Attitudes in mathematical discovery processes: The case of Alex and Milo

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This paper's purpose is to investigate the attitude of students in mathematical discovery processes in terms of the handling of counterexamples. By understanding this attitude as a kind of scientific attitude, it consists of different aspects that become visible in the behaviour during a mathematical discovery process. Since such a process is particularly complex, the author's interest is to use the concept of attitude as an explanation for students' behaviour that occurs when dealing with conflicts such as counterexamples. Semi-structured interviews with sixth graders of a German Gymnasium were conducted and analysed in a qualitative and interpretative way. As a result, the case study of Alex and Milo is presented. Based on the framework that observable behaviour is influenced by an underlying attitude, there are drawn conclusions about Alex's and Milo's attitudes adopted in the mathematical discovery process and their impact on the process is elaborated.

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1 Theoretical framework

1.1 Mathematical discovery process

> "The learning of mathematics is more effective [...] the more it is done in the sense of one's own active experiences [...]."

Winter's (2016, p. 1) quote is based on a constructivist view of learning, namely that learners are supposed to take an active role in the learning process while the teacher provides a suitable learning environment (Kunter et al., 2013). In terms of the teaching and learning of mathematics, this goes along with Freudenthal's (1973) idea of the so-called guided reinvention, so that learners experience mathematics as an activity rather than a ready-made product. In this way, learners are supposed to take an active role and experience the process of discovering and developing mathematics rather than being confronted with just its results. These processes of discovering and developing mathematics is what this paper refers to as *mathematical discovery processes*. Thus, the term *mathematical discovery processes* involves activities that are usually performed by research mathematicians. So, in this way, it is not only about discovering but also about questioning and reasoning to gain new knowledge.

This paper refers to the model for mathematical discovery as quasi-empirical experimentation, which is seeing mathematics as kind of experimental science that deals with abstract objects such as numbers or relations (Leuders & Philipp, 2013). In that sense, by zooming in on the process of mathematical discovery, *mathematical discovery processes* can consist of the following activities: *generating examples*, *structuring* based on relevant characteristics, *developing hypotheses* and *testing and proving* them (based on Philipp, 2013). Within these activities, there are numerous barriers to overcome in order to gain new knowledge and each of those sub-processes can be a great challenge for learners as different study results underline (e.g., Dunbar & Klahr, 1989; Kuhn et al., 1988; Kuhn, 1989):

In mathematical discovery processes, learners tend to propose a hypothesis after only one example. Moreover, they often conduct one single experiment to be convinced that their hypothesis is correct. In contrast to that, learners have difficulties in deciding what evidence is sufficient to reject their hypothesis. At the same time, learners tend to ignore evidence that is inconsistent with their hypothesis or try to gain some evidence that would confirm it. In general, students seem to test their hypothesis in order to find confirming evidence instead of checking the correctness. Tweney (1989) even revealed a general strategy in dealing with hypotheses: people tend to generate evidence that confirms the hypothesis first. Once there is enough evidence gained, people try to look for counterexamples or attempt to disconfirm the hypothesis.

This paper focuses on conflict situations that are most likely to arise during a mathematical discovery process. The way of dealing with those situations is crucial for gaining knowledge. According to Bauersfeld (1985), a conflict is a situation that does not fit into the learner's cognitive frame or "subjective domains of experience" (p. 11) as they refer to it. Therefore, a conflict is a situation, for instance a counterexample, that is not compatible with the existing hypothesis. As mentioned before, some learners tend to ignore evidence that is inconsistent with their hypothesis. Besides of that, studies have shown that counterexamples or contradictions in general also led to a reinterpretation of the evidence and not to a modification of their hypothesis (Kuhn, 1989). Furthermore, when counterexamples were really perceived as counterexamples, they were not considered to be sufficient for disproving a hypothesis (Kuhn et al., 1988). This behaviour seems to be worth analysing in detail with regard to the underlying attitude that learners take in mathematical processes to eventually gain a deeper understanding of its impact on doing mathematics. In order to meet this concern, we will first take a closer look at the concept of attitude in general and the way it is used in this paper.

1.2 Attitudes

As many authors have already stated, there is no universal definition of the term or concept of attitude (e.g., Pepin & Roesken-Winter, 2015; Walsh, 1991). While some earlier studies referred to attitude as a general concept overarching all mathematical topics and activities (e.g., Haladyna et al., 1983), it seems to be common ground nowadays that attitude depends on the objects and situations an individual is faced with (Kulm, 1980). Moreover, attitude can not only be regarded as a single dimensional construct but rather multi-dimensional comprising cognitive, affective, and conative or behavioural aspects (e.g., Di Martino & Zan, 2010). This gave rise to the idea of a "working definition" (Daskalogianni & Simpson, 2000, p. 217), so that the concept of attitude depends on research interest and situations to be studied. With regard to this proposal, I first take a brief look at attitude in mathematics education literature before I then derive an understanding of the concept of attitude suitable for this paper's interest.

In his pioneering work concerning affect in mathematics education, McLeod (1992) described attitude, in addition to beliefs and emotions, as a key affective construct. Later, Goldin (2002) added values, ethics, and morals as a fourth component. When considering stability and intensity, both researchers classified attitudes somewhere in between beliefs and emotions. In this context, beliefs as the most stable and emotions as the most intense of the three constructs form the two poles, between which attitudes can be classified as "feelings of moderate intensity and reasonable stability" (McLeod, 1992, p. 581). On the one hand, one's attitude towards an object or a situation seems, therefore, to be a moderately stable construct but, one the other hand, still has the potential to be modified (Liljedahl et al., 2010).

In line with the perspective of social psychology, attitude can be seen as a trait of an individual that influences their behaviour (Allport, 1935). Since attitudes are moderately stable, they manifest in "manners of acting, feeling, or thinking" (Philipp, 2007, p. 259). By considering attitudes as a concept that one's behaviour is based on, "they may involve positive or negative feelings" as Philipp (2007, p. 259) stated, but seem to be more than an evaluative judgement about an object or in this case, a disposition towards mathematics.

As previously mentioned, *mathematical discovery processes* can be seen as a way of conducting an experiment. That view highlights the dynamic character of mathematics as an evolving science like natural sciences. It is therefore worthwhile to look at the concept of attitude from this point of view as well. In the field of science education, Gardner (1975) proposed a fundamental distinction that is also suitable and probably even necessary for the field of mathematics education. He distinguishes the terms "attitude to(wards) sth." (p. 1) and "adjective + attitude" (p. 1). In his case, the adjective in the second term can be replaced with scientific, while in mathematics education we might call it *mathematical attitude*. The first term always includes some attitude object to which the respondent is invited to react favorably or unfavorably, for instance attitude towards mathematics or attitude towards problem solving. The second term is understood as ways or styles of thinking, acting or behaving, which influence the way we behave in certain situations and it's the meaning which this paper is based on. In this way, attitude has an influence on behavior and the other way around, conclusions can be drawn about attitude from behavior.

In the field of science education, great efforts have been made to characterise a desirable scientific attitude due to its importance for supporting scientific learning and enhancing the performance of students' scientific activity. According to the American Association for the Advancement of Science (1993) scientific attitude generally includes: *curiosity*, *honesty*, *open-mindedness* and *doubt*. Other researchers add further characteristics such as *respect for data*, *diligence*, *creativity*, *cooperation*, and *confidence* (Harlen, 1996; Anderson, 1980). Transferring these considerations to the field of an idealised attitude in mathematical discovery processes, it becomes clear that attitude in this case is a multi-dimensional construct. In this case, the term *scientific attitude* is used in a normative way, so that it is understood in the sense of a desirable attitude. In the study presented in this paper, the term *attitude* will be used in a descriptive way in order to characterize attitudes that students actually adopt in mathematical discovery processes. Thus, students' behaviour in a mathematical discovery process is seen in this paper as the outward expression of an attitude, so that attitude itself is not a directly measurable construct. However, it is possible to draw conclusions about underlying attitudes based on observable behaviour.

1.3 Research questions

As pointed out before, this paper assumes that a learner's observable behaviour in a mathematical discovery process is based on the attitude the learner adopts during the process. In order to gain a deeper understanding of learners' mathematical discovery processes, one aim of the study this paper is based on is to draw conclusions about those different attitudes. As space is limited, this paper especially focusses on a typical situation that might arise in the course of a discovery process: the emergence of counterexamples, contradictions or objections and how learners deal with it. Therefore, this paper addresses the following specific research questions:

1. What is the behaviour of the two students Alex and Milo when dealing with conflicts (such as counterexamples) in the shown excerpt of the interview?
2. To what extent can conclusions be drawn from the behaviour about the students' attitudes in dealing with counterexamples during a mathematical discovery process?

2 Method

The data was collected in an exploratory semi-structured interview with twelve sixth graders of secondary school (German Gymnasium). This paper focusses on the case study of the two students Alex and Milo, who were interviewed together. Their interview took place in October 2020 and was conducted by the author. The interview took about 70 minutes and was designed to simulate a mathematical discovery process with low level of interviewer intervention. For gaining an insight into the students' thinking process, the *think aloud method* was used (Ericsson & Simon, 1993). The students worked in tandem in an interactive situation on an explorative task about sums of successive natural numbers adopted from Leuders et al. (2011). To be more precise, the students' task was to develop a 'trick' how to easily decide whether a given number is a so called *staircase number* (a number, that can be represented as a sum of successive natural numbers). The task requires basic mathematical knowledge but, at the same time, it offers a lot of opportunities for making discoveries, conjecturing and reasoning. For instance, students could assume that **all** numbers are staircase numbers, all **odd** numbers are staircase numbers, all **even** numbers are staircase numbers, **not all** even numbers are staircase numbers or that **all** numbers are staircase numbers, except 2, 4, 8, 16, 32, ... and so on.

From a mathematical perspective, the characteristic this task is looking for is a number (not) being a power of two. So, numbers that are power of two are **not** staircase numbers, all the other numbers are staircase numbers. With that in mind, some of the previously presented hypotheses are wrong or at least need to be modified. Of course, the task does not want the students to use the term *power of two*, since it has not yet been part of their mathematic class so far. However, this characteristic can be discovered and justified, for instance, by using the small round plates (see Figure 1). Nevertheless, it was not intended for the students to solve the task completely but to evoke the aforementioned mathematical processes, so that activities like generating and exploring examples, structuring, developing hypotheses as well as testing and proving them can take place.

At the beginning of the interview, the term *staircase number* was clarified by using enactive representations with small round plates, iconic representations with a dot pattern on squared paper and arithmetic representations of the number 25 (see Figure 1). The students could optionally use all of them during their discovery process.

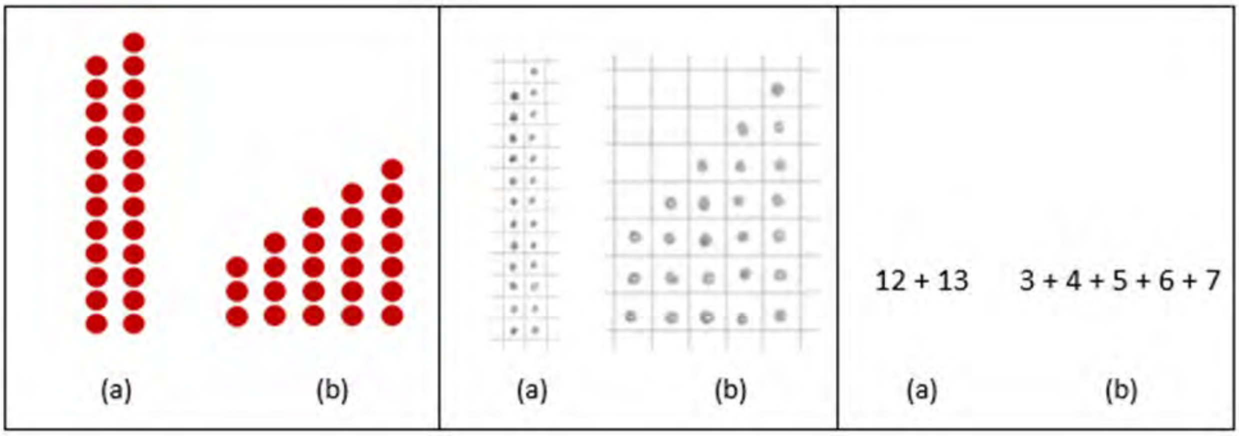

Figure 1. Enactive (with small round plates), iconic (dot pattern) and arithmetic representation of the number 25 as a (a) two-step and (b) multi-step 'staircase number' (own representation).

The interviews were videotaped and transcribed. Following a qualitative research approach the aim of the study is to draw conclusions about students' attitudes adopted in a mathematical discovery process. For analysing the data, a structuring qualitative coding method was initially used to categorize the behaviour in dealing with conflicts to get an overview of the different kinds of students' reactions (Mayring, 2015). Categories have been gained both deductively on the basis of the theoretical background and inductively to further differentiate them in terms of research interest (see Table 1 for an excerpt of the category system). The coding was carried out twice by the author. In order to draw conclusions about the attitude of the students from their behaviour, crucial scenes were analysed by a turn-by-turn analysis following an interpretative research paradigm (Voigt, 1984). The aim of this approach is to generate hypotheses that explain phenomena in teaching and learning mathematics in the sense of *abduction*. This means that, starting from a phenomenon, a general rule is set up that together with the recognition of the case at hand causes the phenomenon (Peirce, 1958, as cited in Meyer, 2018). The overall aim is to "make sense" (Eisenhart, 1988, p. 103) in accordance with the method of objective hermeneutics by making cognitive processes visible. The aim of this approach is to generate hypotheses that can be further investigated in future research.

To answer the research questions, an analysis with particular focus on each learner was first carried out and then the interaction and joint mathematical process were considered. For the sake of clarity, this paper only presents the results that have proven to be plausible within the analysis (Krummheuer & Brand, 2001, p. 90).

Table 1. Category system as a result of the qualitative content analysis

Category	Anchor example	Coding rules
Review of conflict trigger	Milo: so first of all, here's one. that's two that's three. (points at first two steps of 1|2|3)	The conflict trigger (e.g., counterexample) is checked.
Rejection of hypothesis	Milo: I think this one is right. (points at the first hypothesis) there must always be three or more small plates- but not this. (points at the second hypothesis)	The hypothesis is completely rejected and is not pursued further in a modified form (otherwise: modification of hypothesis)
Modification of hypothesis	see subcategories	Also includes a rejection of the original hypothesis (in this way the hypothesis is false), but the hypothesis is pursued in a modified way.
Classification	Alex: so there are different forms of staircase numbers. namely this one (points at 1|2|3) and then this one. (lays 1|2)	A classification takes place with regard to a characteristic, which specifies the hypothesis.
Exclusion of cases	Alex: […] twelve is an exception.	The cases that contradict the original hypothesis are excluded or named as exceptions.
…	…	…
Cancellation	Alex: how difficult is that? […] eh? i don't understand it anymore.	No specific rejection of the hypothesis, but termination of the entire process.
…	…	…

3 Results

In the following I will take a closer look at the case study of Alex and Milo (names are pseudonyms). An excerpt of the interview with Alex and Milo and its corresponding interpretation is presented. In the transcript, the coding is added as well. In the same way as the analysis was carried out, here the individual students Alex and Milo are considered first, before a brief comment is made on the joint mathematical discovery process. At this point, it is important to note that the analysis of one single scene of the interview does not give enough information about the students' attitudes. For this, the behaviour of the students during the entire process must be included to draw conclusions about an underlying attitude. However, this scene and its interpretation can at least give an impression of it.

Note on notations in transcript: the expression *lays 1|2|3* is the written representation of the act to lay the small plates as a staircase with three steps of height one, two and three small plates.

3.1 Excerpt from the interview

Here, Alex and Milo have formulated and noted two hypotheses: *(1) There must always be three or more small plates* and *(2) Only odd numbers can be formed into a staircase*. Immediately before the excerpt begins, Milo has placed the arrangement 1|2|3 with small plates. Then the following scene takes place (see Figure 2). The code *conflict arises* is not a code of the category system but to make the conflict situation clear to the reader.

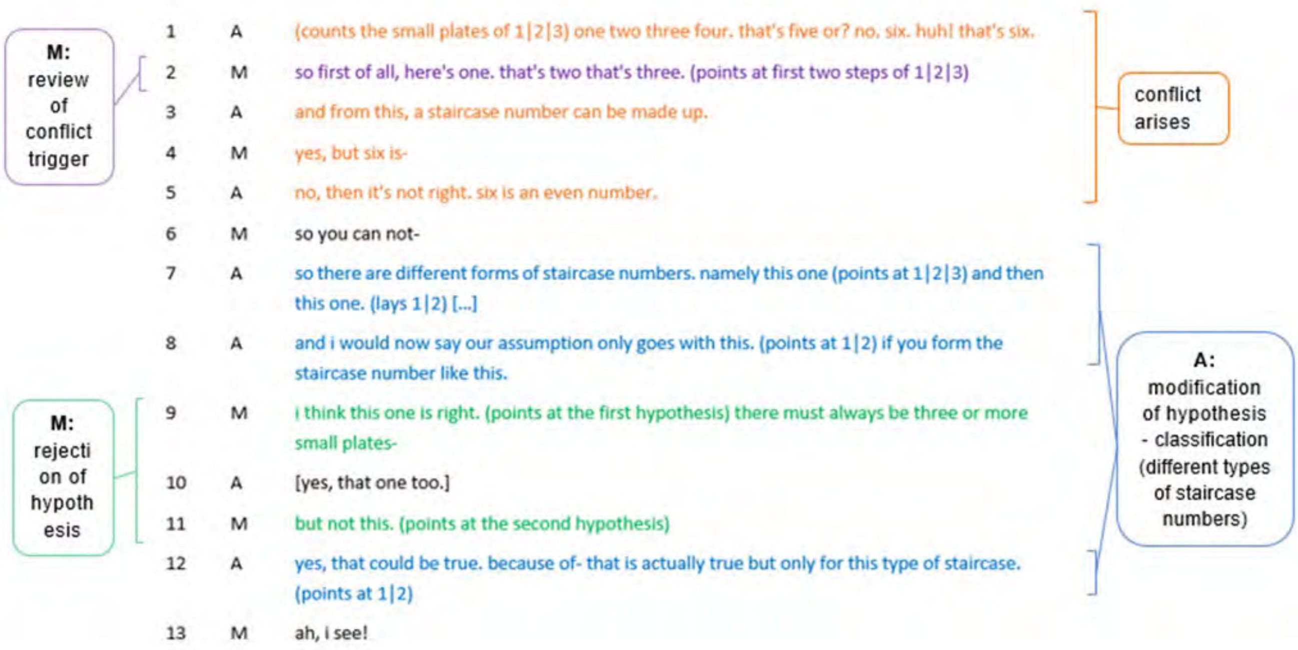

Figure 2. Excerpt from the interview with Alex and Milo with coding (own representation).

3.2 The case of Alex

In turn 1, Alex seems to be surprised when he recognises that the staircase Milo constructed adds up to six. He is convinced of the hypothesis that only odd numbers can be staircase numbers, so that the counterexample six does not fit into his theory. Nevertheless, in turn 3, he states the counterexample to be correct so he perceives six as a counterexample. On that basis, he puts the counterexample in relation to their hypothesis and states the hypothesis to be incorrect (“then it’s not right”, turn 5). He justifies the disconfirmation with the parity characteristic of six. It is striking, that at this point for Alex the occurrence of a counterexample is the trigger to make a *classification of different types of staircase numbers*. He distinguishes between two-step staircases (the type of staircases that has occurred up to the present scene) and the type of staircases to which he assigns the counterexample six. In this situation it is not totally clear which type of staircases he refers to by the latter: it could be multi-stage staircases starting with the height of one plate as well as multi-stage staircases in a more general way. It could also be the case that Alex himself is not quite clear about it.

In the following, Alex’s classification is the starting point for a specification of the hypothesis *only odd number can be formed into a staircase*. Although he has previously falsified the hypothesis (turn 5), he still maintains and even further develops it by specifying the class of types of staircases to which the hypothesis refers (turn 8). Thus, for Alex, the hypothesis *only odd number can be formed into a staircase* turns into *only odd number can be formed into a two-step staircase*. The specified form of the hypothesis shows Alex's way of resolving the conflict created by the counterexample. His conviction of this approach is shown in the fact that he defends it against Milo's objection (turn 12).

Alex's behaviour in this excerpt shows some characteristics that indicate a more general attitude he adopts in mathematical discovery processes. He shows great conviction with regard to the hypothesis that has been made. In the course of the scene it also becomes clear that Alex literally sticks to it. When he recognises that the counterexample contradicts the hypothesis, he does not reject it but accepts the counterexample and modifies the hypothesis to integrate it. It is remarkable that Alex uses the typical mathematical activity of classification for this purpose. In summary, Alex's attitude in this excerpt can be described as persistent, which is also confirmed in the further course of the interview. A counterexample does not make him abandon the hypothesis but rather taking it as a trigger to develop the hypothesis further. For this attitude, counterexamples have a great potential for mathematical discovery processes.

3.3 The case of Milo

Milo is the one who has placed the arrangement 1|2|3 with the small plates. When Alex detects it as a counterexample to their hypothesis *only odd number can be formed into a staircase*, Milo's first reaction is to *check the counterexample by* accurately recounting the small plates of 1|2|3 (turn 2). In the following (turn 4 and 6), he does not really get a chance to verbalise all his thoughts, but due to his further behaviour one can assume that he accepts the counterexample as such, just like Alex does. What is striking is that Milo's handling of the counterexample differs from Alex's. Milo refers directly to the two hypotheses they had previously made. By reinterpreting the counterexample as a confirmation example for the first hypothesis (*There must always be three or more small plates.*), he approves it. In contrast to that, six as a counterexample is the decisive point for *rejecting the second hypothesis* (*only odd number can be formed into a staircase*). He does not make an attempt to resolve the conflict other than strictly disconfirming the hypothesis. Because of the counterexample, the hypothesis has come to an end for Milo at this point. This is particularly clear in the way he contrasts the two hypotheses: in turn 9 he starts his sentence by confirming the first and then clearly ends in turn 11 with the statement that differentiates the second hypothesis as incorrect. The possibility of a further development does not seem to be given until Alex suggests it. The surprise in Milo's statement confirms this interpretation (turn 13). Although for him this solution was not an option as a way out of the conflict, he accepts Alex's proposal and supports the specification in the following.

Like Alex, Milo's behaviour also indicates a certain attitude he adopts in the course of the mathematical discovery process. Although Alex and Milo have set up the two hypotheses together in advance, Milo does not show the same persistent behaviour that Alex does. On the contrary, Milo shows a sceptical attitude towards the hypothesis that is made clear in the significance of the hypothesis for him. As soon as a counterexample occurs, the hypothesis is rejected and not pursued. Thus, the view of hypotheses is a scientific one: a hypothesis as a verifiable or falsifiable assumption that can be disproved by a single counterexample. For Milo, counterexamples seem to be highly significant in the mathematical discovery process (which also becomes clear at several points in the further course of the interview) and consequently he insists on them. Moreover, Milo's attitude can be characterised as a doubtful one: the counterexample makes him doubt the hypothesis, but first he also doubts the counterexample and checks it once more. It can be said that he takes the role of a supervisor or controller, which also becomes apparent in the further course of the interview. In this way, he ensures the necessary precision and elaboration of the hypothesis.

3.4 A short remark on the common mathematical discovery process of Alex and Milo

Since the mathematical discovery process that is previously shown in excerpts takes place in an interactive situation, one cannot disregard the mutual impact that both students have on each other. On the contrary, the interaction of students of different attitudes can bring great potential but also difficulties to their mathematical discovery process. In the case of Alex and Milo, the focus here is on the potential that arises from the interactive process.

Both students contribute to the advancement of the mathematical discovery process. On the basis of their attitudes, the students take a certain role in the process. In the case of Alex and Milo, we see an interplay of both attitudes that has a positive effect on the mathematical discovery process. The attitudes complement each other: Alex's persistent attitude ensures maintenance of the hypothesis by progressively specifying it in response to conflicts that arise. In contrast to that, Milo takes a doubting attitude. Conflicts seem to have a high priority for him so that they make him actually sceptical about the hypothesis. This critical attitude serves as a catalyst for the common mathematical discovery process since it triggers the further development of the hypothesis. By complementing each other, the mathematical discovery process serves as a learning opportunity. Due to the differences in the handling of counterexamples and hypotheses in general, each student individually taken would probably have reached an end beforehand. It is thus the interaction of both attitudes that makes the joint process successful. At the same time, they can learn from each other that the other's attitude in combination with their own helps them to progress in the mathematical discovery process.

4 Discussion and conclusions

It was the purpose of this paper to relate the concept of attitudes to students' mathematical discovery process in terms of the handling of counterexamples. In order to answer the research questions, the behaviour of both students was first analysed. On this basis, an attempt was made to draw conclusions about two general attitudes, which the students adopt in the shown excerpt.

1. *What is the behaviour of the two students Alex and Milo when dealing with conflicts (such as counterexamples) in the shown excerpt of the interview?*

By presenting the results of the analysis, it became clear that the behaviour of both students in dealing with the counterexample is fundamentally different. While Alex holds to their hypothesis, Milo becomes extremely sceptical about it and even rejects it. As a way out of conflict, Alex specifies their hypothesis by introducing a classification of staircase types so that the counterexample can be integrated and no longer contradicts the hypothesis.

2. *To what extent can conclusions be drawn from the behaviour about the students' attitudes in dealing with counterexamples during a mathematical discovery process?*

The behaviour of the students can possibly be explained by underlying attitudes that differ in essential points: on the one hand a persistent and on the other hand a doubting attitude. The persistent attitude expresses itself in the defence and maintenance of the hypothesis while the latter rather doubts and contests it.

With respect to the state of research, the counterexample triggers different behaviour at this point. In contrast to the results of Dunbar and Klahr (1989) and Kuhn (1989), the counterexample was neither ignored nor did it lead to a reinterpretation of the evidence. As we could see in the case of Alex and Milo, there are different attitudes that cause different handlings of the counterexample. Concerning Milo, unlike the study results of Kuhn et al. (1988), the counterexample actually has the value of disproving a hypothesis. In his case, this occurs even to such an extent that the second hypothesis would no longer be pursued by him. Alex, on the other hand, takes the counterexample as an opportunity not to reinterpret the evidence as in Kuhn (1989) but to develop the hypothesis further by modifying it.

With regard to a general scientific attitude, both go with some of the desired characteristics in the shown scene. Alex's attitude stands out because of his *confidence*, with which he maintains the previously made hypothesis. In order to resolve contradictions that are contrary to it, he shows a kind of *creativity* that is crucial for problem-solving. In contrast to Alex's, Milo's attitude is characterised by *doubt* and *diligence*. With his way of behaving like a supervisor or controller, he ensures that the joint mathematical discovery process is appropriately accurate and adequately attention is paid to the counterexamples. Due to both students' ability to *cooperate*, the combination of attitudes works like a symbiosis. The challenging mathematical process is thus shared in a kind of cognitive task distribution so that together they act like a mathematician.

Since this paper focuses only on the part of attitudes which become visible in the process of dealing with counterexamples, the ongoing research will further characterize them on the basis of other categories and situations. For instance, the particular role of hypotheses will be further evaluated in this research project. Moreover, a more precise analysis of the mutual impact of students of different and as well of students of similar attitudes is the aim of the study that includes the presented case study. As already mentioned, with regard to the interaction of students of different attitudes, the only case considered here is the one which has a positive effect on the process; it could also be the opposite. Of course, there is also the possibility that the students' attitudes do not influence each other positively, but rather negatively by hindering each other. This can be caused by unfavourable combinations of attitude characteristics.

I conclude that different attitudes in the mathematical discovery process, manifesting in different behaviour, can be gained out of the data. This leads to the hypothesis that the behaviour of students in the process is not arbitrary but influenced by a fundamental attitude, which in interaction with other attitudes, can have a positive or negative impact on the mathematical discovery process. It can thus be stated that for both research and teaching it is worth taking a closer look at attitudes and their impact on mathematical discovery processes. The case of Alex and Milo already gives an insight into diverse manifestations of attitudes. In further research, more case studies will be taken into account to derive concrete and repetitive attitudes, that are consistent over the course of a mathematical discovery process.

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