Formal definitions in mathematics

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The use of language in both oral and written forms is an important part of learning mathematics, as well as most other things worth learning. Students need to listen and read effectively, as well as learn to communicate their mathematical understanding in oral and written forms. The use of language by learners provides a window on their thinking, both for themselves and their teachers. The formal definition is one language form that is part of the mathematical language register, and is particularly apparent in textbook presentations of mathematics. Mathematics learners need to be able to interpret definitions that they meet in their textbooks. Composing a definition after experiences with a mathematical concept can also assist students to build understanding (Shield & Swinson, 1997). The discussion that follows explores the language structure of a definition then addresses the benefits for students in learning about definitions. The paper also considers the limitations of this language form in conveying the full meaning of many mathematical ideas.

The structure of a definition

Formal definitions normally have a format consisting of three parts. For example, a triangle might be defined in the following way.

> A triangle is a closed plane shape with three straight sides.

The three parts of the definition are:

- *item* — triangle, the idea being defined;
- *class* — closed plane shapes, the group to which the item belongs;
- *features* — three straight sides, the things that distinguish the particular item from other items in the class.

In such a definition it is assumed that the reader already has knowledge of the class.

This 'definition by class' form is often used in mathematics textbooks and other materials. The item in a definition is often highlighted by the use of bold or italic type. Having three parts, there are, at least theoretically, six different ways a definition can be stated, and it is not uncommon to see definitions written as follows.

> A closed plane shape with three straight sides is called a **triangle**.

> A closed plane shape is called a **triangle** if it has three straight sides.

Due to the occurrence of different orders in definitions, it can be helpful for students to be made aware of the structure to avoid possible confusion.

To add to the difficulty of decoding definitions, sometimes they are rather loosely stated without the class being made explicit.

> A shape with three straight sides is called a triangle.

> A triangle is a 3-sided shape.

> A triangle has three straight sides.

In the context of learning about closed plane shapes, this may not be an immediate problem. However, such 'loose' definitions do not help in consolidating a coherent knowledge of plane shapes.

Learning from definitions

As discussed in an earlier paper (Shield & Swinson, 1997), having students compose a definition after explorations of a concept can assist in the process of reflection and building their understanding. It should be noted that sections in mathematics textbooks often present a definition near the start. Such an order is generally not helpful to students in constructing a meaningful representation of the concept.

There are also two important aspects of constructing a definition that can help develop students' understanding of the mathematical concepts involved. Firstly, a definition is based on the least number of features required to establish the item uniquely, that is, the necessary and sufficient conditions. For example, with the definition of a triangle above, it is not necessary to say anything about there being three angles or the angles adding to 180°. These are properties of a triangle that follow from the definition but are not necessary to establish the item uniquely. Any closed plane shape with three straight sides can be nothing but a triangle.

The second aspect of defining that assists in the development of understanding is the idea of 'nesting' within definitions. Having students realise that a particular shape is a member of a class of shapes can help them to build important links and structural knowledge. As a example, in working with triangles students would soon meet the equilateral triangle.

> An **equilateral triangle** (item) is a triangle (class) with all three sides equal in length.

Rather than repeating the description of a triangle and all its properties, by signifying the class as 'triangles', all those properties are included in the definition of an equilateral triangle. A triangle can also be thought of as a polygon and so its definition could have been nested in the definition of a polygon.

> A **polygon** is a closed plane figure with straight sides.

> A **triangle** is a polygon with three sides.

The point here is that there are often several different ways of defining any particular item and thinking in terms of the classes involved in the definitions can help develop students' awareness of important links in the ideas. In the case of a study of plane shapes, defining them in various ways can help learners to build up an understanding of the hierarchical structure in the classification of shapes. A concept map such as the one composed by a Year 8 student shown in Figure 1 could be used in conjunction with the writing of the definitions as students work through the exploration of the various shapes.

The study of geometric shapes is a useful topic through which students can learn about definitions. Having students compose a definition after a period of exploring the properties of a geometric shape can help them to reflect on the unique features of the shape and the properties that distinguish it from related shapes. The task of writing a definition is not easy at first. A concept map like the one in Figure 1 can assist students to recognise the class to which a shape belongs. Students experience difficulties in meeting the need to specify the necessary and sufficient conditions, and often include more features than required. One useful activity involves having students compose definitions for several shapes they have been exploring. Students then rewrite their own definitions, leaving the 'item' words blank and swap their definitions with other students. Each can attempt to fill in the missing items. Difficulties in identifying the shapes being defined can be discussed.

Inappropriate definitions

While constructing and knowing definitions can be useful in developing students' understanding of some mathematical concepts, there are other mathematical ideas that do not lend themselves easily to brief definition. However, most mathematics textbooks attempt to provide definitions for most concepts but in many cases these do not convey anything like the full meaning of the ideas.

For example, the concept of ratio is almost always defined in textbooks as follows.

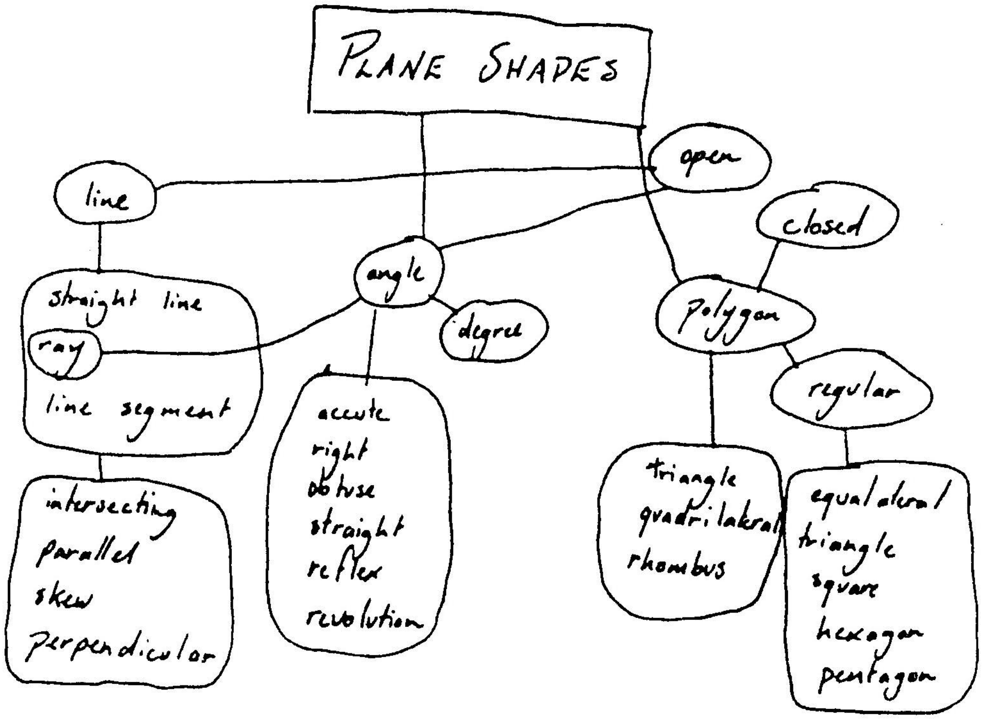

Figure 1. Student's concept map of plane shapes.

A **ratio** is a comparison of two quantities of the **same** kind.

A **rate** is a comparison of two quantities of **different** kinds.

The problem here is the use of the word 'comparison' without further qualification. There are two ways that two quantities can be compared, namely additively or multiplicatively. A ratio is a multiplicative comparison – one quantity is a certain number of times larger than the other. However, in most mathematics textbooks this distinction is not made clear. Some even use the comparison of the heights of two people as an early example of a ratio. If asked, how many people would compare the heights of two individuals multiplicatively? The everyday way to compare heights is to say something like: John is 10 cm taller than Geoff. The simple addition of the words 'by division' after the word comparison would help to clarify the meaning. However, the definition would still not capture the richness of the concept with its proportional nature and links to other rational number concepts.

The concept of rate is usually defined similarly in the textbook a few pages after the materials on ratio.

If there are some problems with comparing quantities of the same kind, it is difficult to imagine what students make of 'comparing' kilometres with hours or dollars with metres. This definition simply does not capture the concept of rate. Such one-line definitions of ratio and rate are not useful in helping students grapple with the richness of the ideas in the multiplicative field. Consideration should be given to the use of other language activities to assist students in understanding such concepts.

As a further example of the difficulties in describing a concept with a definition, consider the concept of variable. Mathematics textbooks have included statements similar to these:

A variable is a quantity, usually represented by a symbol, which can take on a variety of values.

A variable is a symbol that can represent a number. Variables can take on different values.

Variable is generally acknowledged as a difficult concept for students to grasp, but one that can be developed meaningfully over time (White & Mitchelmore, 1993). Definitions like these are usually found early in the sequence of textbook material intended to develop algebraic concepts. Without exploring the correctness or otherwise of the quoted definitions, they are not likely to contribute much to students' understanding of the concept. While language is crucial in the development of the concept of variable, the multi-faceted nature of the concept makes the use of a formal definition problematic.

Learning with definitions

The definition is an important language form in the register of mathematics. Students need to understand the structure of a definition so that they can make sense of the definitions they encounter and so that they can construct their own definitions as part of organising their thoughts about the concepts they have explored. However, I am suggesting in this article that there are many mathematical concepts for which the use of a single formal definition is not helpful in developing students' understanding of the concepts. Rather, more might be gained by having students use language, both mathematical and everyday, in some of the other language tasks that have been suggested (for example, Shield & Swinson, 1997).

Reference

Shield, M. J. & Swinson, K. V. (1997). Encouraging learning in mathematics through writing. *The Australian Mathematics Teacher, 53* (1), 4–9.

White, P. & Mitchelmore, M. (1993). Aiming for variable understanding. *The Australian Mathematics Teacher, 49* (4), 31–33.

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fraction

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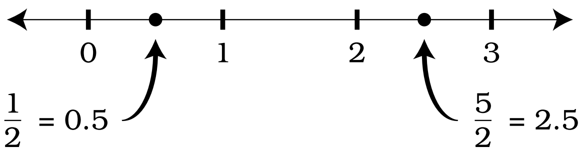
Fractions and decimals are often viewed by students as two separate sets of numbers rather than different representations of the same set of numbers. Often we reinforce this misconception with instruction that first focuses on computations with fractions only, or computations with decimals only. We can help students develop a clearer understanding of the concept of 'representation' of a number as a fraction or as a decimal by showing both names for a number on the number line;



To accomplish this, students need to be able to find the equivalent fraction for a decimal representation of a number and vice versa. Thus, we become familiar with the fractional representation of decimals, writing each decimal as a fraction whose denominator is a power of ten. Consider the following examples:

1. $0.1 = \frac{1}{10}$

2. $0.12 = \frac{12}{10^2} = \frac{12}{100}$

3. $0.123 = \frac{123}{10^3} = \frac{123}{1000}$

4. $0.1234 = \frac{1234}{10^4} = \frac{1234}{10000}$

When the decimals are non-terminating, but repeating, there is a procedure sometimes taught in junior high school to find its frac-