed by the centre of the circle when the circle rolls once around the outside of the square? Please take a moment to investigate this.



The answer is, you will have discovered, the sum of the perimeters of the circle and the square. Let us extend this problem to that of a circle rolling around a rectangle, a triangle, and a parallelogram. What happens in each of these situations? What can be concluded? Please envision what a group of teenage students might conclude from such an investigation. I will return to this shortly.

What is the nature of mathematics learning?

I began by talking about the nature of mathematics, and noted that it will always escape being tamed. I will turn now to the nature of learning, to find out whether something similar is manifest here. How can one tell whether or not mathematical learning has occurred? There have been many answers given to this question. John Holt (1983), for instance, reminded us that we can never tell for sure. What is your answer to this question? And, I ask you also, what is implicit in the way mathematics education is administered in your state?

The acclaimed mathematician and educator, Hans Freudenthal (the person whose scholarship in mathematics education led to the Freudenthal Centre in the Netherlands bearing his name) gave an answer to this question that has similarities with my earlier discussion of wild and tame mathematics. In *Weeding and Sowing: Preface to a Science of Mathematics Education* (1987) he argues that it is the element of 'surprise' (p. 180) that allows you to know if learning has occurred. There is that word again: surprise! Are you surprised by this answer? It is interesting that he values this over things like successful performance on assessment tasks, or over students verbally reporting that they understand. Surprise is of utmost importance for him. It is his paramount conclusion, drawn after many years of painstaking observation of people learning mathematics, and it is the central thesis of his book. Elsewhere he substitutes the word 'discontinuities' for surprise. He says: 'what matters in learning processes are discontinuities' (p. 165). What are these 'learning discontinuities' or 'surprises'? They occur when the learner experi-

ences a 'complete reversion from a convinced 'yes' to a convinced 'no" with respect to some mathematical idea (p. 182). And they arise from programmes of work that give centrality to 'open learning situations' (p. 181); that is, those that have 'rich content' that has not yielded 'to over-stressing formal features' (p. 177). Here Freudenthal talks about moving from a convinced 'yes' to a convinced 'no'. Clearly he is talking about a situation where a student has some sort of structure that is initially seen as adequate, later found inadequate, and subsequently — for otherwise the 'no' would not be 'convinced' — replaced by another. And to repeat, this is, in his estimation, of foremost importance for teaching and learning.

Revisiting the illustrative example

Students typically and correctly conclude that, for each of the polygons named, the path length is again the sum of the two perimeters. This is, I hope you agree, a nice result; it has simplicity and elegance. What if this situation were extended to include pentagonal, hexagonal and other polygons? What might these students conclude now?

The understanding of mathematics and its teaching defended in *The Mathematical Experience* and *Weeding and Sowing* does not naturally lend itself to the sorts of approaches that go with the ubiquitous outcomes-based mode of curriculum organisation found in Australia and other countries. Before I go on I need to make it clear what I mean by this term 'outcomes-based'. There are two common interpretations. The first sees it as placing an emphasis upon what students can do mathematically, and not merely on where they fit on a hypothetical statistical curve. The second interpretation — the one to which I am referring and about which I am critical — understands it as a component of the scientific management of education, and linked to a particular notion of teacher accountability associated with this. The scientific management of education is the application to education of what is sometimes called 'the linear planning model'. Within this model the outcomes of the production process are set in advance, and the system — of which the teachers form a part — is monitored and controlled to ensure that these outcomes are achieved.

The linear planning model does not work well in education. There are a number of reasons for this, and principal among them is the fact that the model clashes with the way experienced teachers work naturally in the classroom. It is noteworthy that the model is now being found inadequate even in the industrial and corporate worlds. It is becoming evident that experienced teachers operate in a way that is better described by what is called a 'complexity' model, a radical alternative to the linear model, and one that is also now being embraced in the industrial and corporate worlds. In the complexity model, teachers and students do not walk down a predetermined linear path. They become together a small learning organisation that engages in 'laying down a path while walking'. This expression is used by the authors of the influential book *The Embodied Mind* for the title their final chapter (Varela,

Thompson & Rosch, 1991). It has a meaning similar to that of 'emergence'; a term commonly used in the literature on complexity. That which is emergent is not that which is predefined, but that which is defined in and through the process of engagement. Brent Davis, Dennis Sumara and Rebecca Luce-Kapler (2000) in *Engaging Minds: Learning and Teaching in a Complex World* — a book that discusses the implications of complexity theory for education — use a similar term, 'occasioning', which they say 'refers to the way things and events 'fall together' in complex and unexpected ways' (p. 144). We could also use the term 'structuring', provided we thought organically and not in terms of engineering.

The teacher's role in the complexity model is that of an artist; but not any kind of artist. The teacher is not the sort of artist that turns lumps of clay into pottery, or a blank canvas into a painting. He or she is an improvisational artist who participates in the process of emergence, but in a special way. The improvisational teacher uses an 'attractor' — this is a technical term used by mathematicians when referring to the way some chaotic systems eventually settle to an emergent order, and in teaching can be taken to mean what is called a 'rich mathematical activity', and what Freudenthal calls an 'open learning situation' — and watches what happens when the students engage with it. The teacher both participates with the students, and skilfully aids the process of mathematical emergence. The latter is achieved by the teacher amplifying some elements of emergent thought — those judged to be mathematically fruitful — and bracketing others — those judged not to be mathematically fruitful. The outcome is not predetermined; it emerges through participatory engagement.

Rethinking mathematics teaching using jazz as a model

I hope I have now sufficiently set the scene for me to be able to introduce the jazz metaphor. Much of what I have been referring to about the complexity model is evident in the way jazz improvisation occurs. And there is much, besides, about the jazz metaphor that is of importance; especially for any conception of mathematics education beyond the ubiquitous linear model. To repeat, the jazz metaphor is a useful way of understanding complexity, and it has specific explanatory potential beyond that. My analysis of the jazz metaphor and its relation to mathematics teaching has led me to identify five key components of jazz: structure, improvisation, playing outside, pursuit of ideals, and 'ways of the hand'. A full discussion of these would require a series of articles similar to this one. In this paper I am concentrating on only a small part of the third of these components: playing outside.

Returning to the example

Many students, given enough time, come up with a theorem which they can

more or less see their way to proving: When a circle is rolled around any polygon the path length of the centre is the sum of the perimeters of the circle and the polygon. The proof correctly involves putting the rounded corners of the path together to form the original circle. The students rightly feel pleased with both their theorem and their proof. But you know there is a problem with it, and the problem is not with their logic, not with their proof; it is with their imaginations. Why? By the way, it is noteworthy that Lakatos, in his book, showed that a geometrical problem not dissimilar to this one — it involved polyhedra — caused mathematicians difficulties for much the same reason; their proofs were good but their imaginations were limited.

Back to jazz

I referred, in the above, to the jazz metaphor. This meaning of 'metaphor' is not quite the one with which you will be familiar from your school poetry lessons. Iris Murdoch (1997) argued that metaphors are more broadly influential than is typically acknowledged in school. They 'are not merely peripheral decorations or even useful models, they are fundamental forms of our awareness' (p. 363). The educator Neil Postman (1995) said the same. 'A metaphor is not an ornament. It is an organ of perception' (p. 174). It is important to be aware that our understanding of mathematics education is always shaped by powerful metaphors. It is not a question of whether or not our understanding is shaped by particular metaphorical images, it is a question of which metaphorical images we use and whether we are aware of our using them. We cannot escape them. It is my assertion that the jazz metaphor is of significant explanatory value, and that it gives emphasis to aspects of mathematics teaching that are marginalized by those approaches associated with the scientific management of education.

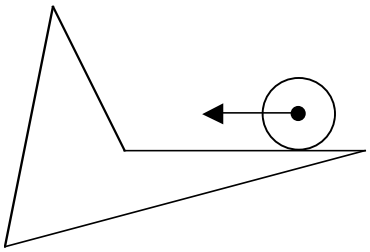
Now a little about jazz structure and playing outside. Briefly, a jazz combo is a complex organisation. It is a small organisation that learns seamlessly as it goes along. It is a thinking organisation. Original improvised music emerges; but what emerges is not predetermined: it emerges only in the playing. There is structure; for instance, the tune around which a given improvisational experience is based has a particular chordal structure known in advance; a chordal structure is a sequence of chords (those things that guitarists use for sing-a-longs, but more sophisticated), each applicable for a set number of musical beats. Incidentally, the tune is much the same as what I called above an attractor. Importantly, those structures that occur in jazz are carefully designed to be neither too small nor too large. Jazz players seek an optimally minimal structure that allows for the best creative improvisation. Too little or too much structure stifles creative improvisation. Again importantly, the structure is always secondary to the primary goal of jazz which is improvisation, not the other way around. Improvisation can, and should, always ask the question of structure: are you the optimal minimum that best serves me?

There are two types of improvisation. The most common is 'playing inside'

the established structures. This is roughly equivalent to the students in our example exploring circles rolling around simple polygons. When musicians play inside they perform freely within the parameters established by the structure. There is play, there is freedom, but nothing happens that is radically unexpected or surprising.

Returning to the example

The students, in forming their theorem and proof, were playing inside a certain structure. They started with squares and triangles, generalised from there, and correctly stated and proved a theorem; one that is true within this structure. But their imaginations did not extend beyond the parameters of this structure to an 'illogical' mathematical truth. They did not have the insight to imagine a polygon with a concave vertex. Such a polygon would be for them what I referred to above as a malformation.

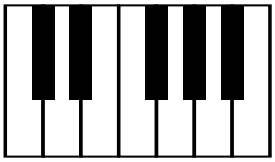


Once this malformation has been encountered the question arises: does the established theorem-proof couplet hold for this new case; or is a new theorem or proof needed? What do you think? Can you justify what you think?

In jazz, 'playing outside' refers to a radical form of improvisation that deliberately transcends the established structures. In one instance of this the player will deliberately play only those notes that lie outside the parameters set by the structure. What does this mean? Briefly put, at every point during a piece, a particular musical chord will be set down for a specified number of musical beats. For each chord, seven of the twelve available notes on the standard musical octave — recall that there are twelve piano notes, some white and some black, in every octave — will fit naturally with the chord, and the other five will not. To use a simple example, if the chord that is set down is roughly the musical equivalent of c-major, then playing some of the white notes on a piano keyboard is equivalent to playing fully 'inside'. In practice, improvisers play a few notes outside these seven, but only a few. These few well-chosen 'outside' notes give the improvised piece some colour and add a little tension.

'Playing outside' fully, in the way I am talking about here, occurs when the only notes chosen by the improviser are from the five that are specifically outside the given structure. The player, as it were, plays deliberately only the wrong notes. You can imagine what this sounds like: surprising, disturbing, exciting, malformed. In our example based on the c-major chord this means

playing only the black notes on the piano keyboard; none of the white notes. Players cannot afford to do this for more than a short space of time, else the whole combo loses its structure. Why is playing outside done at all? It creates a high degree of tension. It is a way of exploring the limitations of the established structure. It is a way of keeping the structure secondary to creative improvisation: if playing outside is impossible, for instance, the structure is too dominant. Finally, playing outside often leads to new innovations in jazz. Playing outside, to put it differently, is playing with the structure, not within it, as happens in normal improvisation. As such, playing outside is essential in the study of mathematics. The case for this was most powerfully argued by Imre Lakatos in *Proofs and Refutations* where he showed that proof is not enough for establishing mathematical truth. It is also critically important that we look also for refutations, for the monsters that lie outside the frame of reference determined by the structures taken as unproblematic. The notion of playing outside can be applied in many other situations in mathematics teaching. Can you think of some others?



If the chord is c-major, the white notes on the piano keyboard will sound 'in' — that is, tuneful — and the black notes will sound 'out' — that is, discordant. Deliberately playing only the black notes over a c-major chord is 'playing outside'.

'Pursuit of ideals' and 'ways of the hand'

I cannot finish this paper without mentioning briefly two other components of jazz, the 'pursuit of ideals' and 'ways of the hand'. 'Pursuit of ideals' is central to jazz, and it requires renewed attention in mathematics teaching if the complexity model is to gain any traction against the scientific management model. Jazz musicians would not be able to play jazz if they did not have some sense of what is good jazz. Interestingly, jazz musicians will typically not be able to put this into words; it remains ineffable. But they recognise it when they hear it, and they know when they fall short. In jazz the notion of 'swing' is an excellent example. All jazz musicians want to swing. But none can tell you what exactly is needed for swing. Swing in mathematics suffers the same limitation. When I'm constructing a proof of a mathematical theorem, for instance, I aim for what mathematicians call 'elegance'. When I come across an elegant proof in a book I recognise it as such without difficulty. But, while I could explain a little of what elegance entails, I could never say exactly what it is. It is part of the background of qualitative distinction that is attendant on mathematics. But it is a background that can never be brought fully into the foreground.

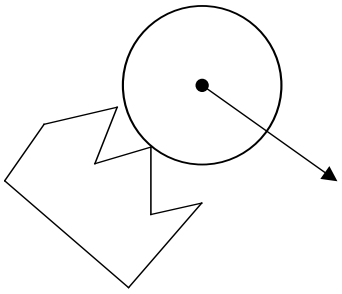
Now to 'ways of the hand'. *Ways of the Hand* is the title of an influential book by John Sudnow (1978), a jazz musician and philosopher. Jazz only occurs properly, he showed, when jazz musicians, as it were, throw themselves

into the playing without self-consciousness, and with a special kind of attentiveness to the music. When this happens, Sudnow observed, the jazz musician finds that her hands do the playing by themselves; they do not consciously follow the command of the mind. 'Ways of the hand' is a powerful way of thinking about competent performance by teachers and students. It makes room for the sort of intuitive and instinctive way that experienced teachers actually operate in the classroom. And it is similar to what teachers recognise in some of their students as 'fluency with mathematics'. In addition, 'ways of the hand' could shed new light on an old problem. 'Ways of the hand' is nearly the opposite of 'mathematics anxiety'. I am hopeful that the study of this feature of jazz playing will eventually lead to a reduction in such anxiety? This is a work in progress.

'Ways of the hand' sharply contrasts with the scientific management model. In particular, it contrasts with the way the latter construes accountability and expertise, and the importance it gives to protocols, procedures and rules in an attempt to ensure the delivery of expert performance. The implications that flow from this way of thinking are, I think, particularly exciting. It seems to me vastly superior to the currently orthodox scientific model. The language of protocols and accountability has always felt misplaced in relation to my experiences teaching mathematics. Generally speaking my best teaching has not resulted from a kind of effortful deliberateness. It has come when I have allowed modes of attentiveness and intuitiveness to come together in a kind of play that goes in its own direction; a direction that, when viewed in retrospect, turned out to be propitious. This is when I was the most effective. This is also when I was the least focussed on predetermined pedagogical paths and on self-conscious carefulness. Good jazz players and good teachers make mistakes, and have bad days, of course. This does not alter my firm belief that we are at our best when we are improvisers, not when we are corporate linear planners.

Back to the example

Imagine that you and your students establish new theorems and proofs that accommodate both 'normal' (convex) polygons and the former malformations with concave vertices. Can you now rest back and assume the job is done? No. There is another monster, or malformation, that will cause surprise when it is first encountered through the equivalent of playing outside. Here it is, and, as you might expect, there are others.



I hope it is apparent that we can never rest back. There is always the possibility that there is some surprise lurking just outside of our field of imagination. We can never be sure that there is not some wild — not tamed — piece of mathematics ready to spring out on us, and that, after our engagement with it, will result in a new emerging structure. And as we deal with each of these refutations, we can never be sure there are no more. But we have to keep looking. This is what keeps mathematics enchanting. This is why playing outside is important. This is why imagination and artistic improvisation are important. This way of thinking about mathematics teaching does not fit well with the orthodox model. But it does fit nicely with the jazz metaphor.

Accessible further reading

If you are interested in reading further I recommend that you start with Davis and Hersh (1981), and perhaps follow this with a look at Freudenthal (1978). If you find each of these interesting and straightforward you might like to try Lakatos (1976); this last one is a bit more challenging but not too challenging for someone who has read Davis and Hersh first. Davis, Sumera and Luce-Kapler (2000) is a very readable introduction to complexity theory and its application to teaching. Humphreys and Hyland (2002) provides a brief discussion of the role of the teacher as jazz improviser. My recent paper, Neyland (2003), is another discussion of playing outside and covers points not mentioned in the above. Holt (1983) and Postman (1995) are two very accessible and thought provoking books that bear reading and re-reading.

References

Davis, P. & Hersh, R. (1981). *The Mathematical Experience.* Boston: Birkhauser.

Davis, B., Sumera, D. & Luce-Kapler, R. (2000). Engaging Minds: Learning and Teaching in a Complex World. London: Lawrence Erlbaum Associates.

Freudenthal, H. (1978). *Weeding and Sowing: Preface to a Science of Mathematics Education.* London: Reidel Publishing Co.

Holt, J. (1983). *How Children Learn (revised ed.).* Harmondsworth: Penguin Books.

Humphreys, M. & Hyland, T. (2002). Theory, practice and performance in teaching: Professionalism, intuition, and jazz. *Educational Studies, 28*(1).

Lakatos, I. (1976). *Proofs and Refutations.* Cambridge University Press.

Murdoch, I. (1997). The sovereignty of the good over other concepts. In I. Murdoch (Ed.), *Existentialists and Mystics: Writings on philosophy and literature* (pp. 363–385). New York: Penguin Books.

Neyland, J. (2003). Re-visioning curriculum: Shifting the metaphor from science to jazz. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds), *Mathematics Education Research: Innovation, networking, opportunities* (Proceedings of the 26th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 547–554). Geelong, Victoria: MERGA.

Postman, N. (1995). *The End of Education: Redefining the value of school.* New York: Vintage Books, Random House.

Sudnow, D. (1978). *Ways of the Hand: The organisation of improvised conduct.* Cambridge: The MIT Press.

Varela, T., Thompson, E., & Rosch, E. (1991). *The Embodied Mind.* Cambridge: The MIT Press.