Connections Between Metric Coefficients, Basis- and Unit Vectors

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(Received 15.03.2023, Accepted 10.07.2023)

Abstract

This work gathers in one place what is pertinent about the connections between metric coefficients, basis- and unit vectors in a four-dimensional relativistic manifold. Some of this material can be found scattered elsewhere; its collection into one place reveals connections that either are not known or are obscure, for example that the metric coefficients are not all independent of each other. It at least should serve as a useful tutorial for those who are not thoroughly familiar with this material.

Keywords: Metric, relativity, basis- and unit vectors.

INTRODUCTION

While there are many details regarding metric coefficients, basis vectors, and unit vectors as applied in Relativity, scattered through major texts on the theories of relativity [1-4], these texts reveal few specifics about the connections alluded to in the title of this work; even less so in older texts [5-6]. It is useful to gather these connections in one place, and that is the aim of this work. Something on basis vectors can be found in section 3.4 of Hobson, et al [3], and in a related although quite different approach in Schutz [4], section 5.6, but not much of what is reported here.

It is also useful to note at the outset that a distinction between unit and basis vectors is not always clearly made in the literature. Sections II and III deal specifically with basis vectors and the definitions there do clarify this difference.

Expanding upon work by de Wolf [7], there even may be some unexpected connections revealed here [e.g. with Eq, (3), and specifically in a number of examples]. We work here with a four-dimensional (4-D) manifold [8-10] or, to put it somewhat less formally, a smooth relativistic four-dimensional (4-D) space with a general coordinate system $\boldsymbol{x} = (\xi^0, \xi^1, \xi^2, \xi^3)$ with unit vectors $\hat{\boldsymbol{\xi}}^\alpha$,and a metric $g_{\alpha\beta}$ or $g^{\alpha\beta}$. The distinction between sub- and superscripts is immaterial for unit vectors. In a space with curvature one or more of the metric coefficients may be negative. As elsewhere, covariant vectors and components have subscript indices and contravariant ones have superscript indices.

At times, a *local inertial frame* (LIF) with Cartesian coordinates and diagonal metric coefficients $\eta_{\alpha\alpha} = (-1, 1, 1, 1)$ will be needed. An arbitrary vector in the more general space can be expanded into its contravariant [11] components **v** $= v^\alpha \hat{\boldsymbol{x}}_\alpha$ or covariant components **v** $= v_\alpha \hat{\boldsymbol{\xi}}^\alpha$. The Einstein summation convention is implied, unless stated otherwise. More particularly, a 4-D path has an infinitesimal line element $d\boldsymbol{l} = \hat{\boldsymbol{x}}_\alpha d\ell^\alpha$. Greek super- and subscripts take on the values 0, 1, 2, and 3 in this work.

The discussion will start $d\boldsymbol{l}$ with the introduction of *basis vectors*, from. which the *metric coefficients* can be formed but inversely, as also will be shown, the basis vectors can be derived from the metric coefficients. Examples are relegated to a set of appendices

COVARIANT BASIS VECTORS AND METRIC COEFFICIENTS

The *covariant basis vectors* $\boldsymbol{e}_\alpha$ can be defined in various ways, the first of which follows from the vectorial line element $d\boldsymbol{l}$ of the space, and is the way basis vectors are usually introduced [3]:

$$d\boldsymbol{l} = \sum_{\alpha=0}^{3} \boldsymbol{e}_\alpha \cdot d\xi^\alpha \;\rightarrow\; \boldsymbol{e}_\alpha = \frac{\partial \boldsymbol{l}}{\partial \xi^\alpha} \tag{1a}$$

This definition shows that the basis vector $\boldsymbol{e}_\alpha$ is tangential to $d\boldsymbol{l}$ but it is not a unit vector if the dimension of $d\xi^\alpha$ is not proper length (see for example Appendix A). Furthermore, Eq. (1a) can be transformed, using the LIF element $d\boldsymbol{l} = \hat{\boldsymbol{x}}_\beta dx^\beta$:

$$d\boldsymbol{l} = \sum_{\alpha,\beta=0}^{3} \frac{\partial \boldsymbol{l}}{\partial x^\beta}\frac{\partial x^\beta}{\partial \xi^\alpha} d\xi^\alpha = \sum_{\alpha,\beta=0}^{3} \hat{\boldsymbol{x}}_\beta \frac{\partial x^\beta}{\partial \xi^\alpha} d\xi^\alpha \;\rightarrow\; \boldsymbol{e}_\alpha = \sum_{\beta=0}^{3} \hat{\boldsymbol{x}}_\beta \frac{\partial x^\beta}{\partial \xi^\alpha} \tag{1b}$$

The last of these equations yields $\boldsymbol{e}_\alpha$ in terms of LIF coordinates x^β (with unit vectors $\hat{\boldsymbol{x}}_\beta$). On the other hand, the square of (1a) yields a relationship that can be compared to the square of the proper-length increment $(d\boldsymbol{l})^2$ in terms of the metric coefficients:

$$(d\boldsymbol{l})^2 = \sum_{\alpha,\beta=0}^{3} (\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta) d\xi^\alpha d\xi^\beta = \sum_{\alpha,\beta=0}^{3} g_{\alpha\beta} d\xi^\alpha d\xi^\beta \;\rightarrow\; g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta \tag{2}$$

On the one hand (2) thus gives the metric coefficients as functions of the basis vectors. On the other hand it follows from (2) that $|\boldsymbol{e}_\alpha| = \sqrt{g_{\alpha\alpha}}$. As it is clear from (1a) that $\boldsymbol{e}_\alpha$ points in the direction of the unit vector $\hat{\xi}^\alpha$ one thus finds a second definition of the covariant basis vectors

$$e_\alpha = \sqrt{|g_{\alpha\alpha}|}\boldsymbol{x}^\alpha \;\rightarrow\; g_{\alpha\beta} = \sqrt{|g_{\alpha\alpha} g_{\beta\beta}|}(\boldsymbol{x}^\alpha \cdot \boldsymbol{x}^\beta) \tag{3}$$

The second of these equations is a condition which nondiagonal metric coefficients must fulfill, i.e. *they are not independent of the diagonal ones*. It also shows that $g_{\alpha\alpha} = |g_{\alpha\alpha}|(\boldsymbol{x}^\alpha \cdot \boldsymbol{x}^\alpha)$ -- no summation here or in (3) -- which yields the correct sign of $(\boldsymbol{x}^0 \cdot \boldsymbol{x}^0)$ when $g_{00} < 0$. It is useful to check that the second of Eqs. (3) indeed holds for a given metric; this is checked in Appendix B for a 2-D set of nonorthogonal coordinates, and in Appendix D for a version of the Kerr metric. It is also of some interest to note that (3) can be used as a definition of a basis vector from that of a metric coefficient and a unit vector, whereas conversely the metric coefficients are obtained from the basis vectors by $g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$,which follows from (2).

CONTRAVARIANT FORMS

The contravariant basis vectors $\boldsymbol{e}^{\beta}$ are defined by the requirement that

$$\boldsymbol{e}_{\alpha}\cdot\boldsymbol{e}^{\beta}=\delta_{\alpha}^{\beta} \tag{4a}$$

Complementary to (1b) is

$$\boldsymbol{e}^{\alpha}=\frac{\partial\xi^{\alpha}}{\partial\boldsymbol{l}}=\sum_{\beta=0}^{3}\hat{\boldsymbol{x}}_{\beta}\frac{\partial\xi^{\alpha}}{\partial x^{\beta}} \tag{4b}$$

Equations (1b) and (4b) have a slight disadvantage in that their application to any metric requires both curvilinear and LIF coordinates. Equation (3) requires only one coordinate system (as well as the metric coefficients); this is often an advantage, At any rate, general expressions, complementing (3), for the contravariant basis vectors are more complicated. To obtain these, one starts with $d\boldsymbol{l}=\boldsymbol{e}^{\alpha}d\xi_{\alpha}=\sum_{\alpha,\beta}\boldsymbol{e}^{\alpha}g_{\alpha\beta}d\xi^{\beta}$ which then gives rise to

$$\boldsymbol{e}_{\gamma}=\frac{\partial\boldsymbol{l}}{\partial\xi^{\gamma}}=\sum_{\alpha}\boldsymbol{e}^{\alpha}g_{\alpha\gamma} \tag{5}$$

Summation signs are shown here for extra clarity. In the 4-D space under consideration this is a set of four linear equations expressing the covariant $\boldsymbol{e}_{\gamma}$ in terms of the contravariant $\boldsymbol{e}^{\alpha}$ vectors (where indices 1, 2, 3, 4 are used instead of 1, 2, 3, 0 (the latter is more usual in relativity). By inversion, the solution for $\boldsymbol{e}^{1}$ in terms of Cramèr determinants is

$$\boldsymbol{e}^{1}=\begin{vmatrix}\boldsymbol{e}_{1} & g_{21} & g_{31} & g_{41}\\ \boldsymbol{e}_{2} & g_{22} & g_{32} & g_{42}\\ \boldsymbol{e}_{3} & g_{23} & g_{33} & g_{43}\\ \boldsymbol{e}_{4} & g_{24} & g_{34} & g_{44}\end{vmatrix}\Big/\begin{vmatrix}g_{11} & g_{21} & g_{31} & g_{41}\\ g_{22} & g_{22} & g_{32} & g_{42}\\ g_{33} & g_{23} & g_{33} & g_{43}\\ g_{44} & g_{24} & g_{34} & g_{44}\end{vmatrix} \tag{6}$$

and the remaining contravariant basis vectors are given by cyclic similar expressions. For a diagonal metric tensor all $g_{\alpha\beta}=0$ for $\beta\neq\alpha$ which results in

$$\boldsymbol{e}^{1}=\frac{1}{g_{11}}\boldsymbol{e}_{1}=\frac{\sqrt{|g_{11}|}}{g_{11}}\boldsymbol{x}^{1} \tag{7}$$

The second equality in (7) results from application of (3) to $\boldsymbol{e}_1$. In particular, it shows that $\boldsymbol{e}^0 = \left(\sqrt{|g_{00}|}/g_{00}\right)\boldsymbol{x}^0 = -\left(1/\sqrt{|g_{00}|}\right)\boldsymbol{x}^0$ when $g_{00} < 0$. For a 2 X 2 metric tensor (with off-diagonal coefficients!) one finds:

$$\boldsymbol{e}^1 = \begin{vmatrix} \boldsymbol{e}_1 & g_{21} \\ \boldsymbol{e}_2 & g_{22} \end{vmatrix} / \begin{vmatrix} g_{11} & g_{21} \\ g_{22} & g_{22} \end{vmatrix} = \frac{g_{22}\boldsymbol{e}_1 - g_{21}\boldsymbol{e}_2}{g_{11}g_{22} - g_{12}g_{21}} = \frac{g_{22}\boldsymbol{e}_1 - \sqrt{g_{11}g_{22}}(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)\boldsymbol{e}_2}{g_{11}g_{22} - g_{11}g_{22}(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)^2}$$

$$= \sqrt{g_{22}}\,\frac{\sqrt{g_{22}}\boldsymbol{e}_1 - \sqrt{g_{11}}(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)}{g_{11}g_{22}[1-(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)^2]} = \frac{g_{22}\sqrt{g_{11}}\boldsymbol{x}^1 - \sqrt{g_{11}g_{22}}(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)\sqrt{g_{22}}\boldsymbol{x}^2}{g_{11}g_{22}[1-(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)^2]}$$

$$= \frac{\sqrt{g_{11}}\boldsymbol{x}^1 - \sqrt{g_{11}}(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)\boldsymbol{x}^2}{g_{11}[1-(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)^2]} = \frac{\boldsymbol{x}^1 - (\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)\boldsymbol{x}^2}{\sqrt{g_{11}}[1-(\boldsymbol{x}^1 \cdot \boldsymbol{x}^2)^2]} \tag{8}$$

This expression, which does not contain the covariant basis vectors, will be utilized in several of the 2-D examples below. Similar expressions containing the unit vectors can be obtained for (6) but these become unwieldy and are omitted here for that reason.

The metric coefficients also determine the connection between co- and contravariant vectors, e.g. for the basis vectors, as has been applied between (5) and (6). For example

$$\boldsymbol{e}_\alpha = g_{\alpha\beta}\boldsymbol{e}^\beta, \qquad e^\alpha = g^{\alpha\beta}e_\beta \tag{9}$$

where $g^{\alpha\beta}$ is the contravariant metric coefficient [12] with $g^{\alpha\beta}g_{\beta\gamma} = \delta_\alpha^\gamma$. By comparison with (7) one sees that $g^{\alpha\alpha} = 1/g_{\alpha\alpha}$ for a diagonal metric tensor. Furthermore, similar to (1):

$$\boldsymbol{e}^\alpha = \frac{\partial \xi^\alpha}{\partial \boldsymbol{l}} = \sum_\beta \hat{\boldsymbol{x}}_\beta \frac{\partial \xi^\alpha}{\partial x^\beta} \tag{10}$$

This can be simpler to use than (6) but it does require the use of local LIF coordinates. This, in a nutshell, collects most of what is needed to clarify the connections between basis vectors and metric coefficients. Now follow four illustrative examples.

REFERENCES

[1] Misner C. W., Thorne K. S. and Wheeler J. A., *Gravitation*. Princeton U. Press, 2017.

[2] Hartle J. B., *Gravity, An Introduction to Einstein's General Relativity,* Addison-Wesley, 2002.

[3] M. P. Hobson, G. Efstathiou, and A. N. Lasenby, *General Relativity, An Introduction for Physicists,* Cambridge University Press, 2006.

[4] Schutz B. F., *A First Course In General Relativity*, Cambridge U. Press, 1985.

[5] Pauli W., *Theory of Relativity*, Pergamon Press, 1958.

[6] Bergmann P. G., *Introduction To The Theory of Relativity*, Dover Publications, 1976.

[7] De Wolf D.A. , *Basis Vectors in Relativity*, Eur. J. Phys. Ed., 12, 2021.

[8] See ref. 1, §9.7ref. 2, §2.1, or *https://en.wikipedia.org/wiki/Manifold*

[9] See ref. 8 and in particular ref. 3, §2.1

[10] *https://en.wikipedia.org/wiki/Curvilinear_coordinates,*
https://www.physicsforums.com/threads/general-relativity-and-curvilinear-coordinates.818066/

[11] Covariant vectors are indicated with subscripts and contravariant ones with super-scripts. See also ref. 3, §3.4 and §3.5, or *https://www.seas.upenn.edu/~amyers/DualBasis.pdf.*

[12] See ref. 2. §20.2.

[13] *https://en.wikipedia.org/wiki/Kerr_metric*.

[14] Ref. [2], section 8.4

APPENDIX A

Consider spherical coordinates, $\boldsymbol{x} = (\hat{r}, \theta, \hat{\varphi})$ in a three-dimensional space. Then, according to the basis-coordinate definition (1b) and the definitions of Cartesian coordinates in terms of spherical ones, the metric coefficients are $g^{\alpha\alpha} g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2 \sin^2\theta$. The unit vectors are $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \text{and} \hat{\boldsymbol{\varphi}}$.

$$\boldsymbol{e}_r = \hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta = \hat{r}$$

$$\boldsymbol{e}_\theta = r(\hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - z\sin\theta) = r\hat{\theta}$$

$$\boldsymbol{e}_\varphi = -r\sin\theta(\hat{x}\sin\varphi - \hat{y}\cos\varphi) = (r\sin\theta)\hat{\varphi} \tag{A1}$$

These also follow (arguably more simply) from (3):

$$\boldsymbol{e}_r = \sqrt{g_{rr}}\boldsymbol{x}^r, \qquad \boldsymbol{e}_\theta = \sqrt{g_{\theta\theta}}\boldsymbol{x}^\theta = r\hat{\theta}, \qquad \boldsymbol{e}_\varphi = \sqrt{g_{\varphi\varphi}}\boldsymbol{x}^\varphi = (r\sin\boldsymbol{\theta})\hat{\boldsymbol{\varphi}} \tag{A2}$$

Likewise, the contravariant basis vectors follow from (7)and (A1):

$$\boldsymbol{e}^r = \frac{1}{g_{rr}}\boldsymbol{e}_r = \hat{\boldsymbol{r}}, \qquad \boldsymbol{e}^\theta = \frac{1}{g_{\theta\theta}}\boldsymbol{e}_0 = \frac{1}{r}\hat{\boldsymbol{\theta}}, \qquad \boldsymbol{e}^\varphi = \frac{1}{g_{\varphi\varphi}}\boldsymbol{e}_\varphi = \frac{1}{r\sin\boldsymbol{\theta}}\hat{\boldsymbol{\varphi}} \tag{A3}$$

These are an orthogonal set but $\boldsymbol{e}_\theta$, $\boldsymbol{e}^\theta$ and $\boldsymbol{e}_\varphi$**,** $\boldsymbol{e}^\varphi$, are not of unit length. The non-zero spherical metric coefficients follow directly upon forming $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$:

$$g_{rr} = \boldsymbol{e}_r \cdot \boldsymbol{e}_r = 1, \qquad g_{\theta\theta} = \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta = r^2, \qquad g_{\varphi\varphi} = \boldsymbol{e}_\varphi \cdot \boldsymbol{e}_\varphi = (r\sin\theta)^2 \tag{A4}$$

and the contravariant metric coefficients are obtained similarly.

APPENDIX B

Consider the following set of two-dimensional (2-D) linear coordinates, which are not mutually orthogonal, in terms of the two Cartesian coordinates $x^1 = x$ and $x^2 = y$.

$$\boldsymbol{x}^1 = x^1 - \frac{1}{\sqrt{3}}x^2 \qquad x^1 = \boldsymbol{x}^1 + \frac{1}{2}\boldsymbol{x}^2$$

$$\boldsymbol{x}^2 = \frac{2}{\sqrt{3}}x^2 \qquad x^2 = \frac{1}{2}\boldsymbol{x}^2\sqrt{3} \tag{B1}$$

The covariant basis vectors are easily obtained from (1b):

$$\boldsymbol{e}_1 = \hat{x}\frac{\partial x^1}{\partial \boldsymbol{x}^1} + \hat{y}\frac{\partial x^2}{\partial \boldsymbol{x}^1} = \hat{x}$$

$$\boldsymbol{e}_2 = \hat{x}\frac{\partial x^1}{\partial \boldsymbol{x}^2} + \hat{y}\frac{\partial x^2}{\partial \boldsymbol{x}^2} = \frac{1}{2}\left(\hat{x} + \hat{y}\sqrt{3}\right) \tag{B2}$$

The metric coefficients, easily found from $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = g_{\alpha\beta}$, are

$$g_{11} = 1, \quad g_{22} = 1, \quad g_{12} = g_{21} = \frac{1}{2} \tag{B3}$$

In order to check if (3) is satisfied, one needs the unit vectors which are seen from $\boldsymbol{x}^\alpha = \boldsymbol{e}_\alpha / |\boldsymbol{e}_\alpha|$ to be

$$\boldsymbol{x}^1 = \hat{x}, \qquad \boldsymbol{x}^2 = \frac{1}{2}\left(\hat{x} + \hat{y}\sqrt{3}\right) \tag{B4}$$

Thus $\boldsymbol{x}^1 \cdot \boldsymbol{x}^2 = 1/2$ and it is clear from (B3) and (B4) that (3) is satisfied for g_{12} and g_{21}. The contravariant basis vectors are again obtained from (10) and they are

$$\boldsymbol{e}^1 = \hat{x}\frac{\partial \xi^1}{\partial x^1} + \hat{y}\frac{\partial \xi^1}{\partial x^2} = \hat{x} - \frac{1}{\sqrt{3}}\hat{y}, \qquad \boldsymbol{e}^2 = \hat{x}\frac{\partial \xi^2}{\partial x^1} + \hat{y}\frac{\partial \xi^2}{\partial x^2} = \frac{2}{\sqrt{3}}\hat{y} \tag{B5}$$

so that these contravariant basis vectors again are neither of unit length nor orthogonal to each other. As a result: $g^{11} = \boldsymbol{e}^1 \cdot \boldsymbol{e}^1 = 4/3$, $g^{22} = \boldsymbol{e}^2 \cdot \boldsymbol{e}^2 = 4/3$, $g^{12} = g^{21} = \boldsymbol{e}^1 \cdot \boldsymbol{e}^2 = -2/3$.

The generalization of (8) for this 2-D metric is

$$\boldsymbol{e}^\alpha(\boldsymbol{x}) = \frac{1}{\sqrt{|g_{\alpha\alpha}|}} \frac{(\boldsymbol{x}^\beta \cdot \boldsymbol{x}^\beta)\boldsymbol{x}_\alpha - (\boldsymbol{x}^\alpha \cdot \boldsymbol{x}^\beta)\boldsymbol{x}^\beta}{(\boldsymbol{x}^\alpha \cdot \boldsymbol{x}^\alpha)(\boldsymbol{x}^\beta \cdot \boldsymbol{x}^\beta) - (\boldsymbol{x}^\alpha \cdot \boldsymbol{x}^\beta)^2} \tag{B6}$$

It is readily seen that this yields (B5).These basis vectors are sketched in Fig. 1 to illustrate the differences here between co- and contravariant vectors . Note in particular that $\boldsymbol{e}^1$ is perpendicular to $\boldsymbol{e}_2$ and $\boldsymbol{e}^2$ is perpendicular to $\boldsymbol{e}_1$.

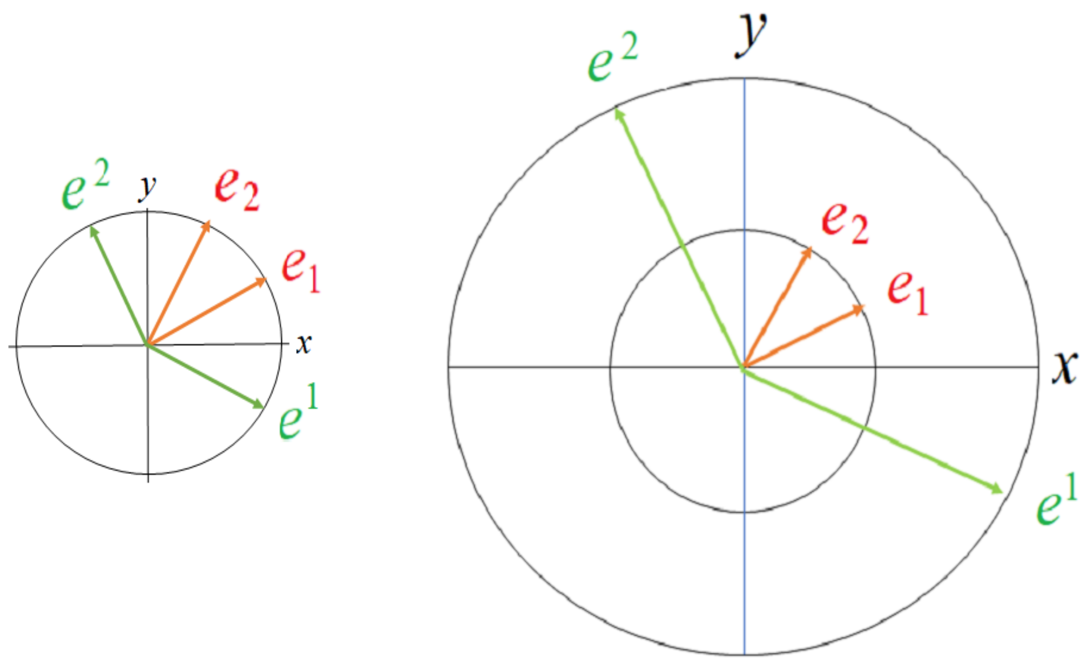

Figure 1. *Illustration of the co- and contravariant vectors for the curvilinear hyperbolic coordinates*

APPENDIX C

In this example, 2-D hyperbolic coordinates u and v are chosen; they are neither linear nor orthogonal to each other, and are given in terms of the Cartesian coordinates by

$$u = \ln\sqrt{x/y} \qquad x = ve^{u}$$
$$v = \sqrt{xy} \qquad y = ve^{-u} \qquad \text{(C1)}$$

Here $\xi^1 = u$ and $\xi^2 = v$. The differential elements are $dx = e^u(dv + vdu)$, $dy = e^{-u}(dv - vdu)$ and consequently the 2-D line element (squared) is given by

$$d\mathbf{l}^2 = (2\cosh 2u)dv^2 + (4v\sinh u)dvdu + (2v^2\cosh 2u)du^2 \qquad \text{(C2)}$$

from which the metric coefficients follow:

$$g_{vv} = (2\cosh 2u),\ g_{uu} = 2v^2\cosh 2u,\ g_{uv} = g_{vu} = 2v\sinh u,\ g_{uu}g_{vv} = 4v^2\cosh^2 2u \qquad \text{(C3)}$$

In order to check that (3) holds one needs to obtain the basis vectors, as stated above from (1), which is feasible because x, y are equivalent Cartesian coordinates. The covariant basis vectors then are:

$$\boldsymbol{e}_u = \hat{x}\frac{\partial x}{\partial u} + \hat{y}\frac{\partial y}{\partial u} = \sqrt{|g_{uu}|}\,\hat{u}$$

$$\boldsymbol{e}_v = \hat{x}\frac{\partial x}{\partial v} + \hat{y}\frac{\partial y}{\partial v} = \sqrt{|g_{vv}|}\,\hat{v} \qquad \text{(C4)}$$

and the unit vectors follow after some algebra from (C1) and (C4) with use of the definitions $\cos\varphi = e^{u}/\sqrt{e^{2u}+e^{-2u}}$, $\sin\varphi = e^{-u}/\sqrt{e^{2u}+e^{-2u}}$; they are

$$\hat{u} = \frac{v}{\sqrt{|g_{uu}|}}(\hat{x}e^{u} - \hat{y}e^{-u}) = \hat{x}\cos\varphi - \hat{y}\sin\varphi$$

$$\hat{v} = \frac{1}{\sqrt{|g_{vv}|}}(\hat{x}e^{u} + \hat{y}e^{-u}) = \hat{x}\cos\varphi + \hat{y}\sin\varphi \qquad \text{(C5)}$$

It is readily confirmed that $\hat{u}\cdot\hat{v} = \cos 2\varphi = \tanh 2u$ from which it follows that indeed $g_{ur} = \sqrt{g_{uu}g_{vv}}\,(\hat{u}\cdot\hat{v})$. Furthermore, a useful corollary of (C4) and (C5) following from this, with (3), is

$$\boldsymbol{e}_1(u,v) = v\sqrt{2\cosh 2u}\,(\hat{x}\cos\varphi - \hat{y}\sin\varphi)$$

$$\boldsymbol{e}_2(u,v) = \sqrt{2\cosh 2u}\,(\hat{x}\cos\varphi + \hat{y}\sin\varphi) \qquad \text{(C6)}$$

The contravariant counterparts follow in a similar fashion from $\boldsymbol{e}^{\alpha} = \hat{x}_{\beta}(\partial \boldsymbol{x}^{\alpha}/\partial x^{\beta})$:

$$\boldsymbol{e}^1(x,y) = \hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} = \frac{1}{2v}(\hat{x}e^{-u} - \hat{y}e^{u}) = \frac{1}{2v}\sqrt{2\cosh 2u}\,(\hat{x}\sin\varphi - \hat{y}\cos\varphi)$$

$$\boldsymbol{e}^2(x,y) = \hat{x}\frac{\partial v}{\partial x} + \hat{y}\frac{\partial v}{\partial y} = \frac{1}{2}(\hat{x}e^{-u} + \hat{y}e^{u}) = \frac{1}{2}\sqrt{2\cosh 2u}\,(\hat{x}\sin\varphi + \hat{y}\cos\varphi) \qquad \text{(C7)}$$

APPENDIX D

The Kerr metric [3] (in Boyer-Lindquist form [13]), given here in $\boldsymbol{x} = (ct, r, \theta, \varphi)$ coordinates, is

$$ds^2 = g_{00}(cdt)^2 - (g_{03}+g_{30})(cdt)d\varphi + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 \qquad \text{(D1a)}$$

with

$$g_{00} = -\left(1 - \frac{\mu r}{\rho^2}\right),\ g_{11} = \frac{\rho^2}{\Delta},\ g_{22} = \rho^2,\ g_{33} = \left(r^2 + a^2 + \frac{a^2 \mu r \sin^2\theta}{\rho^2}\right)\sin^2\theta,$$

$$g_{03} = g_{30} = -\frac{a\mu r \sin^2\theta}{\rho^2} \tag{D1b}$$

with $\rho^2 = \left(r^2 + a^2\cos^2\theta\right)$, $\Delta = \left(r^2 + a^2 - \mu r\right) = \left(\rho^2 + a^2\sin^2\theta - \mu r\right)$, $\mu = 2GM/c^2$, $a = J/Mc$, ,and r, θ, φ are standard oblate spheroidal coordinates (see below, J is the spin angular momentum). It is not easily directly seen that (3) holds, but one can infer this indirectly as follows. Hypothesize the 4-D vectorial line element to be

$$\boldsymbol{ds} = \sqrt{|g_{00}|}(c d\boldsymbol{t}) + \sqrt{|g_{11}|} d\boldsymbol{r} + \sqrt{|g_{22}|} d\boldsymbol{\theta} + \sqrt{|g_{33}|} d\boldsymbol{\varphi} \tag{D2}$$

with the $g_{\alpha\beta}$ as in (D1)and specifically with $d\boldsymbol{t} \cdot d\boldsymbol{\varphi} < 0$. Form the dot product with itself under the assumption that the only nonzero cross product is $d\boldsymbol{t} \cdot d\boldsymbol{\varphi}$:

$$ds^2 = g_{00}(cdt)^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 + 2\sqrt{|g_{00}g_{33}|}(cd\boldsymbol{t} \cdot d\boldsymbol{\varphi}) \tag{D3}$$

This would be the squared line-element expression from (D2) for the Kerr metric if $(g_{03} + g_{30})cdtd\varphi = 2\sqrt{|g_{00}g_{33}|}(cd\boldsymbol{t} \cdot d\boldsymbol{\varphi})$ .To make this plausible use $d\boldsymbol{t} \cdot d\boldsymbol{\varphi} = dtd\varphi(\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{\varphi}})$ and replace (D3)by

$$ds^2 = g_{00}(cdt)^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 + 2cdtd\varphi\sqrt{|g_{00}g_{33}|}(\hat{\boldsymbol{t}} \cdot \hat{\varphi}) \tag{D4}$$

Eq. (D4) then does imply that (3) holds and specifically that $\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{\varphi}}$ is the cosine of the angle between these two unit vectors (both of which are perpendicular to $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$).

One might think that $x^0 = ct$ and the oblate spheroidal coordinates $x^1 = x = \sqrt{r^2 + a^2}\sin\theta\cos\varphi$, $x^2 = y = \sqrt{r^2 + a^2}\sin\theta\sin\varphi$, $x^3 = z = r\cos\theta$ are LIF coordinates, but they are not. There are various ways to see this, but the following is easiest. Interpreting *ct, x y z* as LIF coordinates, one obtains

$$\boldsymbol{e}_1 = \hat{x}_0 \frac{cdt}{\partial r} + \hat{x}\frac{\partial x}{\partial r} + \hat{y}\frac{\partial y}{\partial r} + \hat{z}\frac{\partial z}{\partial r}$$

$$= \frac{r\sin\theta}{\sqrt{r^2+a^2}}(\hat{x}\cos\varphi + \hat{y}\sin\varphi) + \hat{z}\cos\theta = \frac{r\sin\theta}{\sqrt{r^2+a^2}}\hat{\rho} + \hat{z}\cos\theta \qquad \text{(D5)}$$

and also from (3) that

$$\boldsymbol{e}_1 = \frac{R}{\sqrt{\Delta}}\hat{r} \qquad \text{(D6)}$$

The expression (D5) does not agree with (D6), which would indicate that the oblate spheroidal coordinates are not LIF ones because there otherwise would be agreement.

LIF coordinates [3, 14] can be found from an alternative to (D1a):

$$ds^2 = -\frac{R^2\Delta}{\Sigma^2}(cdt)^2 + \left(\frac{\Sigma\sin\theta}{R}\right)^2 d\psi^2 + \frac{R^2}{\Delta}dr^2 + R^2 d\theta^2 \qquad \text{(D7a)}$$

with $\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta\,(r^2+a^2)$ and $d\psi = d\varphi - \omega dt$ with $\omega = \mu cra/\Sigma^2$ .This equation can be considered to be the dot product with itself of

$$d\boldsymbol{l} = \left(\frac{R\sqrt{\Delta}}{\Sigma}cdt\right)\hat{\boldsymbol{t}} + \left(\frac{R}{\sqrt{\Delta}}dr\right)\hat{\boldsymbol{r}} + \left(Rd\theta\right)\hat{\boldsymbol{\theta}} + \left(\frac{\Sigma\sin\boldsymbol{\theta}}{R}d\psi\right)\hat{\boldsymbol{\psi}} \qquad \text{(D7b)}$$

with $d\boldsymbol{\psi} = d\boldsymbol{\varphi} - \omega d\boldsymbol{t}$ and with $\hat{\boldsymbol{\psi}}$ perpendicular to the other three-unit vectors (which also are perpendicular to each other).It is left to the reader as an exercise to work this out further. As a result, $\hat{\boldsymbol{t}}, \hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{r}}$ are four mutually orthogonal unit vectors of a local inertial frame.