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# Cognitive Conflict Based on Thinking Errors in Constructing Mathematical Concept

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**Abstract:** This study aims to evaluate cognitive conflict in constructing mathematical concepts, based on thinking errors. The data were collected through observations of words or sentences, leading to the derivation of qualitative outputs. Furthermore, the results showed that the two selected subjects experienced cognitive conflicts regarding thinking errors in mathematical concept construction. This selection process was conducted through the provision of the questions obtained in the first semester of mathematics, at Universitas Negeri Makassar, Indonesia. These questions consisted of several indicators, with each of them having three components, namely the main, tracking, and supporting items. Based on the results, two subjects experienced thinking errors in constructing the concept of algebraic root form addition. This emphasized the errors in placing concepts (misplaced), pseudo-thinking, and misanalogy. A conflict was also observed between the initial concept knowledge of using variables in algebraic addition and the new principles understood through the root summation in algebra. Furthermore, there was also a conflict between the concept understanding he had in terms of the addition operation of the root form and a new understanding of the addition operation of the root form in algebra.

**Keywords:** *Cognitive conflict, misplaced concepts, misanalogy, pseudo thinking, thinking errors.*

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## Introduction

A contradiction in students' understanding of a specific mathematical concept is one problem related to the knowledge of mathematics as a cognitive process. This shows the capability of understanding mathematical concepts through various related information (Faradiba et al., 2019). This information is mostly emphasized as the definitions explaining the concepts, visual representations, examples, or applications. When these understandings do not integrate into students' thinking processes or contradict each other, mental imbalances are often observed. This conceptual disintegration or dispute prioritizes a condition known as cognitive conflict. According to Lee and Kwon (2001), cognitive conflict was a perceptual condition where people considered the differences between the mental structure and its environment (external information). It was also emphasized as the differences between the components of one's cognitive structure, including conceptions, beliefs, sub-structures, etc.

Cognitive conflict is the differences encountered by students between new information or ideas and existing mental elements, such as knowledge, perceptions, behaviours, and attitudes (Waxer & Morton, 2012; Zazkis & Chernoff, 2006). For example, conflict often arises during mathematics education, where students have preconceived ideas about solving mathematical problems that have different solution methods (Maume & Matthews, 2000). Moreover, cognitive conflict is an interactive, inspiring, exciting, and challenging learning strategy for students (Lee & Kwon, 2001). In mathematics education, this conflict is found to play an essential role in mathematical concept understanding (Baddock & Bucat, 2008; Fraser, 2007; Lee et al., 2003; Maharani & Subanji, 2018; Susilawati et al., 2017; Tall, 1977; Zazkis & Chernoff, 2006).

Cognitive conflict is widely recognized as an important factor in conceptual change and is used effectively as a teaching and learning strategy to promote theoretical development in students (Mufit et al., 2018). The concept of this conflict has recently obtained several considerations in the teaching and learning process, especially in mathematics education. Based on relevant literature, many students often encountered contradictions between conceptual descriptions and explanations, as well as construction patterns. Therefore, present mathematics educators are interested in the conceptual

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change problem through cognitive conflict implementation, to promote deeper learning and theoretical understanding of the subject (Proudfoot, 2016; Watson, 2002, 2007; Xie et al., 2022).

Several previous studies on cognitive conflict mostly emphasized the patterns of helping students reflect on their mathematical understanding. These experiments were commonly performed when handling knowledge contradictions and mainly using cognitive conflict as a teaching strategy (Maharani & Subanji, 2018; Shahbari & Peled, 2015). According to Lin (2007), teaching conflict in mathematics education was mainly used to help students reflect on their present understanding of the subject. It also helped in handling inappropriate knowledge contradictions and recognizing the importance (need) of modifying understanding to solve a different problem. However, the drawback is that they do not assess whether students actually experience cognitive conflict in their research. It could be that they only suspect that students may experience cognitive conflict because they have designed various types of situations that may appear as contradictory information to students. So, it is possible that there is a gap between what researchers expect students to experience and what students actually experience.

Cognitive conflict-based mathematics learning is also used to improve math students' problem-solving abilities, develop critical thinking, and increase knowledge, especially their communication skills (Gal, 2019; Hermkes et al., 2018; Webb et al., 2019; Widada et al., 2018; Winarti et al., 2019). However, its implementation as a teaching and learning strategy in mathematics education is not adequately understood. From this context, several studies have recommended that mathematics teachers need to be aware of cognitive conflict and its role in teaching and learning mathematics (Baglama et al., 2017; Ngicho et al., 2020). This awareness is for the better position of teachers in the learning environment and the improvement of students' critical thinking skills (Baglama et al., 2017; Gavronskaya et al., 2022; Makonye & Khanyile, 2015; Ngicho et al., 2020).

Several elements are found to cause cognitive conflicts, including information accessibility difficulties. From this context, the mathematical material information about distance is not directly related to the provided and available resources in the cognitive structure. According to Pratiwi et al. (2019), cognitive conflict was observed during information processing when the data obtained by a sensory memory and transferred to a short-term storage system were indirectly related to the resources in a long-term capacity. Moreover, the causal elements of this strategy were often prioritized in misanalogy analyses, where most of them underwent a transformation process, such as in Wyrasti et al. (2019). The misanalogical construction also transformed into cognitive conflict when the students realized contradictions to their initial beliefs. This construction was subsequently exhibited due to the anomalous situations encountered by the students when solving algebraic tasks, such as $(xy)^2 = x^2y^2$ and $(x + y)^2 = x^2 + y^2$. In this case, the anomalous situation was realized, leading to a contradiction between the initial misanalogical construction belief and the existing anomalous situation.

The causes of cognitive conflicts, such as misanalogy, are a form of thinking errors in constructing concepts, as shown by Subanji and Nusantara (2013). In the study, the characteristics of students' thinking errors in this concept included (1) pseudo-true & pseudo-false thinking, (2) analogy thinking, (3) placing concepts, and (4) logical thinking. This was in line with Mahharrini et al. (2020); Wibawa (2019); Wibawa et al. (2017, 2018), where thinking errors were the deviant activities carried out by students in building a mathematical concept. In this case, four errors were observed in the construction of mathematical concepts by students.

According to Subanji (2015), thinking errors in conceptual construction were deviations from formal principles in mathematical concept development. For example, the pseudo-thinking errors developed by students are caused by the construction of mathematical concepts being different from existing principles. The study also emphasized the provision of algebraic problems such as $2x + 3x = 5x$ to students, which used their perspectives to answer the question appropriately. However, a mistake was observed in the answer justification when the interview was conducted. This was due to their assumptions that $2x + 3x = 5x$ was true due to x being illustrated as objects such as “books” and “oranges”. In this process, $2x + 3x$ was not interpreted as numbers, with the nature of the addition operation subsequently ignored. Based on these problems, students experienced true pseudo-thinking constructions. This led to the following questions, *"What happens when these errors become cognitive conflicts? and how does the thought process occur?"*, as illustrated by Wyrasti et al. (2019). From these descriptions, such misanalogical constructions are capable of transforming into cognitive conflicts.

A preliminary report was also conducted through observation and interviews with two students, which were the selected participants at Universitas Negeri Makassar, Mathematics Education Study Program. Initially, the participants were provided with questions in the form of a true-false logic statement. This was accompanied by the need to explain their understanding of the entire concept, by thinking aloud during problem-solving. These analyses emphasized the provision of the thinking process with indications of conceptual construction errors and cognitive conflicts. The problem provided is as follows, $\sqrt{2x} + \sqrt{2x} = 2\sqrt{2x}$. In this case, the participants were expected to provide true or false answers with appropriate reasons for the statement presented. Based on the results, all the participants provided similar answers to the question, regarding the statement being "true". This indicated that the variables were immediately summed up since they were the same, namely $x + x = 2x$ (True).

Similar to the previous question, a form of intervention was also provided, namely $\sqrt{2x} + \sqrt{3x} = \sqrt{5x}$. From this context, the same question was asked, regarding the statement being true or false with the provision of adequate reasons. Based on the results, the participants answered that the statement was "true" due to having similar variables, namely $x$, where $\sqrt{2x} + \sqrt{3x} = \sqrt{5x}$. This was in line with the previous question, where appropriate answers were obtained because the coefficients and variables in the roots were the same as $x + x = 2x$, with $x$ being the example of $\sqrt{2x}$. When different coefficients are observed with similar variables, the direct addition of $2 + 3 = 5$ was sufficient despite having the roots as $2x + 3x = 5x$. In this case, the data obtained from the thought process of the participants assumed that similar root coefficients and variables led to the application of $p\sqrt{x} + q\sqrt{x} = (p + q)\sqrt{x}$. When the variables are the same with different coefficients, the direct addition of $2 + 3 = 5$ was also observed and unassumed despite being in the roots.

Based on the results, a solution was unachieved when the variables were different with similar coefficients. This occurrence was observed when considering the errors in placing concepts and analogy. When students were working on the aforementioned questions, previous indications of cognitive conflicts were also found based on the theory described. Regarding the investigation of the theory, the participants were asked the reasons they expressed confusion when provided with the questions. The results showed that a difference or conflict was also observed between the concepts understood by the students in the algebraic addition operations carried out with similar variables. In contrast, the concept encountered was responsible for summing the algebraic root form. To be used as an initial indication from preliminary studies, more deep analyses are expected on identifying cognitive conflicts regarding thinking errors in constructing concepts. Therefore, this study aims to evaluate cognitive conflicts in mathematical concept construction, based on thinking errors.

## Methodology

*Study Design*

In this study, a qualitative approach was used due to the production of comprehensive descriptions (Creswell, 2002; Fraenkel et al., 2012; Miles et al., 2014). The approach was also used to frame data about the subject's in-depth perception, through observation (attentiveness), empathic understanding (empathetic understanding), and grouping preconceptions (bracketing preconceptions), regarding the topic evaluated. Furthermore, the data analysis method was inductive because the experimental activities used field facts to determine the identification of cognitive conflicts. In this case, direct observations were critically carried out directly, accompanied by the collection of data from various sources, including interview transcripts and field notes, as well as the review and interpretation of relevant information. This inductive data analysis was subsequently conducted by constructing patterns, categories, or themes into more abstract information units, to provide a holistic account of the analyzed problem.

*Participants*

This analysis was carried out in Mathematics Education Study Program for first-semester students at Universitas Negeri Makassar, Indonesia. As many as 207 students from 4 classes in the first semester. The number of male students was 73, and 134 female students. Based on the following criteria, the students were selected as the participants, (1) they have obtained material on algebra and fundamental concepts, (2) the students were able to communicate the mathematical idea of the present problem, as well as (3) they have the opportunity to develop a perspective on the problems encountered, design construction, and reflect on previous experiences. These criteria enabled the easier identification of cognitive conflicts in constructing concepts, regarding thinking errors. From this context, a total of 20 students were used as the prospective study subjects. There are 8 male students and 12 female students. This is based on the results of discussions and suggestions from lecturers in algebra courses.

*Data Collection and Analysis*

The expert is the main instrument acting as a planner, which directly handles experimental subjects, obtains data, performs information processing, analyzes outputs, concludes, and reports analytical results. This was shown by the researcher giving questions to the participants first, then analyzing the results of the problem solving, then conducting more in-depth interviews based on the results of the problem solving. Based on identifying a cognitive conflict, the thing that needs to be observed when students have solved the given problem is the intervention of the expert. This analysis was carried out before meeting several characteristics indications of thinking errors in mathematical concept construction. In the early stages of the analysis, 20 questions were provided to 20 students, whose selection as the prospective subjects emphasized the previous relevant criteria. This question prioritized algebra expression, regarding the concept of basic algebraic and arithmetic operations on the roots containing several indicators. In this case, each indicator had three components namely the main, tracking, and supporting questions. The main question is often determined for the understanding possessed by the student. This is accompanied by the tracking which are commonly used to observe thinking errors in constructing concepts and supporting questions which are commonly used to observe cognitive conflict, respectively. This cognitive conflict can be seen based on the components of cognitive conflict, namely (a) recognizing an anomalous situation, (b) expressing interest or anxiety in overcoming cognitive conflict, and (c) engaging in cognitive reappraisal.

Based on the results, the indications regarding the experience of the participants with thinking errors in constructing concepts are expected to be observed. Interviews are also needed to clarify invisible elements when the subject solves the problem. This depends on the discrepancy between the expressed and written elements. So that, for the validity of the data using the triangulation method. Therefore, any indications should be explored by subsequently tracing and confirming the existence of an error in constructing the concept. The identification process was also carried out through a think-aloud approach, to determine the indications of cognitive conflicts caused by thinking errors. From the results, only two selected participants experienced a cognitive conflict based on an error in constructing the concept. This was accompanied by in-depth interviews, to ensure the dispute levels of the participants before identifying the occurrence of the cognitive conflicts.

## Results

Based on the results, only two selected participants experienced cognitive conflicts based on thinking errors in constructing concepts. The two selected participants represent students who experience cognitive conflict who have the same characteristics based on errors in constructing concepts consisting of 1 male student and 1 female student. In this case, the provided questions consisted of several indicators, each having three components, namely the main, tracking, and supporting items. The several indicators, only one was observed where the participants experienced cognitive conflict, namely the indicator of Completing Algebraic Root-form Operations. The following is a presentation of the results obtained,

*Participant UAA*

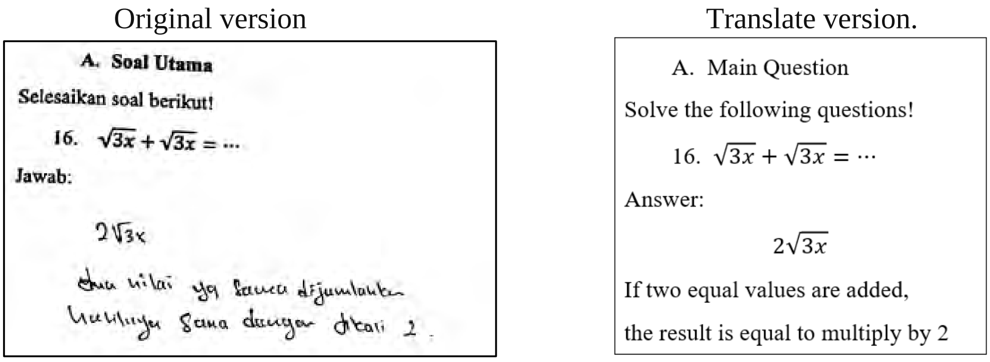
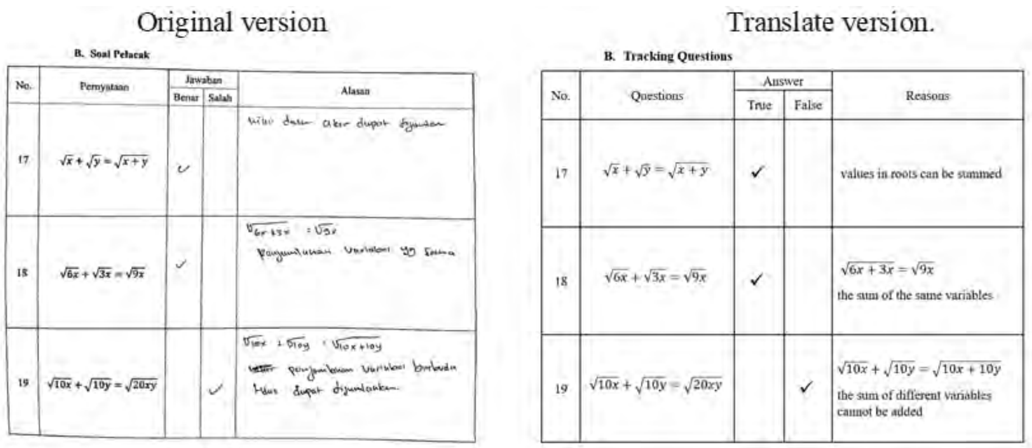



*Figure 1. Understanding Possessed by Participant UAA Based on Answer to the Main Question*

According to Figure 1, the understanding possessed by this participant indicated that the problem was solvable. In this case, the result obtained was $2\sqrt{3x}$, proving that the participant's answer was corrected. Since the reason provided emphasized the addition of similar values, the output obtained was then the same as multiplied by 2. After tracing, the statement also assumed a multiple of $\sqrt{3x}$, leading to two times of $\sqrt{3x}$. Similarly, the thrice addition of the algebraic operation produced three times the value of $\sqrt{3x}$.



Original version

B. Soal Pelacak

| No. | Pernyataan | Jawaban Benar | Jawaban Salah | Alasan |
|---|---|---|---|---|
| 17 | $\sqrt{x}+\sqrt{y}=\sqrt{x+y}$ | ✓ | | nilai dalam akar dapat dijumlahkan |
| 18 | $\sqrt{6x}+\sqrt{3x}=\sqrt{9x}$ | ✓ | | $\sqrt{6x+3x}=\sqrt{9x}$ penjumlahan variabel yg sama |
| 19 | $\sqrt{10x}+\sqrt{10y}=\sqrt{20xy}$ | | ✓ | $\sqrt{10x}+\sqrt{10y}=\sqrt{10x+10y}$ penjumlahan variabel berbeda tidak dapat dijumlahkan |

Translate version.

B. Tracking Questions

| No. | Questions | Answer True | Answer False | Reasons |
|---|---|---|---|---|
| 17 | $\sqrt{x}+\sqrt{y}=\sqrt{x+y}$ | ✓ | | values in roots can be summed |
| 18 | $\sqrt{6x}+\sqrt{3x}=\sqrt{9x}$ | ✓ | | $\sqrt{6x+3x}=\sqrt{9x}$ the sum of the same variables |
| 19 | $\sqrt{10x}+\sqrt{10y}=\sqrt{20xy}$ | | ✓ | $\sqrt{10x}+\sqrt{10y}=\sqrt{10x+10y}$ the sum of different variables cannot be added |

*Figure 2. Thinking Errors in Constructing Concepts by Participant UAA Based on Answer to the Tracking Questions*

Based on Figure 2, the statement was "true" for Question 17, with reasons subsequently provided for adding up the root values. When continuously explored, the interpretation of the summation process emphasized combination into one root, where $\sqrt{x}+\sqrt{y}$ became $\sqrt{x+y}$, which was assumed to be $(x)+(y)=(x+y)$. In this case, the "true" option was selected when answering Question 18, $\sqrt{6x}+\sqrt{3x}=\sqrt{9x}$, because the sum of the variables was the same. For Question 19, $\sqrt{10x}+\sqrt{10y}=\sqrt{20xy}$, the "false" option was selected by the participant, due to the unsolvable summation of the different variables, which should be expressed as $\sqrt{10x + 10y}$. Figure 3 is then obtained when the structure of thinking is described.

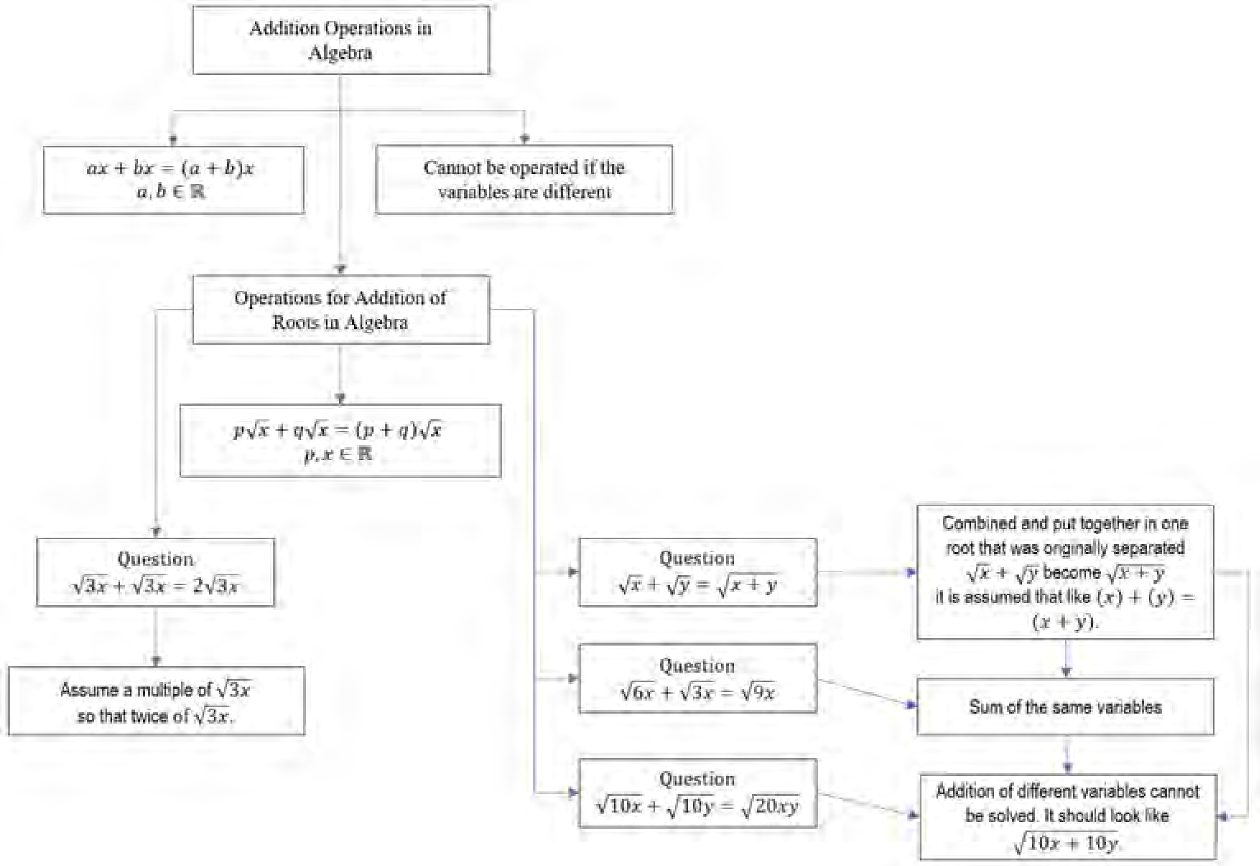
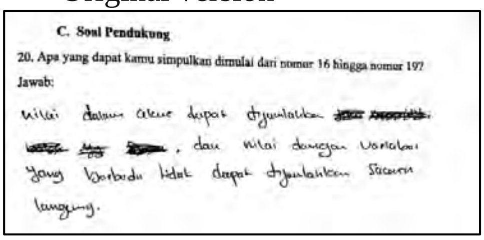



*Figure 3. Thinking Structure of Participant UAA*

In Figure 3, an error was found in Question 17 during the mathematical concept construction. This error was observed in placing the concept as misanalogical because the problem analogy and applied theory were incorrect and inappropriate in the addition operation of the root form. For Question 18, an error was also found in placing the concept, with the answer provided still using the principle of the algebraic addition operations instead of in the root form. In this case, the student only perceived that the variables were similar to the operable ones. Meanwhile, pseudo-corrected thinking was observed for Question 19, where the correct answer was provided with the wrong reasons. This was because the student was linking the answer to Question 17, where an error or misanalogy was encountered.

Original version


C. Soal Pendukung

20. Apa yang dapat kamu simpulkan dimulai dari nomor 16 hingga nomor 19?

Jawab:

nilai dalam akar dapat dijumlahkan ~~...~~, dan nilai dengan variabel yang berbeda tidak dapat dijumlahkan secara langsung.

Translate version.

C. Supporting Question

20. what can you conclude starting from number 16 to number 19?

Answer:

values in roots can be added and values with different variables cannot be added directly

*Figure 4. Cognitive Conflict Experienced by Participant UAA Based on Answer to the Supporting Question*

Figure 4 showed the conclusion of the student, where the root and different variable values were directly and indirectly summed up. Based on the previously described answers, a cognitive conflict was observed due to the opinion differences on the main and tracking questions. Besides this, the conflicts occurring through interviews were also determined. The following is the interview session conducted with a participant,

P : Try to explain. What do you know about addition operations in algebra?

UAA : For the addition operation in algebra, sir, what needs to be considered is that the variables are the same, and then they can be operated.

P : How about the sum of the roots without variables?

UAA : What do you mean, sir? How about an example, sir?

P : $\sqrt{3} + \sqrt{3} = ..$? what is the answer?

UAA : $2\sqrt{3}$ sir.

P : Why? why not $\sqrt{6}$

UAA : Because it can be operated when the one inside the root is the same. So, it can be assumed that $x$ is $\sqrt{3}$

P : How about these $3\sqrt{5} + 2\sqrt{5} = \cdots$ ?

UAA : It's the same as before, sir. So $5\sqrt{5}$ because $(3 + 2)\sqrt{5}$

P : So, when it is not the same as inside the root, then it can't be operated?

UAA : Hmm. Yes Sir.

P : Try to pay attention to your answer on number 16 with 18. Is it appropriate?

UAA : Yes, sir, it is appropriate.

P : Why is the answer to question 16 correct and not $\sqrt{6x}$ ?

UAA : Oh. Yes, yes. (think) but right, sir, it should be a multiple of the same as $x + x = 2x$, where $x$ is $\sqrt{3x}$. Same with the previous problem, sir.

P : How about numbers 17 and 18? Can it be like that?

UAA : I'm confused, sir. But in my opinion, sir, it is appropriate when the different variables cannot be added up. Just put them together in the root. So, case number 18 is directly added up based on question 17.

P : So, how about number 19?

UAA : It seems, sir, you can't be like that hehehe (while scratching head and laughing), because only the root can be operated when it's not the same. When the variables are different, it can't be like a number. 19. So $\sqrt{10x + 10y}$, that's enough for the answer, sir, and should not be $\sqrt{20xy}$

P : What do you think is the meaning of $x$ and $y$ from the questions?

UAA : Is it variable, sir? A number can be substituted.

P : Now, try to change to the number 18.

UAA : When $x = 3$ then $\sqrt{6 \times 3} + \sqrt{3 \times 3} = \sqrt{18} + \sqrt{9} = 3 + \sqrt{18}$, here it is, sir.

P : The result is not the same as when you add up the roots $\sqrt{9x} = \sqrt{9 \times 3} = \sqrt{27}$, or you initially change the value and then sum it together.

UAA : $\sqrt{18 + 9} = \sqrt{27}$ same sir with $\sqrt{9x} = \sqrt{27}$. But the result is different from $3 + \sqrt{18}$ (looks doubtful with the answer and starts to get confused)

P : So, how?

UAA : I am still confused, sir. So, that means I'm wrong, sir.

P : Try to note this statement, $\sqrt{5} + \sqrt{5} = 2\sqrt{5}$ and $\sqrt{5x} + \sqrt{5x} = 2\sqrt{5x}$. Both of these statements are. "true", because they are multiples. For the first case, when assuming $x = \sqrt{5}$, then $x + x = 2x$.

Likewise, with the second case, when assuming $a = \sqrt{5x}$, then $a + a = 2a$. What do you think about that?

UAA : Right, sir, because the example has the same variable, it can be operated.

P : How about this? I said the number 18 was wrong because it is not operable. When, for example, the variables are different.

UAA : Hmm, yes, sir. (thinking) but sir, you can still add up the ji directly because it is the same as the variable.

P : Try to relate it to a concept that you understand.

UAA : I understood from the start that, in addition, the variables had to be the same before they could be operated. When the sum of the forms of the roots inside the roots, they should be the same.

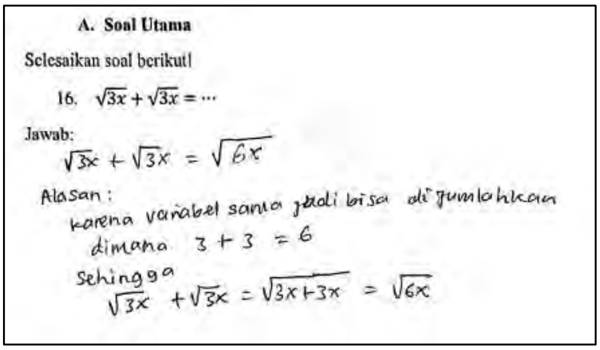
P : How about the operation of adding roots in algebra?

UAA : It turns out that everything in the root needs to be the same for the algebraic case. Then it can be operated even though only the variables are the same. But it is also necessary to pay attention to the coefficients. So, when the coefficients are different, it cannot be, for example, when it is converted into an algebraic addition operation, where the variables should be the same.

From this interview, the participant showed signs of experiencing cognitive conflict, regarding the emergence of doubt and confusion. According to the subject's understanding of addition in algebra, the point of consideration indicated that the variables were operable when similar. For the addition of the root forms without variables, the operation was possible when the values within were the same. This aligned with the following statement, *"I started to get confused when I looked back at the answers to questions 17 and 18"*. From the results, the answer obtained was appropriate because the variables were not operable when different. This was in line with the following statement, *"Just put them together in one root"*. Therefore, a conflict was observed between the existing understanding of the student and the new knowledge acquired on the use of variables.

*Participant FRA*

Original version



**A. Soal Utama**

Selesaikan soal berikut!

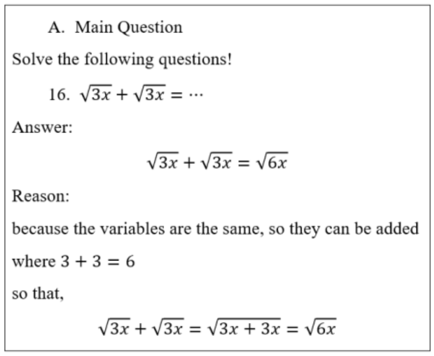
16. $\sqrt{3x} + \sqrt{3x} = \cdots$

Jawab:

$\sqrt{3x} + \sqrt{3x} = \sqrt{6x}$

Alasan :
karena variabel sama jadi bisa di jumlahkan
dimana $3 + 3 = 6$
Sehingga
$\sqrt{3x} + \sqrt{3x} = \sqrt{3x + 3x} = \sqrt{6x}$

Translate version.



A. Main Question

Solve the following questions!

16. $\sqrt{3x} + \sqrt{3x} = \cdots$

Answer:

$\sqrt{3x} + \sqrt{3x} = \sqrt{6x}$

Reason:

because the variables are the same, so they can be added

where $3 + 3 = 6$

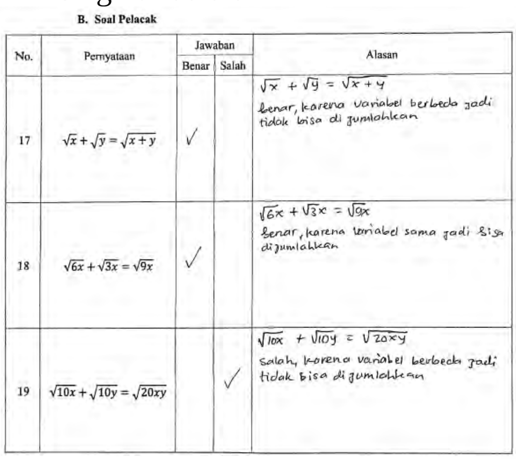
so that,

$\sqrt{3x} + \sqrt{3x} = \sqrt{3x + 3x} = \sqrt{6x}$

*Figure 5. Understanding Possessed by Participant FRA Based on Answer to the Main Question*

Based on Figure 5, the understanding possessed by this participant showed that the problem was solvable when the variables were the same. This proved that $\sqrt{3x} + \sqrt{3x} = \sqrt{6x}$ was solvable due to the similar variables being the same, where $3 + 3 = 6$. In this case, the expression, $\sqrt{3x} + \sqrt{3x} = \sqrt{3x + 3x} = \sqrt{6x}$. After subsequent exploration, the problem was found to be generally identical to the algebraic addition concept. This indicated that the variables were solvable and inoperable when similar and different, respectively. Therefore, the coefficients were highly sufficient for summation with similar variables, proving that the determinant is capable of being added up to $6x$ when $3 + 3 = 6$.

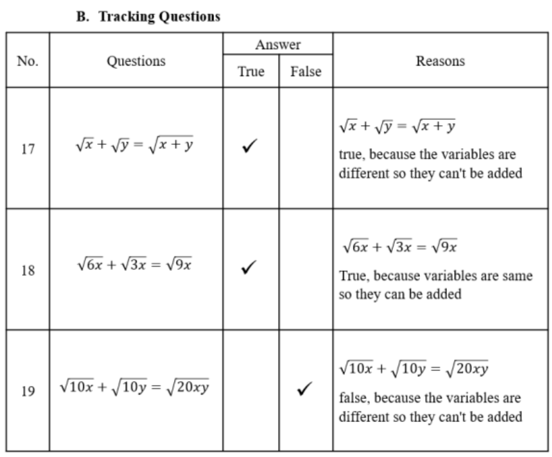
Original version



**B. Soal Pelacak**

| No. | Pernyataan | Jawaban Benar | Jawaban Salah | Alasan |
|---|---|---|---|---|
| 17 | $\sqrt{x} + \sqrt{y} = \sqrt{x + y}$ | ✓ | | $\sqrt{x} + \sqrt{y} = \sqrt{x + y}$ benar, karena variabel berbeda jadi tidak bisa di jumlahkan |
| 18 | $\sqrt{6x} + \sqrt{3x} = \sqrt{9x}$ | ✓ | | $\sqrt{6x} + \sqrt{3x} = \sqrt{9x}$ benar, karena variabel sama jadi bisa dijumlahkan |
| 19 | $\sqrt{10x} + \sqrt{10y} = \sqrt{20xy}$ | | ✓ | $\sqrt{10x} + \sqrt{10y} = \sqrt{20xy}$ salah, karena variabel berbeda jadi tidak bisa dijumlahkan |

Translate version



**B. Tracking Questions**

| No. | Questions | Answer True | Answer False | Reasons |
|---|---|---|---|---|
| 17 | $\sqrt{x} + \sqrt{y} = \sqrt{x + y}$ | ✓ | | $\sqrt{x} + \sqrt{y} = \sqrt{x + y}$ true, because the variables are different so they can't be added |
| 18 | $\sqrt{6x} + \sqrt{3x} = \sqrt{9x}$ | ✓ | | $\sqrt{6x} + \sqrt{3x} = \sqrt{9x}$ True, because variables are same so they can be added |
| 19 | $\sqrt{10x} + \sqrt{10y} = \sqrt{20xy}$ | | ✓ | $\sqrt{10x} + \sqrt{10y} = \sqrt{20xy}$ false, because the variables are different so they can't be added |

*Figure 6. Thinking Errors in Constructing Concepts by Participant FRA Based on Answer to the Tracking Questions*

In Figure 6, the statement was "true" for question 17, with the provided reason confirming that the variables were inoperable when different. These variables were combined between the roots of $x$ and $y$ after continuous exploration, with $\sqrt{x}+\sqrt{y}$ becoming $\sqrt{x+y}$, which was considered to be $(x)+(y)=(x+y)$. This was slightly similar to the answers in Question 19, leading to the selection of the "false" option. In this case, the variable was unable to be summed up when different. Based on subsequent investigation, the corrected answer should be separated as observed in question 17, leading to the following expression, $\sqrt{10x}+\sqrt{10y}=\sqrt{10x+10y}$. Although the coefficients were identical, the similarity of the variables was still very important. This demonstrated that the coefficients were capable of being added up when the same, as observed in question 18. From these results, the "true" option was selected for the statement because the variables were the same. Therefore, $\sqrt{6x}+\sqrt{3x}=\sqrt{9x}$, where $6+3=9$. When the structure of thinking was described, Figure 7 is then obtained.

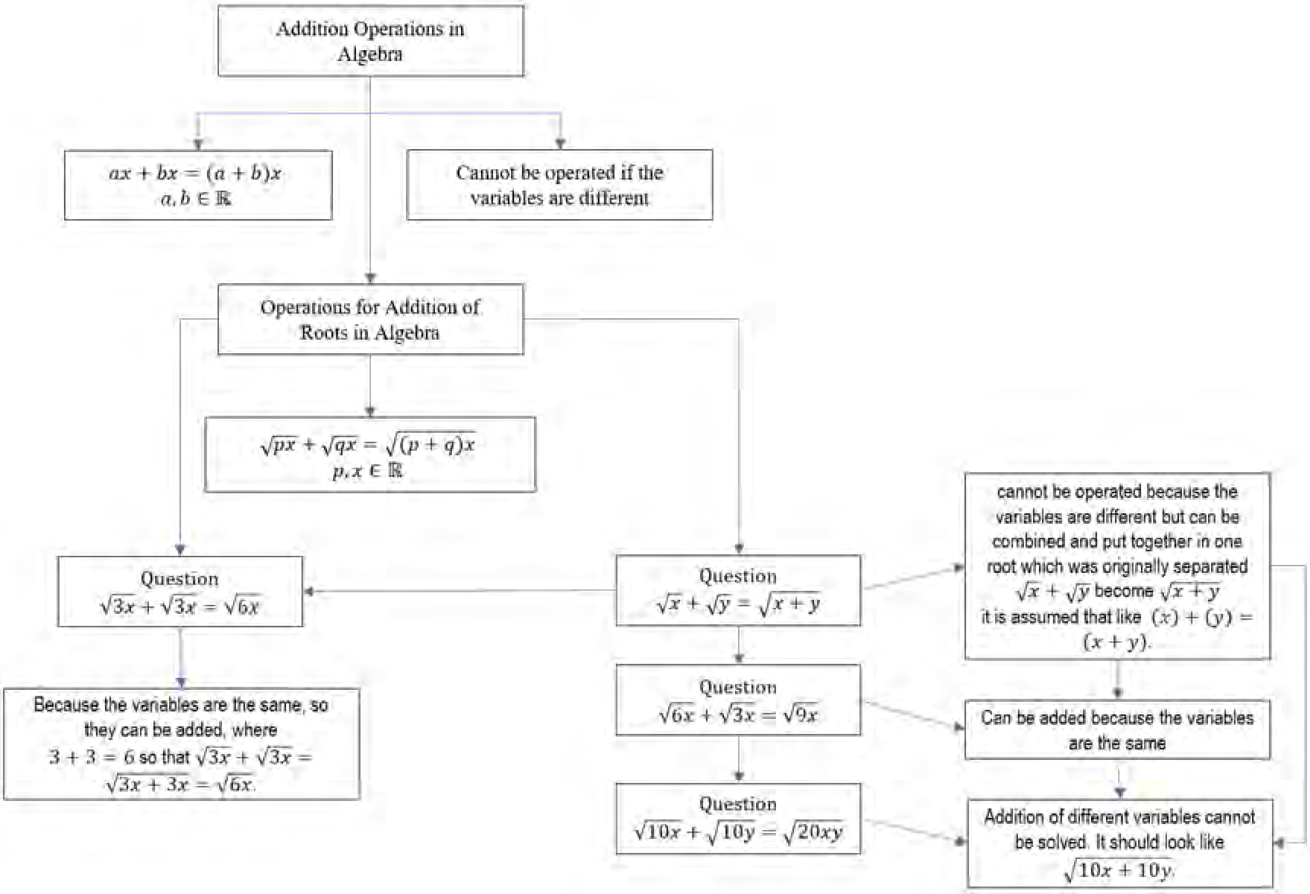



*Figure 7. Thinking Structure of Participant FRA*

According to Figure 7, an error was found in the thinking structure of the mathematical concept construction. This showed that an error in placing the concept as misanalogical was observed in question 17, because the theory applied was limited to algebraic addition. The student also assumed that a similar procedure was carried out in the brackets capable of being concatenated or not operated due to different variables. For question 18, an error in placing the concept with the answer provided still used the principle of algebraic addition operations instead of in the form of roots. However, pseudo-true thinking was observed in question 19, where the appropriate answer was stated with the wrong reason. This was due to the student's answer linkage to number 17, where a misanalogical error was observed.

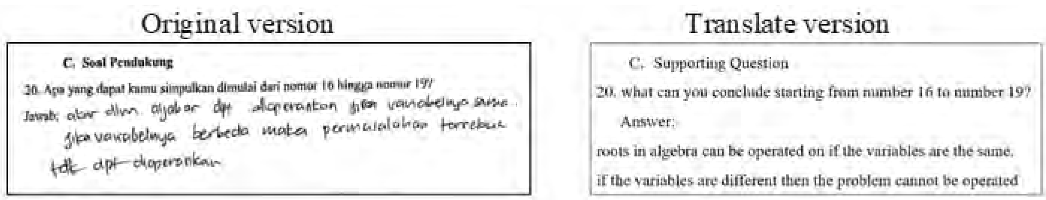



*Figure 8. Cognitive Conflict Experienced by Participant FRA Based on Answer to the Supporting Question*

Based on Figure 8, the algebraic roots were operated and unsolvable when the variables were the same and different. From the previously described answers, cognitive conflicts were also observed due to the opinion differences on the general and written understanding of the algebraic root form. Furthermore, the conflicts occurring through interview sessions were determined, as observed below, P : try to explain. What do you know about addition operations in algebra?

FRA : in addition, operations in algebra can be operated when the variables are the same.

P : how about the sum of the roots without variables?

FRA : when the roots are the same, it can be operated directly, sir.

P : give an example.

FRA : for example, sir, $\sqrt{3}+\sqrt{3}=2\sqrt{3}$

P : why the answers $2\sqrt{3}$, not $\sqrt{6}$

FRA : because it can't, sir.

P : how about $\sqrt{3}+\sqrt{2}=\sqrt{5}$ ?

FRA : That is wrong, sir. It can't be operated, sir.

P : try to pay attention to the number 18. Why answer like that?

FRA : wait, sir (thinking and getting confused). This is suitable, sir because in algebra, it can add up when. the variables are the same

P : You said earlier that when the roots are the same, they can be operated, but your answer to number 16 is the same as in question 18. You add up only in the roots, unlike the form without variables.

FRA : It's different, sir. When there is a variable, we don't know whether the value is the same. So it's enough that the roots are added up according to the algebraic sum, where similar variables can be operated.

P : then, how about this $\sqrt{3}+\sqrt{2}=\sqrt{5}$ Do you say this is wrong?

FRA : yes, sir, hehehe (scratching head and laughing). Hmm. (thinking) but, sir, because there is no variable element, you can't add it up directly, in my opinion.

P : in your opinion, what is a variable?

FRA : a variable is an example or a substitute for a number.

P : so, try to change it to number 18

FRA : how much do you change, sir?

P : suppose $x=1$

FRA : $\sqrt{6(1)}+\sqrt{3(1)}=\sqrt{6}+\sqrt{3}$. This is it, sir.

P : just like that?

FRA : hmm. Yes, sir, (doubtfully) it's only there because it's not the same

P : why not the answer $\sqrt{9}=3$

FRA : the roots are different, sir.

P : then why do you say number 18 is true despite the changes to the x-value showing that it can't be operated?

FRA : oh yes, sir (scratches head and looks confused)

P : how about number 19?

FRA : because the variables are different, they can't be operated, especially when the value of $x$ and $y$ are changed. The answer should be $\sqrt{10x+10y}$

P : Why put them together in one root?

FRA : similar to question number 17, sir. It can't be operated, but its form can be like that.

P : if it can be like that, what's the difference with the number 18?

FRA : Yes, sir. (doubtful)

P : try to explain your understanding now. How about adding algebra with the form of roots?

FRA : At first, I thought algebraic addition could be operated when it has the same variables. In contrast, for the addition of roots, it is important to pay attention to the similarity of the roots before they can be operated.

P : How about the addition of roots in algebra?

FRA : which in the root must also be the same even though the variables are similar, but the variable is replaced with a number. When a different value is produced, it cannot be operated.

From this interview session, the participant showed signs of experiencing cognitive conflict, through the emergence of doubt and confusion. According to the subject's understanding, the addition of algebra was operable when the variables were the same. For the addition of the root form without variables, the operation was conducted when the internal values were similar. When the participant was instructed to replace the variable with a number, confusion was immediately observed while relating the expression to the answers in numbers 18 and 19. Regarding the subject's answer, the root form operation was generally added up when the root was the same, although when replacing the variable with different numbers. For number 19, the result was also appropriate because the difference in the variables led to the inability to operate. Therefore, a conflict was observed between the existing understanding of the student and the new knowledge acquired in the use of algebraic root form operations.

## Discussion

Based on the results, the two participants experienced thinking errors in constructing the mathematical concept. This indicated that both subjects experienced errors in placing the concept of algebraic addition to the operation of the root form. The concept understood by these students was also the operations with similar variables, where differences often lead to unsolvable summation. When this concept was applied and combined with the concept of root form addition, they only assumed that operation was possible with similar variables. Moreover, the two participants experienced errors in analogy/misanalogy and pseudo-true misconceptions when providing the correct answers that were then proved wrong through subsequent explanations. During this process, the reasons provided only considered the form or writing, not the final result.

From the results, the two subjects experienced cognitive conflicts between their initial beliefs and the anomalous situations encountered. The errors in placing the concept, analogy, and pseudo-true thinking were also identified in the conceptual construction and were the initial beliefs held by the subjects. Based on the first subject, the concept of variables in algebraic addition operations was highly understood. In this case, coefficient summation procedures were only directly possible when the variables were similar. For example, $6x + 3x$ was directly added together because of the similar x-variable. This was marked by doubts and confusion when the student realized that the initial possessed understanding contradicted the situation where the coefficients need to be considered for the addition operation of the root form, namely $\sqrt{6x} + \sqrt{3x}$. Both of these signs are examples of cognitive conflict behavior that has been described in Lee et al.'s (2003) research. Doubt occurs when recognizing one's conceptions are not consistent with the results of the experiment/discourse/textbook/etc., whereas confusion occurs when feeling anxious about the anomalous situation.

For the second subject, the possessed initial understanding of root form operations was contrary to that of the algebraic expression. This was subsequently marked by doubts and confusion when the algebraic root operations were identified by only considering some direct variables, such as $\sqrt{6x} + \sqrt{3x}$, because they both contained the $x$-variable, subsequent analysis provided $\sqrt{6x + 3x} = \sqrt{9x}$ , which was different from the root-form addition operation. Therefore, the roots such as $\sqrt{6} + \sqrt{3}$ should not be operated directly, unless similar coefficients including $\sqrt{6} + \sqrt{6}$, were involved.

According to these results, the situations occurring due to thinking errors led to cognitive conflicts when the subjects realized contradictory elements between the initial beliefs and the anomalous conditions encountered. This is in line with the results of Wyrasti's (2019) research which states that errors in conceptual construction in the form of misanalogy can turn into cognitive conflict. Besides misanalogical constructs transforming into the conflicts suggested by Wyrasti et al. (2019), other construction errors mentioned by Subanji (2015); Subanji and Nusantara (2013), were also capable of such transformation.

To assess whether students really experience cognitive conflict or not, someone can look at it in terms of thinking errors in constructing the concept. So, someone does not need to deliberately design various types of situations that may appear as contradictory information to students such as research Baglama et al. (2017) and Ngicho et al. (2020), but, unexpectedly based on thinking errors in constructing concepts can also create situations of cognitive conflict.

## Conclusion

Based on the results, cognitive conflicts were observed between the initial understanding of the concept and the new knowledge acquired, specifically regarding algebraic variable and root addition operations. A conflict between the concept understanding possessed in terms of the addition operation of the root form and a new understanding of the addition operation of the root form in algebra was also found. Furthermore, the initial understanding was identified as a thinking error in the mathematical concept construction. In this case, errors in placing the concept, pseudo-thinking, and misanalogy were observed.

From the results, the thinking errors experienced by the participants in constructing the mathematical concepts were transformed into cognitive conflicts. Moreover, the analogy, pseudo-thinking, and placing the concept errors were also observed as the sources of the conflict. This was observed when the subjects realized contradictory conditions between the initial beliefs and the anomalous situations encountered.

## Recommendations

Based on the results, misanology and other errors such as pseudo thinking and placing the concept were capable of transforming into cognitive conflict. Despite these conclusions, the experiment still needs to be explored more deeply regarding the four thinking errors in mathematical concept construction. This is because the study is only limited to algebra, especially in the material using square roots. Therefore, the use of another thinking error, namely logical, is recommended for subsequent analysis. From these conclusions, future studies should also consider various materials asides from algebra, to develop the thinking errors that are unable to be transformed into cognitive conflict.

## Limitations

The limitations of this study are only focused on the algebraic material, specifically on the use of square roots. The selection of participants was also limited to one class in the early semester, where 20 students and two representatives were used in the analysis. These representatives were selected due to the following criteria, (1) the ability to communicate mathematical ideas and develop perspectives on the problems encountered, (2) capable of carrying out constructions, and (3) the ability to reflect on previous experiences.

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### Authorship Contribution Statement

HR: Concept and design, data acquisition, data analysis/interpretation, and drafting manuscript. Purwanto: Critical revision of the manuscript, technical or material support, supervision, and final approval. Sukoriyanto: Admin, technical or material support, reviewing, supervision, and final approval. Parta: Concept and design, critical revision of the manuscript, technical or material support, supervision, and final approval.

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