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Research Article

Uncovering student errors in measures of dispersion: An APOS theory analysis in high school statistics education

Chiew Leng Ng [1]
0000-0002-1370-3683

Cheng Meng Chew [1,2*]
0000-0001-6533-8406

[1] Universiti Sains Malaysia, MALAYSIA
[2] Wawasan Open University, MALAYSIA
[*] Corresponding author: cmchew@usm.my; cmchew@wou.edu.my

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ABSTRACT

Despite statistics learning becoming more important during this information explosion era, many students still deem the subject complex and challenging. Measures of dispersion, a critical component of statistical knowledge that students often struggle with, have received little attention in research on statistics education. The goal of this study was to uncover students' errors in solving problems involving measures of dispersion by examining students' response in the diagnostic test through the lens of APOS theory. The participants consisted of 85 grade 11 high school students and were then divided into three groups according to their performance to better understand the difficulties and errors made by students from different cognitive levels. The findings revealed that majority of low achievers operate at the action level, as indicated by the numerous conceptual errors discovered during the test. These students have limited conceptual understanding on the topic which required proper remedial from the educators. The study's results are discussed, as well as potential implications for education.

Keywords: measures of dispersion, statistical learning, APOS, error analysis

INTRODUCTION

Technological democracies require statistical literacy (Gigerenzer et al., 2007) and the ability to comprehend data and graphs is indispensable in all facets of modern life (Lee et al, 2022). Reseachers urge for statistical education enhancements in secondary school curricula to focus on complex data (Engel, 2017), data skills (Boaler & Levitt, 2019), and develop student's literacy for using data in their everyday lives (Lee & Campbell, 2020; Pangrazio & Selwyn, 2021). Comprehending complex concepts such as sample distribution, inference, and ρ-values necessitates an understanding of statistical variation and measures of dispersion (Chance et al., 2000; Saldahna & Thompson, 2002). Notwithstanding reforms, statistics education remains challenging (Garfield et al., 2008). Students with diverse backgrounds and skill levels could exacerbate the difficulties associated with teaching statistics (Tishkovskaya & Lancaster, 2012). Students' grasp of advanced statistical issues is hampered by a misconception and poor understanding of measures of dispersion, which can lead to errors while solving statistical problems (Delmas & Liu, 2005). Kula and Kocer (2020) argued that students were taught the logic application while missing the essential steps in the direction of construction in learning statistical inference. Many students lack statistical reasoning as a result of the traditional emphasis on procedural and computational skills over conceptual understanding (Moore, 1997).

Past research showed that students with insufficient basic math skills are more likely to have arithmetic misconceptions, so teachers must notice students' errors and misconceptions and alter teaching instructions accordingly (Riccomini, 2005). Diagnostic assessment of student error patterns is an effective method to review students' responses and allow teachers to deliver meaningful instruction to every student (Horn et al., 2015). Ingram et al. (2015) assert that students' errors are the foundation for reasoning and arguments. Many studies in the past have focused on exploring students' errors in various mathematical topics such as algebraic manipulation, geometry, logarithm, calculus, etc. Similar study in the domain of statistical topics is however still uncommon. Clark et al. (2007) examined how grade 'A' students thought about the mean, standard deviation, and the Central Limit Theorem immediately after finishing an elementary statistics course. Cooper and Shore (2008) explored students' misconceptions on how to interpret histograms and stem-and-leaf plots to determine the centre and variability of data. Teachers' responses to questions about the normal distribution were examined by Bansilal (2014), who discovered that most teachers lack statistical reasoning in the context of the topic. Very few attentions have been devoted to spotting errors and misconceptions in statistical issues, particularly among high school students solving problems related to measure of dispersion. Error analysis can help teachers and educators focus on developing pedagogical strategies that can help students learn measures of dispersion more effectively. Teachers will be better able to help students reorganise their mathematical knowledge towards a more cohesive body of information if they have a firmer grasp of the most common errors (Swan, 2001).

In Malaysia, measures of dispersion is one of the statistical topics learned by grade 10 and 11 students. Under this topic, students will learn about range, interquartile range, quartile deviation, variance, and standard deviation for both grouped and ungrouped data. Unfortunately, statistics has become a challenging subject for students, and they are frequently confused, resulting in errors when handling statistical issues. Chan et al. (2014) claimed that high school students from Malaysia showed poor level of statistical reasoning ability and another research conducted by Zaidan et al. (2012) which analyzed 122 postgraduates reported the same findings. Hence, this study focused on gaining insights into the high school students' errors, and more specifically, to identify the kinds of conceptual errors, procedural errors and technical errors committed by students in solving problems involving measures of dispersion, according to the APOS theory.

LITERATURE REVIEW

Constructivism

Constructivism conceives that knowledge construction is a cognition process driven by mental self-regulation in each individual. Piaget (1968) asserts that individuals develop cognitive conflict when they encounter circumstances that differ from their understanding. This disruption in thought compels the individual to reconcile and settle his or her perturbation. Assimilation, accommodation, and equiliberation are constructivist components of knowledge development (Siegler, 2007). Assimilation is a process by which an individual incorporates new experience into existing mental models. Cognitive conflict occurs in mathematics learning when assimilation results from overgeneralization. Students recognise differences between some of their mathematical concepts and expert concepts held by teachers or in mathematics textbooks. When students identify and resolve conflicts, they gain new knowledge through the process of adaptation in response to perturbation (Cobb, 1994). von Glasersfeld (1989) stressed that knowledge can only be actively constructed by an individual. A teacher can only help students construct their own abstract mathematical knowledge by providing physical or mental models. Individuals learn mathematics not by memorising external rules, but rather by internal construction as a result of their own cognitive development (Piaget, 1968).

Many scholars believe that constructivism is related to student's mathematical misconception (Nesher, 1987). Misconception happens as a result of students acquiring knowledge rather than being taught by teachers (Confrey & Kazak, 2006). When a student recognises the limitations of a concept, he or she is more likely to resolve the misconceptions. However, this is not always the case, as some students never resolve their cognition conflicts. Confrey and Kazak (2006) argued that the act to demolish and rectify misconceptions will disrupt the good concepts that students have generalised to form the misconceptions. Instead, teachers should help students review and refine them. Thus, constructivism helps teachers to understand the pattern of thinking in students' formation of misconceptions which may not be easily revealed to teachers. Analysis of students' works will help to understand the erroneous patterns. This outlook underlined this study, which sought to study errors from the students' perspectives.

Dubinsky's APOS Theory

Dubinsky (1991) reckon that formation of concepts occurs first as an activity, then a process, an object and lastly a schema (APOS). It is an explanation of how individuals learn based on Piagetian epistemology (Arnon et al, 2014). When students learn a concept, they first perform actions that implement the concept, which is referred to as the action stage. At the process stage, the student sees the concept as a semi-external process or procedure. The student internalises the concept at the object stage, and at the schema stage, the student coordinates these individual mental constructs into a schema for a concept (Dubinsky & McDonald, 2002). Mathematics educators believe that an individual's early schema for any given concept may be disorganised, incomplete, and inconsistent. APOS-described maturation may occur as the individual experiences disequilibrium due to conflicts between expectations and outcomes, and engages in serious reflection (Mathews & Clark, 2003).

Action: At the action level, a student can perform rote procedures; in other words, students must act on the information they are given. For example, a student acting mathematically on σ= $\sqrt{\frac{\sum f_i(x_i-\bar{x})^2}{\sum f_i}}$ is expected to interpret and assign meaning to the expressions as descriptors of sequence of computations, e.g. to interpret $\frac{\sum f_i(x_i-\bar{x})^2}{\sum f_i}$ as sum of the product of f_i and the squared of the diffferences between x_i and $\bar{x}$, divided by sum of f_i. At this stage, a student is aware of and uses basic properties of operations, especially the distributive property.

Process: The stage of process occurs when an action is repeated until a student can transform the same type of action and create an internal mental construction without the external stimuli. At this stage, for example, the student is able to see the process as a whole and confident that manipulation of $\sigma = \sqrt{\frac{\sum f_i(x_i-\bar{x})^2}{\sum f_i}}$ is a process that delivers equivalent expressions.

Object: Object is a stage of understanding a mathematical concept in which the student can compose transformations to form cognitive objects through the application of activities at the action and process stages. For example, if a student can apply procedures of standard deviation as a whole and understands that transformations such as using the alternative method (i.e., $\sigma = \sqrt{\frac{\Sigma f x_i^2}{\Sigma f} - \bar{x}^2}$) can be performed on it, then a student is at an object level of the APOS theory. Mathematicians generally agree that condensing processes into objects is challenging (Sfard, 1991).

Schema: A schema is a set of processes and objects that occurs when a student is able to organise and comprehend new and unfamiliar mathematical concepts. For example, at a schema level, a student can connect the properties of standard deviation and is able to make conclusion and understand the concept based on the connections.

APOS is a description of the mental activities and mental constructions that students tend to make when formulating their understanding of mathematical concepts (Trigueros & Oktac, 2019). This study employed the APOS theory because it focuses on how students acquire mathematical knowledge and how to apply this knowledge to prescribe instructional strategies that optimise and stimulate mathematical concept learning (Salgado & Trigueros, 2015). Instead of merely reporting whether or not a student failed or passed a test, the APOS theory may pinpoint the precise challenges or knowledge that students demonstrated (Bansilal, 2014).

Conceptual and Procedural Knowledge

Conceptual knowledge is a product of devising new tactics or modify existing strategies to meet new challenges (Rittle-Johnson et al., 2016) In this web of knowledge, the links that connect the various nodes are just as important as the connections that exist between them (Groth & Bergner, 2006; Miller & Hudson, 2007). The connecting process in mathematics learning is formed when students can identify specific rules or procedures from more abstract concepts (Hiebert & Carpenter, 1992). Procedural knowledge, on the other hand, is the ability to answer a mathematical issue by going through a set of rule-based processes (Canobi, 2009; Sáenz, 2009). Skemp (1987) refers procedural knowledge as instrumental understanding, or rule knowledge. This can involve familiarity and grasp of the symbols used to construct algorithms, and the procedural rules needed to solve problems, without necessarily knowing the underlying mathematical concepts. Both the conceptual knowledge and procedural skills are intertwined in mathematics problem solving (Hurrell, 2021; Nesher, 1987).

In this study, conceptual, procedural, and technical errors made by students were categorised and examined. A conceptual error is an error in understanding the concept in which the understanding is not in accordance with the scientific definition as agrees generally by the experts in that field. In mathematics, this error happens when students fail to relate the initial concept with the newly given one (Russell et al., 2009) Conceptual errors occur when students attach their own interpretations to concepts they do not fully grasp (Chamundeswari, 2014). A lack of comprehension of the problem's concepts or a misunderstanding of the link between the problem's concepts are both examples of conceptual mistakes. For example, students use the improper formula for calculating the standard deviation.

Procedural error refers to errors in the process of executing algorithmic procedures, which include operations, algorithms, placements, and incorrect step, as well as missing steps in problem solving (Herholdt & Sapire, 2014; Raghubar et al., 2009; Siyepu, 2013). Procedural error occurs may be due to misgeneralization, where students generalise an existing concept wrongly (Andriani & Nurhasanah, 2021). Student lack of the basic knowledge in mathematical operations, and might have memorized formulas and properties without understanding how to apply them to problem solving. For example, students misinterpreted " $\Sigma(x_i - \bar{x})^2$ " as $[\Sigma(x_i - \bar{x})]^2$ in calculating standard deviation.

As for technical errors, it is possibly due to either a lack of attention to detail or a misinterpretation of an already-known method or notion that is being employed in a new context (Godden et al., 2013). These were errors resulting from carelessness and inattention during the execution of an appropriate or pertinent procedure. Students, for example, mistakenly took the value of variance as standard deviation.

RESEARCH DESIGN

The study involved eighty-five grade-11 students at a private school in Penang, Malaysia. All the students were taught about the calculation and simple application of measures of dispersion such as range, interquartile range, quartile deviation, variance, and standard deviation before participating in this study. Students' errors in measuring dispersion were investigated using a diagnostic test that provided no formula. The test used for the study consisted of a total of 11 sub-questions, which were carefully adapted from past-year questions from The United Examination Independent Chinese Secondary Schools Malaysia (UEC) senior middle level examinations and reference books with similar syllabus. The test instruments were validated by the four experts, using a five-point scale (1 = not related, to 5 = most relevant) (Almanasreh et al., 2019). Both the item-level content validity index (I-CVI) and the universal-CVI (UA-CVI), based on the proportion of items on an instrument that achieved a rating of 4 or 5 by all the content experts, were 1.0, which revealed a 100% agreement among the experts (Lynn, 1986; Polit & Beck, 2006). In the test instrument, content knowledge of measures of dispersion, such as range, interquartile range, quartile deviation and standard deviation in ungrouped and grouped data, were assessed.

Methodology

The data collection involves both quantitative and qualitative research methods. A diagnostic test with 11 subquestions was administered to a sample of 85 students. Students' errors on each test item were the focus of the study. The test had a time limit of 40 minutes for the students. Before the test, students were informed and had plenty of time to study. Prior to administering the test, a scoring system was devised. The researcher marked the scripts based on the scoring system. A second marker marked the scripts in order to avoid any inconsistencies. More than 95% agreement was achieved between the two markings. Upon completion of marking, students' answers to the diagnostic test were then analysed.

The data that had been gathered were analysed based on Miles and Huberman (1994) model of qualitative analysis. This model consists of three consecutive phases which involves data reduction, data display, and conclusion drawing and verification. In the data reduction phase, the written responses of each student to each item were examined and the correct or incorrect processes (errors) were categorised. Next, in the data

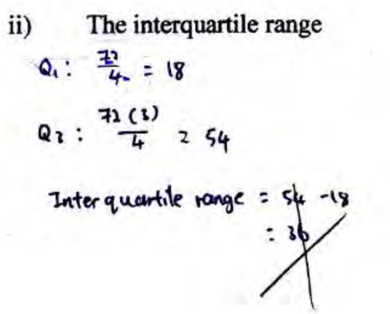

Figure 1. S84's answer to Item 1(ii) (Source: Authors)

display phase, the types of students' errors were analysed and classified based on the APOS theory as well as referring to the literature. In this study, the students' responses were categorised into: no error, blank answer, conceptual error, procedure error, and technical error. Finally in the last phase, a conclusion was derived and verified to ensure that the conclusion is suitable with the objective of the study.

RESULTS

The total score for the test was 20 marks. Each item was allocated one to three marks. For anonymity, the participants were randomly coded S1, S2 and so on up to S85. From the analysis, the students' marks ranged from zero to 18, with zero being the lowest and 18 being the highest. The mean score was 10.64, with a standard deviation of 5.173 among the 85 students tested. This showed that the students on average scored 53.2 percent of the total 20 marks, and the coefficient of variation of 48.6 percent indicated that the range of marks around the mean was quite large. Based on their exam scores, the students were subsequently categorised into three categories: high-achievers (35.3 percent, n=30), mediocre (31.8 percent, n=27), and low-achievers (32.9 percent, n=28). The errors detected in each item were analysed and reviewed.

Common Errors related to Range

In question 1(i), students had to deduce the distribution's range from the graph. This question was correctly answered by 90% of the top students, but only 35.7% of the bottom students were able to do so. Conceptual errors made by the low-achievers were largely due to incorrectly interpreting frequency as hours.

Common Errors related to Interquartile Range

In question 1(ii), students were required to find the interquartile range by subtracting the value of the third quartile (Q3) from the value of the first quartile (Q1). 63.3% of the high-achievers scored on the question, while just 7.1% of the low-achievers got it right. Approximately 42.9 percent of the low-achievers did not answer the question. There was no evidence that these students had engaged in the process of determining the range of distribution. High percentage of conceptual errors were found in mediocre and low-achievers alike.

There were 16 occurrences of procedural errors in items 1 (ii). The interquartile range formula was successfully applied by most of the students, and they were able to accurately identify the Q1 and Q3 positions. In other words, this shows a commitment at the process level. They however wrongly took the Q1 and Q3 positions instead of going further to find the Q1 and Q3 values. This response indicated a process-level understanding of interquartile range.

Figure 1 shows the responses of student S84 who managed to find the Q1 and Q3 position, and arriving at the wrong answer by subtracting the Q1 position from the Q3 position.

Another item related to interquartile range was question 2a(i) where students were required to find the interquartile range for the ungrouped data. There were 52 accurate answers from a total of 85 students. Moreover 83.3% of the high-achievers correctly answered the question, compared to only 25% of the low-achievers. Majority of the errors were conceptual where students had applied the interquartile range formula incorrectly and thus hindered further progress. They had very little knowledge about interquartile range and were considered as action level. There were 3 occurrences procedural errors in Item 2a(i). These students

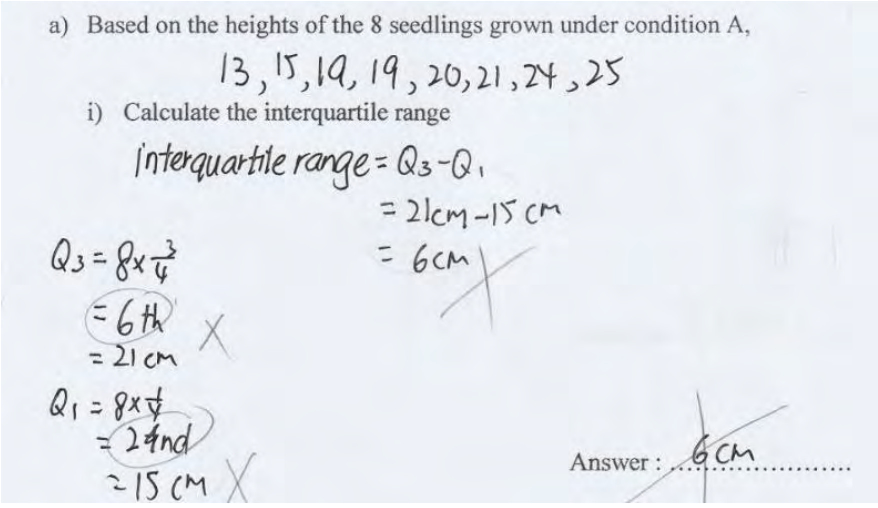

Figure 2. S63's answer to Item 2a(i) (Source: Authors)

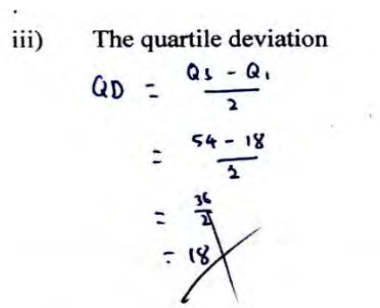

Figure 3. S84's answer to Item 1a(iii) (Source: Authors)

rearranged the ungrouped data in ascending order and applied the interquartile range formula correctly. This is indicative of a process-level engagement. They however substituted the wrong Q1 and Q3 values respectively which attributed to the wrong answer.

In **Figure 2**, student S63 did not add one to the total number of data set and thus attributed to the wrong Q1 and Q3 positions.

There were 2 technical errors in both questions 1(ii) and 2a(i). These were careless errors due to slips. Both students could have avoided the mistakes if they did the checking.

Common Errors related to Quartile Deviation

Question 1(iii) required students to work out the quartile deviation using the graph. Out of the 85 students, there were 25 correct answers. Students who answered item 1(ii) correctly did well on question 1(iii), because the quartile deviation was used to calculate the half-difference in the interquartile range. Similar to question 1(ii), 60% of the high-achievers got the answer correctly. As for the low-achievers, only 10.7% got the answer correctly, while 46.4% of them did not attempt to answer. Nearly a third of the mediocre and low-achievers made conceptual errors by applying unrelated formulas to find quartile difference. There was no evidence that the proper formula had been recognised in the responses, thus they were interpreted as pre-action. There were 15 students committed procedural errors in this question where the students applied the quartile deviation formula correctly which is indicative of a process-level engagement. It was, however, because of their incorrect use of the Q1 and Q3 positions that they came up with the inaccurate result.

In **Figure 3**, student S84 wrongly took the Q1 and Q3 position as the Q1 and Q3 values.

In item 1(iii), there were two technical errors which were mainly due to carelessness in calculation.

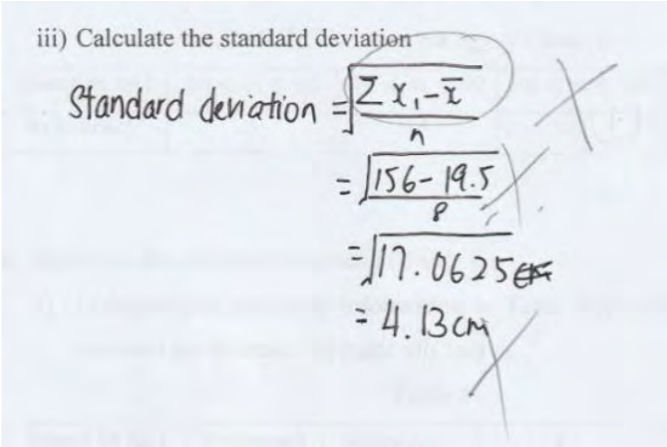

Figure 4. S63's answer to Item 2a(iii) (Source: Authors)

Common Errors related to Standard Deviation

Standard deviation is a measure of variability regardless of the distribution. The standard deviation of a sample estimates the variability of the population (Altman & Bland, 2005). In this study, students were required to find the standard deviation for both the ungrouped and grouped data in questions 2a(iii) and 3a(iii) respectively. The analysis shows that more that 90% of high-achievers were able to answer both questions correctly, while the percentage of the mediocre who scored in these two questions was in the range of 33%-56%. In contrast, only 7.1% the low-achievers got the answer of question 2a(iii) correctly while no low-achievers scored in question 3a(iii). In question 2a(iii) which was about the standard deviation of ungrouped data, majority of the low performers left the answer blank (i.e., 42.9%) or committed conceptual errors (ie 46.4%). These students were at the pre-action level and had not developed action conception of statistical standard deviation as they did not even use the formula for standard deviation in the solution. As for other conceptual errors, some students wrote the standard deviation formula incorrectly which hindered further progress. The illustrations of these students' works were evidence of action conception of standard deviation.

In **Figure 4**, student S63 wrote the formula of standard deviation for ungrouped data as $\sqrt{\frac{\Sigma x_i - \bar{x}}{n}}$ instead of $\sqrt{\frac{\Sigma (x_i - \bar{x})^2}{n}}$, which had attributed to the wrong substitution of data, and thus the wrong answer.

There were 5 occurrences of procedural errors in Item 2a(iii), committed by the mediocre and low-achievers. This group of students were at the process-level where they had applied the formula of standard deviation correctly but substituted the values of the data wrongly. This evidence of errors shows that students had not progressed further from the process level because they were confused and lack of the knowledge in mathematical symbols.

In **Figure 5**, the alternate formula was written by student S45 correctly with the correct substitution of "n" and "$\bar{x}$". The student had however taken the "$\Sigma x_i{}^2$" as the squared of the sum of the data (i.e., 156^2) instead of the sum of the squared of the data (i.e., $13^2 + 15^2 + 19^2 + 19^2 + 20^2 + 21^2 + 24^2 + 25^2$).

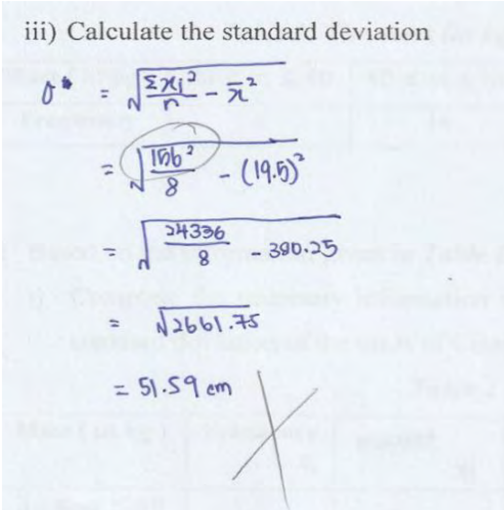

Figure 5. S45's answer to Item 2a(iii) (Source: Authors)

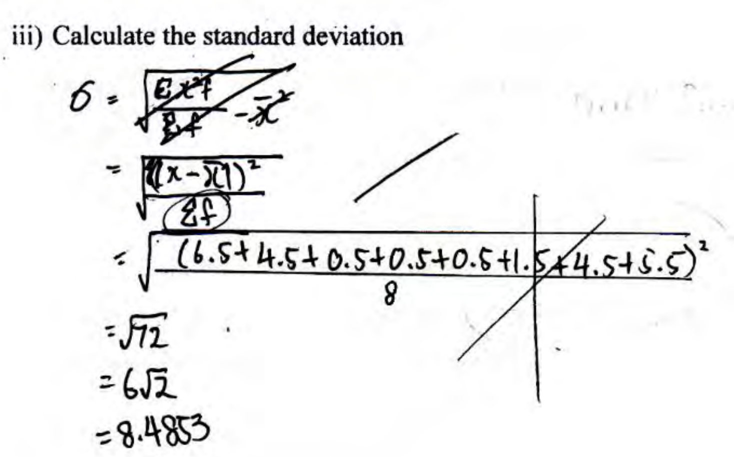

Figure 6. S56's answer to Item 2a(iii) (Source: Authors)

As for Student S56, he had applied the formula of standard deviation correctly which was an indication of action level but carried out the procedure wrongly (see **Figure 6**). He did not understand the algorithm of " $\Sigma(x_i - \bar{x})^2$ " and wrongly calculated the value as " $[\Sigma(x_i - \bar{x})]^2$ ".

As for the solution for standard deviation of grouped data in question 3a(iii), 90% of the high-achievers and 40.7% of the mediocres answered the question correctly. 53.6% of the low-achievers left the answer blank while 35.7% and 7.1% of them had committed conceptual and procedural errors respectively. Responses containing an inappropriate or irrelevant answer were taken as pre-action level. These students revealed poor understanding and lack of action conception of standard deviation.

In **Figure 7**, student S64 had wrongly applied the formula of standard deviation for grouped data.

Student S49 showed another example (see **Figure 8)** of conceptual error where the formula of standard deviation was written wrongly as $\sqrt{\frac{x_i^2 f_i}{F}} \times \bar{x}^2$ instead of $\sqrt{\frac{\Sigma f x_i^2}{\Sigma f} - \bar{x}^2}$. Student S49 was confused with the alternate method of calculating standard deviation.

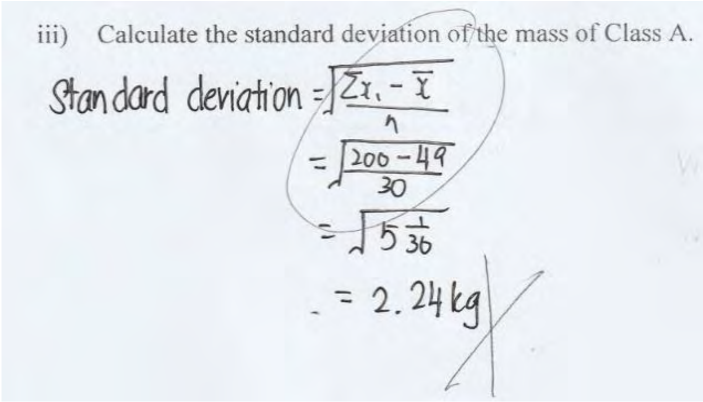

Figure 7. S64's answer to Item 3a(iii) (Source: Authors)

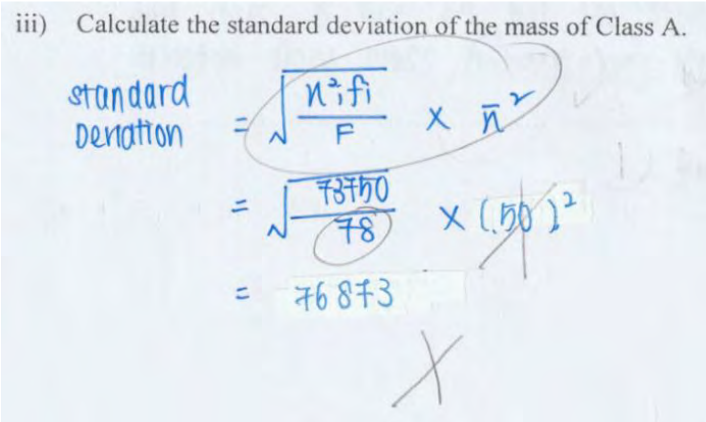

Figure 8. S49's answer to Item 3a(iii) (Source: Authors)

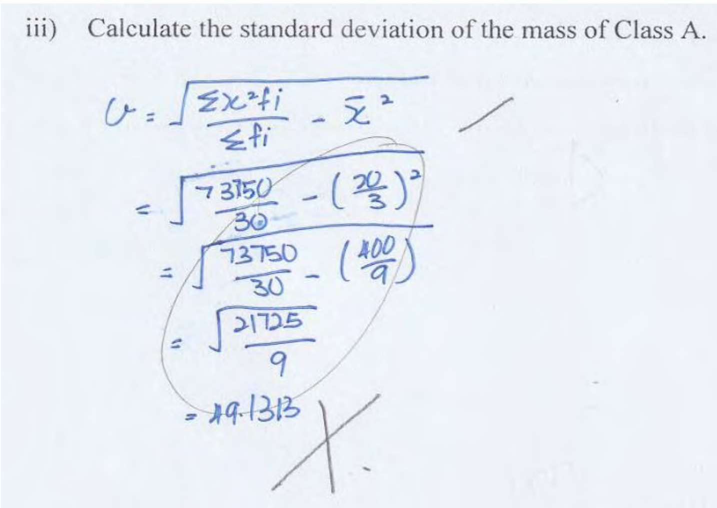

Figure 9. S57's answer to Item 3a(iii) (Source: Authors)

There were five students who committed procedural errors in Item 3a(iii). These students could be at the action stage by merely memorising the formula without much understanding.

Figure 9 showed an example of the procedural errors, where student S57 had written the alternative formula of standard deviation correctly and substituted the correct values of "$\Sigma f_i x^2$" and "Σf_i". Student S57 had however substituted the wrong value of "$\bar{x}$", which resulted in the wrong answer.

Another example of procedural errors is showed in **Figure 10**, where student S38 had wrongly applied the formula of standard deviation for ungrouped data (ie. $\sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n}}$) to calculate the standard deviation for

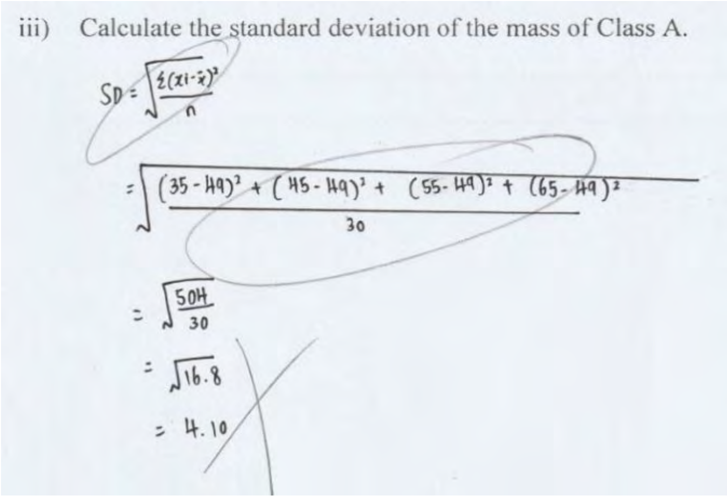

Figure 10. S38's answer to Item 3a(iii) (Source: Authors)

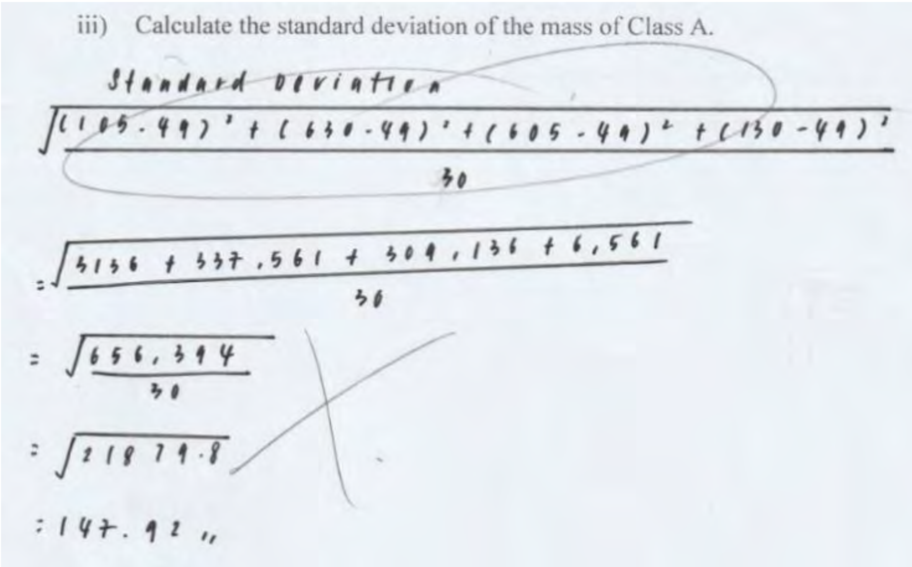

Figure 11. S35's answer to 3a(iii) (Source: Authors)

grouped data. The element of grouped frequency was not taken into consideration in the calculation, and thus resulted in the wrong answer.

As for student S35, she had mistakenly applied the formula of standard deviation for grouped data as $\sqrt{\frac{\Sigma(fx_i-\bar{x})^2}{n}}$ instead of $\sqrt{\frac{\Sigma f(x_i-\bar{x})^2}{\Sigma f}}$. Student S35 was confused with the mathematical symbols and formula (see **Figure 11**).

There were nine technical errors in Item 3a(iii). Majority of these students had applied the standard deviation formula for grouped data correctly. These students were at the process stage as they knew the procedure of finding the standard deviation. The substitution of the values was correctly done except for the value of $\bar{x}$, which was wrongly calculated in Item 3a(ii), due to the wrong mid values or miscalculation in the "Σfx".The wrong value of mean in item 3a(ii) was then carried down in item 3a(iii).

Another technical error was showed in **Figure 12**, where student S76 had applied the formula of standard deviation correctly, with the correct substitution of values. The student had however forgotten to find the square root of the value, leaving the answer as the value of variance instead of standard deviation. She is at the object stage as this is a careless mistake which can be avoided if the student had done the checking.

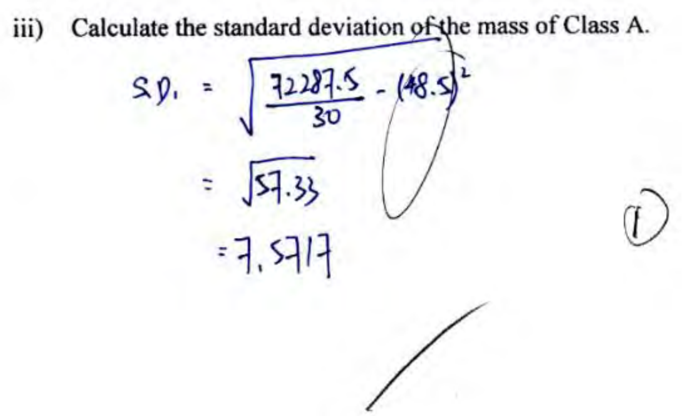

Figure 12. S76's answer to 3a(iii) (Source: Authors)

Common Errors related to Reasoning

Both items 2(b) and 3(b) tested students' reasoning skills related to variation of data. Out of the 85 students, there was no correct answer in question 2(b) while two in question 3(b). There was no procedural error as these questions do not involve algorithm calculation. There was a higher percentage of technical errors in both questions due to incomplete answers. These students had answered the question by merely giving comparison between the mean and standard deviation of both classes, without stating the reasons. Majority of the students were at the pre-action or action level with limited reasoning skills. The percentage of high-achievers who committed conceptual errors and technical errors (i.e., incomplete answers) in these two questions were high. In contrast, majority of the low-achievers had either left the answer blank or committed conceptual errors. As for the mediocres, about 40% of them did not attempt the questions while other errors committed were majority conceptual.

DISCUSSION

The study found that many students struggle through the process of statistical problem solving and made errors because they applied past knowledge (conceptual errors) that was not relevant to the task at hand (procedural errors). APOS theory asserts that students are engaging at different levels with the concepts and therefore they would need different interventions in order to help them develop the necessary mental constructions. The study showed that there were large numbers of responses at the pre-action and action levels. It is clear from the pre-action responses that a huge percentage of students have very rudimentary understanding of the task at hand. The basic properties of measures of dispersion, as well as a simple computation, might need to be reviewed with these pupils. Students who were at the action level could not internalise procedures into processes. They need more understanding with concepts or even repeated instruction in order to move on beyond the action level. This study revealed that standard deviation is a concept that is difficult for most students, particularly those who performed poorly. Low and mediocre achievers had not moved beyond action conceptions of standard deviation in more than 90% of cases. For example, only two out of 85 students in questions 2(b) and 3(b) were able to demonstrate relational grasp of how to calculate a standard deviation and proceeded to engage in rich objection of this notion. To help students develop a deeper knowledge of relationships, educators should implement lessons and activities that emphasise the concept of spread (i.e., $x_i - \bar{x}$).

The finding of the study showed that students were lack of the conceptual understanding of measures of dispersion such as properties, axioms, and other structural features. There was an average of 20.5 out of the total 85 students committed conceptual errors in each item, which was the most common error found in the study. Almost every question had the highest occurrence of conceptual errors as compared to other types of errors. Students demonstrated conceptual errors that originated from the poor understanding of some of the statistical concepts related to measures of dispersion. This might be because the majority of mathematics instructors rarely elucidated the concepts underlying statistical terms. This suggests that these students did not understand some of the concepts of measures of dispersion and thus applied an irrelevant formula to solve the problem. An average of 15.5 students, mainly from the low achievers, submitted blank reponses to each item tested. This revealed that they did not make sense of the question at all or found the items too difficult to attempt. This situation was particularly prominent in item 2b and 3b, where students were tested on their statistical reasoning skills. Although some students were able to determine the correct standard deviation value for questions 2a(iii) and 3a(iii), their answers to questions 2b and 3b demonstrated a lack of understanding of the concept. Students might just applied the algorithm correctly by memorising the formulas.

The main procedural errors revealed were students' failure to manipulate the calculation of standard deviation correctly. Some were able to apply the correct formula but substitute the wrong values due to lack of understanding of the symbols. This suggests that students were weak at the algebraic manipulation and confused over the representation of symbols. Students had difficulty recognizing the differences between symbols and thus wrong interpretation or improper use of symbols. Procedural methods of solving mathematical problems might had been drilled into students during their time in school. Clark et al. (2007) discovered that students were unable to move from a process to an object perception of standard deviation under the traditional education which emphasized procedural method. For example, the calculations of standard deviation are somewhat complex and the risk of making mistakes is high. The learning of the first method (ie $\sigma = \sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n}}$ for ungrouped data; $\sigma = \sqrt{\frac{\Sigma f(x_i-\bar{x})^2}{\Sigma f}}$ for grouped data) will give us insight into how standard deviation really works, which is a procedure of getting the squared root to the average squared differences between each data and the mean. However, both the alternative methods (i.e., $\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \bar{x}^2}$) for ungrouped data; $\sigma = \sqrt{\frac{\Sigma f x_i^2}{\Sigma f} - \bar{x}^2}$ for grouped data) do not have any meaning in this procedure, and students were unable to understand how the concept of standard deviation relates to the formulas. Algorithms that do not carry with them the process probably promote instrumental understanding and hinder relational understanding (Clark et al., 2007). Students are further confused by the variety of the formulas and thus hinder them from learning statistics. This could be one of the reasons students chose to use rote learning strategy in calculating standard deviation, because they simply could not understand what they were doing. Observational skills and reflective practice among teachers are essential if they are to gain a deeper understanding of their students' learning challenges and make appropriate adjustments to their lesson plans. Teachers' ability to observe and reflect on students' errors and misconceptions help them better understand students' difficulties in statistics learning.

CONCLUSION

The finding suggests that students encounter substantial challenges in the learning of measures of dispersion particularly related to conceptual and procedural knowledge. Analysis of students' thinking, understanding, and misconceptions can be gained through error analysis (Busi & Jacobbe, 2014). The early detection of error patterns through error analysis is an effective way to help students with the learning of mathematics and reduce the gap between struggling and performing students (Riccomini, 2005). As a result, teachers must provide students with instruction that addresses their difficulties by examining the incorrect concepts and/or procedures of students. Strategies may include the design of learning activities such as classroom discussion that encourage students to investigate and discuss their errors. Many mathematics and statistics teachers may lack confidence, content knowledge, and pedagogical content knowledge (Lee & Harrison, 2021; Lovett & Lee, 2018). It is possible for teachers to enhance their students' statistical abilities by identifying and resolving the conceptual and procedural challenges that students face. The finding of the study will be useful in devising approaches that will help with the identification and minimization of errors in measures of dispersion. It is helpful to provide guidance on appropriate teaching strategies that will remedy and rectify such errors. Future research could focus on teaching pedagogies and remediation to overcome students' difficulties in the same area.

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