Is "Fruit Salad Algebra" Still a Favorite Menu in Introducing Algebra in Schools?

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Abstract: Students' difficulties in algebra are generally caused by the use of algebraic notation, the meaning of letters, as well as the types of relationships or methods used. Therefore, interpreting letters in algebra is one of the critical points in the transition from arithmetic to algebra. However, many students have misconceptions in interpreting letters in algebra. One of the most well-known misconceptions in interpreting letters in algebra is the "letter as object" where the term "fruit salad algebra" is sometimes used to name this misconception. The aim of this research is to explore further information about this misconception, its causes, and alternative solution. This study used the case study method with 35 grade 7th junior high school students as respondents. Data collection was carried out through written tests, interviews, and documentation. The results showed that students interpreted letters in algebra in various ways but tended to lead to "letters as objects", the "fruit salad algebra" approach is found in several textbooks and is rooted in the teaching culture, and there is a gap between the development of operations on letters as unknown and the idea of equality within equation.

Keywords: algebraic notation, letter, misconception, symbols, variable.

INTRODUCTION

Almost all children know algebra after arithmetic in learning mathematics at school. This happens for historical reasons, algebra was created long after the discovery of arithmetic (Carraher et al., 2006). Although, arithmetic is considered a prerequisite for algebra because the basis for algebra manipulation uses four arithmetic operations and maintains its meaning, but algebra cannot be considered as an extension of arithmetic because the problem-solving approach is different (Dettori, Garuti & Lemut, 2006). Knowledge of arithmetic rules that have worked well, to some point does not apply. For example, the equation of the form of $Ax + B = Cx + D$ does not apply to arithmetic ideas because it involves operations with 'unknown' which is outside the arithmetic domain. Herscovics & Linchevski (1994) revealed the cognitive gap between arithmetic and algebra, which can be characterized as the inability of students to work spontaneously with or in unknown. This inability is because students see literal symbols as static positions and operational aspects can only be understood when letters are replaced by numbers (Linchevski &

Herscovics, 1996: 41). To operate on unknown or in general quantities in general (for example variables or parameters), one must think analytically, that is one must consider the uncertain amount as if they are something known or as if it is a specific amount. Student understanding of arithmetic is the prerequisite ability but is not enough to deliver students to understand algebra. Therefore, a smoother bridge is needed to support the transition of students from arithmetic to algebra.

The transition from arithmetic thinking to algebraic thought has become a topic of substantial interest in mathematical education research throughout the world for more than the last two decades (for example, Anniban, et al., 2014; Ann van Amerom, 2003; Herscovics & Kieran, 1980; Kiziltoprak &Kose, 2017; Malisani & Spagnolo, 2008; Onal, 2023; Panorkou, 2013). The main difficulty mentioned in their report is a significant difference between arithmetic and algebra. Arithmetic is a systematic process in mathematics about addition, reduction, multiplication, and division in its primitive form (Akkan, et al., 2011; Mason, 1996; NCTM, 1991). In general, many researchers agree that arithmetic is a procedural and concrete system that produces numerical answers in certain numbers, manipulation of fixed numbers, letters are measurement labels or abbreviations of an object, symbolic expressions represent the process, and the same sign as a signal to calculate (Christou & Vosniadou, 2012; Kieran, 1992; Linchevski & Herscovics, 1996; Stacey and MacGregor, 2000; Ann van Amerom, 2002). Therefore, thinking arithmetic is done with a known quantity.

Meanwhile, algebra requires reasoning about unknown or variable amounts and recognizes differences between certain and general situations. There are differences regarding the interpretation of letters, symbols, expressions, and concepts of equations. For example, in arithmetic letters are usually abbreviations or units, while algebra letters are standing for variable or unknown. Nathan &Koellner (2007) states that algebra has two core concepts, namely equations, and variables. Usiskin, (1999) states that an understanding of "letters" (variables) and operations must be owned by students in studying school algebra. Students tend to believe that the variables are always in the form of letters and that letters always represent numbers. In fact, the values taken by variables are not always numbers. This is because variables have many definitions, referrals, and symbols. The use of variables is determined by or related to the conception of algebra and correlates with different interests. Variables can be interpreted as the generalization of fundamental patterns in mathematical modeling, unknown or constants, arguments (i.e., abbreviated domain values of function), or parameters (i.e., abbreviations of numbers that depend on other numbers). According to Radford (2006) using letters is not the same as algebra because not all symbolizations are algebra, as well as all patterns of pattern lead to algebraic thoughts. Therefore, algebraic thinking can be defined as an approach to quantitative situations that emphasize general relational aspects of tools that are not always symbolized by letters, but which ultimately can be used as cognitive support to introduce and maintain a more traditional school algebraic discourse (Barerjee, 2011; Kieran, 2004).

The body of mathematical knowledge is seen as a result of a long historical construction process, formulation, and clarification so that it cannot be fully understood through its formal dimensions

(Gallardo, 2002). According to Booker & Windsor (2010), algebra does not begin with symbolic reasoning but has been separated into three phases, namely: 1) Rhetorical Algebra which involves the use of words and sentences, 2) Algebra The form of syncopation in which words and actions are expressed in form The abbreviation means the words and sentences used previously, 3) Symbolic algebra, namely modern conception involving special symbols, functions, and structures. It seems that many students follow the same sequence of developments as the historical development of algebra. Therefore, the historical foundation needs to be discussed as a basis for understanding the development of students in algebra and compiling the right steps in helping students according to the development stage.

Historical-critical analysis provides facilities to construct teaching and learning sequences in which students and teachers are involved in reflecting progress in theoretical investigation. The concept and use of symbolization marks the difference between arithmetic thinking and algebra. Viète's work entitled Analitic Art and Development of Experimental Teaching Sequences according to Gallardo (2002A) marked the existence of didactic discipline in the algebraic historical evolution line in connection with the symbolic representation of 'Unknown' and the possibility of operating on "unknown". Ely & Adams (2012) has given a good explanation of the development of variable ideas, which starts from Unknown and Placeholder. Unknowns had been used for thousands of years before placeholders appeared. The beginning of the symbolic algebra was marked by the emergence of Placeholder in 1591 and opened the way for the development of the complete ideas of variables in 1637. When representing and manipulating unknown, the Babylonians and Greece generally used words rather than symbols such as those carried out by Islamic and Indian mathematicians In the Middle Ages. The quantity of unknown is usually referred to as "thing" or "number," or "root". The use of symbols as a placeholder, requires changes in thinking that allows symbols to refer to more common types of objects. This is what allows modern mathematicians to place letters as a substitute for quantity. Manipulation carried out on these letters will work the same way for certain numbers that can represent them. FranciscusVieta (François Viète) in his work in 1591 proposed a new practice representing the values given in problems with letters that could represent them. Therefore, it can be concluded that the general idea in the heart of the placeholder (a letter can represent a set of unknowns quantity) and covariational reasoning (how to represent and measure the way of change of one quantity to another) are two important things for the development of variable ideas.

METHOD

Research Design

This research uses qualitative research with a case study approach because it is considered appropriate to the purpose of this study, namely to explore students' misunderstandings through an in-depth approach to find out the causes of these misunderstandings in order to prevent, reduce or correct misconceptions about interpreting letters in algebra. There are three research questions used, namely "how do students in grade 7 junior high school understand the letters in algebra?"; "what are the misconceptions experienced by grade 7 junior high school students in understanding letters in algebra?"; and "what kind of learning experiences do grade 7 junior high school students have in understanding letters in algebra?".

Setting and Participants

This study took place at a junior high school in Bandung, West Java, Indonesia. Purposive sampling technique was used to determine participants in this study because of its suitability in advancing research objectives. Grade 7 students of junior high school semester 1 were chosen because they are in the transition stage from arithmetic to algebra where students at this level have just been introduced to algebraic material so they are prone to misconceptions. There were 35 students who agreed to become participants in this study.

Instruments

Data collection techniques used were written tests, interviews, and documentation. Written tests are used to identify students' misconceptions. The written test questions consist of five description questions as shown in Table 1.

No	Question Description
1	Ani has a basket of fruit in which there are 4 apples and 3 bananas so that all of Ani's fruits are 7. Can the sentence be written as "$4a + 3b = 7ab$"?
2	Based on question number 1, what is the meaning of $4a$, $3b$, and $7ab$?
3	Budi bought 1 box containing several strawberries, 2 boxes containing several pears and 5 melons. Can the sentence be written as "S+2P+5M"?
4	Based on question number 2, can S+2P+5M also be written as □+2△ +5⬠ ?
5	The price of a watermelon is three times the price of a mango. While the price of 3 watermelons and 2 mangoes is Rp. 55,000. How much does a watermelon cost?

Table 1: Written Test Guidelines

The interview was conducted after the students' responses in the written test were analyzed. Students who became informants in interviews were selected based on written test answers that were considered to be able to provide relevant information. The interview was conducted using a semi-structured interview guide with three standard questions, namely: (a) What do you think about the letters? Why?; (b) Does an algebraic form have to have letters in it? Why?; and (c) How is the process of teaching and learning algebra in class? While other questions were developed based on the responses given by students during the interview process. Each question asked aims to confirm students' answers, the conceptions they have, and their learning experiences. The duration of the interview was between 10 and 15 minutes. Document analysis was carried out by analyzing the mathematics textbook used in the teaching-learning process in class and reading interview transcripts. During the process of collecting, storing and analyzing data, researchers maintain the confidentiality of sources and anonymity.

RESULTS

Student performance in completing the written test is described based on each question number as follows.

Question 1

Question number 1 is a question specifically designed to see how students in class 7th junior high school's translation understanding of letters as symbols (variables) in algebra. Understanding translation according to Sudjana (1995) is the ability of students to understand an idea that is expressed in another way from a known or previously known original statement or sentence in terms of translating sentences in word problems in mathematical form, for example mentioning known or asked variables, the ability to translate symbols, as well as the ability to translate into symbolic forms and vice versa. Usiskin (1997) states that the adaptation of arithmetic thinking to algebraic thinking can be done by algebraic representation of a variable where numbers can be represented by words, blank marks such as "____" or "……", boxes, question marks, or letters. Chick (2009) argues that it is very important for students to understand that letters represent numbers in algebra either as common numbers, unknown numbers, or variables.

There were 23 students in this study who agreed that "$4a + 3b = 7ab$" is a representation of the sentence "Ani has a fruit basket in which there are 4 apples and 3 bananas so that all of Ani's fruits are 7". One of the students interviewed read the equation "$4a + 3b = 7ab$" as "four apples plus three bananas is 7 apples and bananas". This indicates a misunderstanding known as the "letter-as-object misunderstanding" which is explained by Küchemann (1981) as a misunderstanding whereby students view letters as objects derived from abbreviated words such as a for apple rather than as representing a number.

Meanwhile, 12 other students did not agree that $7ab$ was the result of $4a + 3b$ because they could not add apples and bananas, but they agreed that $4a + 3b$ was the sum. Letters as a misunderstanding of objects according to Chick (2009) can also be strengthened by applying letters in formulas such as $L = p \times l$, where L is the area. According to Herscovics & Kieran, (1980) "non-conservator" students, namely students who do not realize that an unknown value does not depend on the letters used, have problems performing arithmetic operations on algebraic expressions. The difficulty they have is thinking of letters representing numbers. This shows that some students failed to develop the symbolism and notation used in the equation.

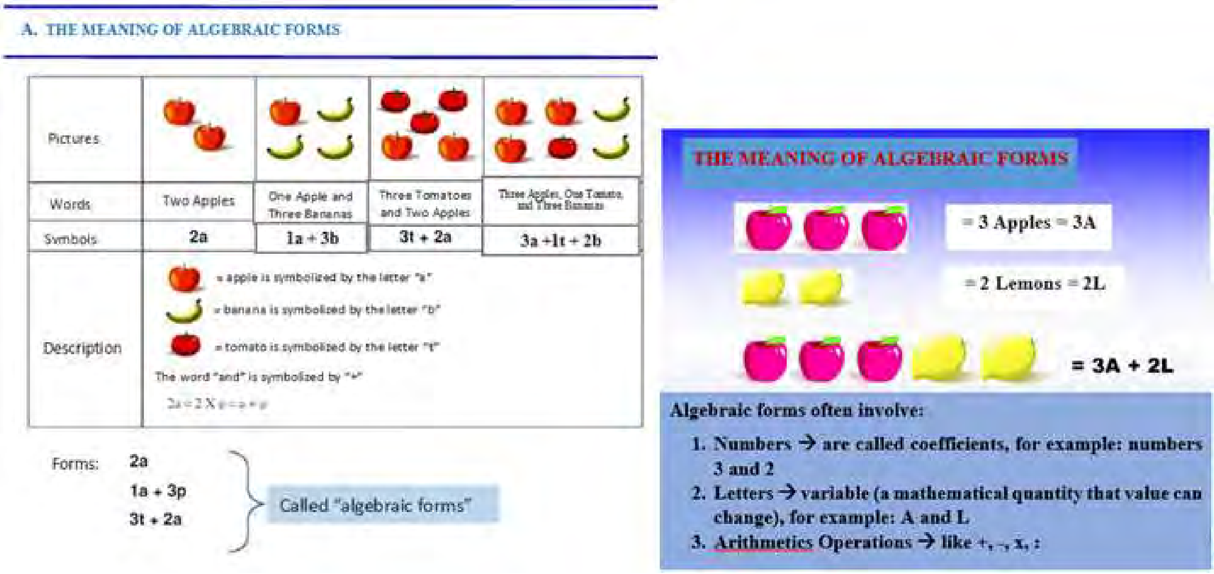


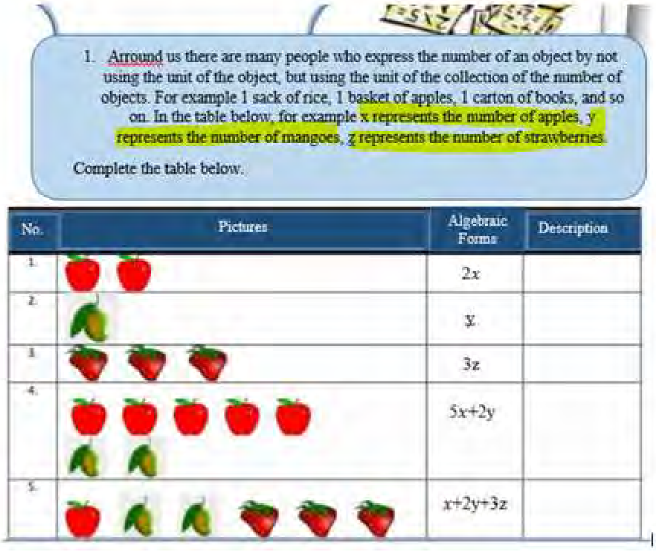
Figure 1: The "Fruit Salad Algebra " where the Letters are Object Name Abbreviation

Based on the results of an analysis of the mathematics textbooks used by students during the teaching and learning process in the classroom, there is evidence that the “fruit salad algebra” approach is present in textbooks as shown in Figure 1. If textbooks are considered as authorities in mathematics, then this evidence gives legitimacy to the meaning of variables in algebra, namely "letters stand for objects" where letters stand for named objects, such as the letter "a" for apple, "L" for lemon, "b" for banana, and so on. This strategy according to Thomas & Tall (2004) can provide short-term success, but is misleading in the future. Like adding $3a + 2b$ with $4a + 3b$ to get $7a + 5b$ by imagining apples and bananas put together, then how to explain an expression like $3ab$ being used? Is it three apples and a banana? Definitely not 3 times applying bananas. This shows that many fail to give a meaning that is in accordance with the meaning of mathematics that should be.

Question 2

All students in this study stated that $4a$, $3b$, and $7ab$ were "four apples", "three bananas" and "seven apples and bananas" respectively. The shift from arithmetic in everyday situations to synthetic arithmetic and algebraic symbolism involves more complicated expressions that cause difficult transitions for many students. This transition is made more difficult by the change in the meaning of the symbolism. In arithmetic, the expression $4 + 3$ is an operational rule in the sense that it has a calculation procedure that shows the result. One of the difficulties found in the context of problem number 2 is that $4a$ in algebra does not represent 4 apples, but four times the number of unknowns. Algebraic complexity is related to syntactic inconsistencies in arithmetic, for example: the invisible multiplication sign such as $4a$ which is $4 \times a$, a variable can simultaneously represent many numbers, letters can be chosen freely, the equals sign as an equality relationship, different concepts and rules exist in arithmetic and algebra (Breiteig & Grevholm, 2006). In algebra, however, the symbol $4 \times a$ is the first expression for the evaluation process, which cannot be executed until the value of a is known. This is one of the hardest things for some seventh graders to deal with, which is "but how can I multiply 4 by a, when I don't know what a is?". The difficulty of imagining algebraic expressions as solutions to problems has been described as a category closure misconception by Thomas & Tall (2004).

Student textbooks not only display the "fruit salad algebra" method which gives the meaning of a letter or variable in algebra which stands for the named object, such as the letter "a" for apple, but also to refer to a quantity or represent a certain number (value) such as shown in Figure 2.


1. Arround us there are many people who express the number of an object by not using the unit of the object, but using the unit of the collection of the number of objects. For example 1 sack of rice, 1 basket of apples, 1 carton of books, and so on. In the table below, for example x represents the number of apples, y represents the number of mangoes, z represents the number of strawberries.

Complete the table below.

No.	Pictures	Algebraic Forms	Description
1.		2x	
2.		y	
3.		3z	
4.		5x+2y	
5.		x+2y+3z	

Figure 2: The "Fruit Salad Algebra " where Letters Represent Numbers

There is a practice question page in the textbook that begins with examples of problems and their solutions. In this example, a contextual problem is given which reads "Around us many people express the number of an object by not using the units of the object, but using the unit of the collection of the number of objects. For example 1 sack of rice, 1 basket of apples, 1 box of books, and so on. In the table below, for example x represents the number of apples, y represents the number of mangoes, z represents the number of strawberries. Complete the table below."

After the problem description is given, there is a table consisting of four columns where the first column shows the serial number, the second column contains contextual images, the third column contains the algebraic form of the image shown in the second column, and the fourth column contains information that describes the image and algebraic form in columns two and three. There is an ambiguous concept between the statements given, for example in the introductory sentence it is stated that "...declare the amount of an object with not the unit of the object, but using the unit of the group of the number of objects...for example 1 basket of apples,...", then proceed with the statement which starts the problem, namely "...x represents the number of apples..", while the images and algebraic forms displayed in the table do not reflect the given statement. It can be seen that the picture presented is in the form of fruit in seed quantity, not in "basket" size quantity as stated in the initial statement. Also, if "...x represents the number of apples...", wouldn't it make more sense if the algebraic form shown was "$x = 2$" instead of "$2x$"?. It can be seen that this "fruit salad algebra" approach provides easy access to students' misconceptions.

Question 3

The third question aims to evaluate the effectiveness of introducing letters as"unknown" in a way that often appears in students' math books on algebraic forms material in grade 7 junior high school. In the context of question number three, this adopts the problems in textbooks that are often taught to students as shown by Figure 3.

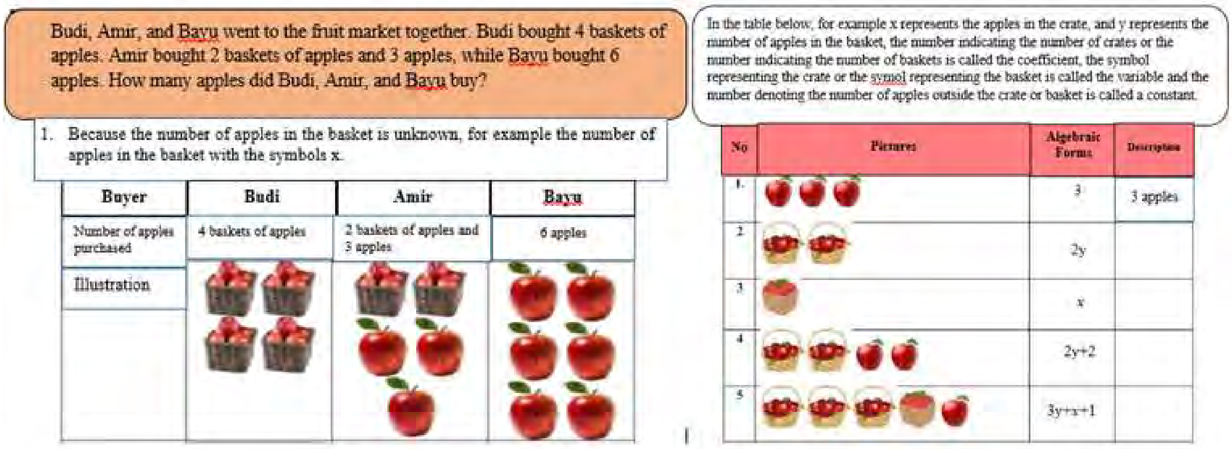


Figure 3: The "Fruit Salad Algebra " where Letters are Unknown

The algebraic representation of the sentence "Budi bought 1 box containing several strawberries, 2 boxes containing several pears and 5 melons" namely "$S + 2P + 5$" where the number of strawberries and pears is unknown, while the number of melons is known so letters are generally used, for example "S" and "P", to represent unknown values. However, the number of "several" as "unknown values" and the number of "units" refer to known values is a logically ambiguous concept when it comes to the concept of equality. The word "some" used in the context of the amount of fruit in a basket, box or bag does not have an equivalent standard unit reference, in contrast to the context referring to "price" which can be used through a "barter" approach which has been reported in Streefland's experimental research (1995) shows the steps in the conceptual development of variables. In this case the letters "S" and "P" represent the fruit itself, not its price value.

One example of the difficulty in the ambiguity of the unknown concept is through the "number of fruit in a basket" approach, for example in the algebraic form in number 4 in Figure 3 namely "$2y + 2$" refers to "the number of apples in two baskets and 2 apples". Thus, if we want to find out the number of apples (each fruit) in each basket using the algebraic form, we get: $2y = -2$ then $y = -1$. Even though y is defined as "the number of apples in the basket", this makes no sense. This can make the letters in algebra (variables) meaningless so that it can cause students to base their interpretation of algebraic letters and expressions on intuition and guesswork, on analogies with other symbol systems they know, or on false foundations made by misleading teaching materials. They are often unaware of the general consistency of mathematical notation and the power it exerts. Their misinterpretation causes difficulties in understanding algebra and can persist for several years if not recognized and corrected.

This ambiguity is proven to cause misconceptions among most students in this study, namely 17 students believe that letters represent objects (abbreviations of object names), such as the letter "a" which stands for the word "apple" represents the apple itself either in a basket or standing alone (outside the basket). So according to these students $S + 2P + 5M$ is a correct or reasonable representation of "1 box of strawberries, 2 boxes of pears and 5 melons".

Question 4

The fourth question aims to explore students' misconceptions that tend to believe that variables are always letters. All students in this study agreed that the symbols □, △, and ⬠ are not variables because they are not letters, so $S + 2P + 5M$ cannot be written as □+2△+5⬠. They are also not used to working with symbols like $3 + 5 =$△ or $5 +$⬠$=15$ while studying arithmetic in elementary school. This view is supported by many textbooks and reinforced by many educators as shown in Figure 1 where algebraic forms are described as "combinations of letters and numbers separated by arithmetic operations" such as $2a$, $1a + 3p$, $3t + 2a$, or $3A + 2L$. It also gives a definition that "numbers in algebraic form are called coefficients" with an example of the coefficients in $3A + 2L$ being 3 and 2. In addition, it also states that letters in algebra are variables, namely "a quantity in mathematics whose value can change" with examples of letters A and L is a variable of $3A + 2L$.

Based on the definitions and examples of these variables, it can be concluded that variables are "letters that represent numbers". Though the values that variables take aren't always numbers, even in high school math. Usiskin (1999) clarifies this concept by mentioning the variety of variables in several areas of mathematics such as: variables in geometry often represent points where the variables A, B, and C are used when we write "if $AB = BC$, then ΔABC is isosceles"; in logic, the variables p and q often represent propositions; in real analysis, the variable f often represents a function; in linear algebra, the variable A can represent a matrix or the variable v for vectors;

and in higher algebra, the variable can represent an operation where the variable is not necessary represented by letters, because variables have many definitions, references, and symbols.

Thus, if you refer to the definitions and examples of algebraic forms in the textbook, $5 + x = 8$ is usually considered algebraic, while $5 + \_\_ = 8$; $4 + \Delta = 7$; $3 + ? = 6$ is not considered algebraic even though the blank, triangle and question mark in this context want a solution to an equation that is logically equivalent to x. The definitions given in the textbooks seem to try to fit the notion of variables into a single conception by simplifying the ideas and in turn actually change the goal of algebra.

Question 5

The problem in the fifth item adopts the concept of the Chinese barter problem that inspired Streefland (1995) as a naturally and historically formed starting point for the teaching of linear equations, claimed by Streefland to represent the steps in the conceptual development of variables. Streefland (1995) found in his teaching experiments that literal symbol meanings are important constituents of students' progressive formalization. Furthermore, Streefland reported that students need to be aware of the changes in meaning experienced by letters because in this way the level of students' mathematical thinking can develop. In the concept of this barter problem, according to Ann van Amerom (2003) students are required to be able to compose not only the form of an equation (from the amount of fruit to money) but also the meaning of the unknown (from the object related to the quality of the related object). These considerations show the steps in the conceptual development of variables.

The first step that must be taken by students to solve problem number five is to represent the sentence "the price of a watermelon is three times the price of a mango" and "the price of 3 watermelons and 2 mangoes is Rp. 55,000" into algebraic form. There were 4 students who failed at this stage representing "the price of a watermelon is three times the price of a mango", as shown in Figure 4 below.

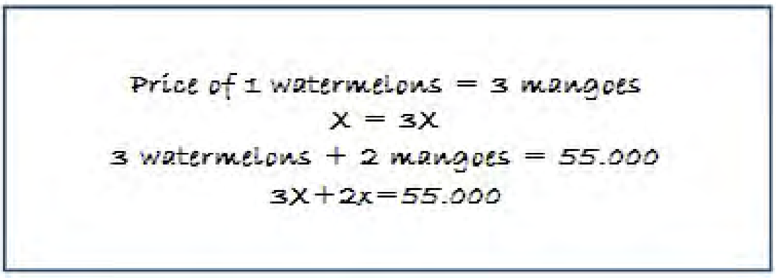


Figure 4: Example of Student Answers in Question 5

As many as 27 other students have been able to make the correct representation "the price of a watermelon is three times the price of a mango" but failed in the next process, namely substituting the first equation (i.e., $s = 3m$ where s refers to the price of a watermelon and m refers to the price of a mango) to the second equation which is $3s + 2m = 55{,}000$ becomes $3(3m) + 2m = 55{,}000$. Figure 5 shows an example of student failure in solving problem number 5.

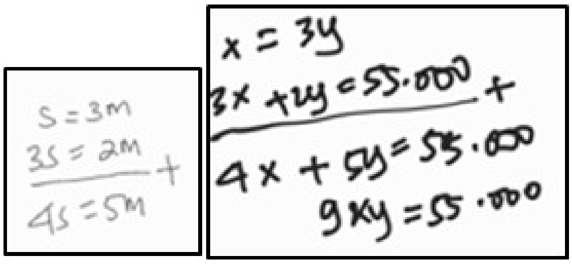


Figure 5: Examples of Student Misconceptions in Question 5

Student errors in solving question number 5 support Ann van Amerom's (2003) statement that if students continue to interpret letters as objects or labels or abbreviations, not variables, they experience difficulties both in dynamic (procedural) conceptions and in static conceptions of ideas algebra known as 'reversal error' when converting verbal descriptions into formulas for example to translate word problems into equations.

DISCUSSION

Algebraic abstraction is one of the biggest problems for students in learning mathematics at the high school and college levels. Students' difficulties in algebra are generally caused by the use of algebraic notation, the meaning of letters and variables as well as the types of relationships or methods used. The generality of algebraic ideas makes semantics weak so that there are deadlocks experienced by students regarding the use of algebraic notation. During the process of learning mathematics in elementary schools, those who had so far seen arithmetic symbolism as a representation of processes that could be carried out by arithmetic procedures suddenly discovered that this "universal law" did not apply. For example, there are 2 apples and 3 bananas on the table. They think of $2a + 3b$ as "2 apples and 3 bananas", then think of it as "5 apples and bananas" and write 5ab what makes sense to them. However, this does not apply to algebra. Expressions with letters cannot be worked out unless the values are known and if the values are known why use algebra? According to algebra students is an unnecessary and irrelevant difficulty.

Breiteig and Grevholm (2006) explained that algebraic complexity is associated with syntactic inconsistencies with arithmetic, such as: a variable can represent many numbers simultaneously, letters can be chosen freely, there is no positional value, equality as an equivalence relationship (equivalence), multiplication sign is not visible, priority rules and the use of brackets. Many students cannot connect arithmetic and algebra because classroom learning treats these two topics as if they were completely different from one another. The concept of equivalence is difficult for a student because they see statements as arithmetic problems such as $2 + 7$ being interpreted as adding 7 to 2 yields 9. This makes expressions such as $a + b$ unintelligible. If a or b is not known, then it is impossible to calculate in $a + b$. So finding the sum of a and b rather than calculating a to b is a more meaningful emphasis. For example, in the fraction 4/7, the numbers 4 and 7 cannot be seen as separate numbers, because 4/7 itself is a number.

Misinterpreting algebraic letters as object names (e.g. interpreting the letter "b" as "banana", so "$5b$" means "five bananas") is agreed upon by researchers (e.g., Ann van Amerom, 2003; Chick, 2009; MacGregor and Stacey, 1997; Malisani & Spagnolo, 2009; Usiskin, 1999) as a well-known and serious obstacle when writing expressions and equations in certain contexts. Also, the conveyance of concepts in applied mathematics is usually denoted by the initial letter of their name (such as A for area, m for mass, t for time, etc.). It is quite possible that this use of letters reinforces the belief that letters in mathematical expressions and formulas stand for words or things, not for numbers. The results of research conducted by Edo & Tasik (2022) show that students tend to interpret variables as "labels" and as "objects" rather than numbers. The use of letters as abbreviations for words or labels such as the "fruit salad algebra" approach is still very much found in student math textbooks or student worksheets compiled by teachers.

If students are properly taught in early grades about some of the important parts of algebra such as equivalence, patterns, expressions, and functions, they will not experience much difficulty in transitioning from arithmetic to algebra or in understanding algebraic notation. Onal (2023) states that students must learn that there are many meanings associated with arithmetic symbols, this is because certain interpretations will suit different contexts and solving procedures. Equality is a relationship that expresses the idea that two mathematical expressions have the same value and must be well understood by students so that it does not become a major stumbling block for them moving from arithmetic to algebra (Oksuz, 2007). Students need an understanding of the equals sign to be able to see the relationships expressed by a number of sentences.

CONCLUSION

The use of the "fruit salad algebra" approach has proven to be still a favorite "menu" in introducing variables in algebraic form in junior high schools. However, despite being a favorite, this approach was reported as a "misguided early presentation" of developing algebraic thinking. Some researchers such as (Edo & Tasik, 2022; Gunawardena, 2011; Widodo et al., 2018) also report that misinterpreting letters as labels is a fundamental misunderstanding that will lead to many other errors in algebra. Different interpretations in different contexts in interpreting letters can cause students to be confused and misinterpret the use of variables (Edo & Tasik, 2022). The student misconceptions found in this study support the report. Algebraic reasoning involving variables and symbolic notation appears to be a cognitive barrier for students learning algebra at school. Students have difficulty recognizing the structure of the problem when they try to represent the problem symbolically. They can recognize the solution procedure (e.g., reverse computation) but they cannot give reasons for the unknown itself.

Responses from discussions with teachers indicated that the “fruit salad algebra” approach was rooted in the teaching culture, reinforced by several textbooks, and influenced by how teachers themselves were taught in the past. Its continued use occurred for a number of reasons, including that the "apple and banana" analogy was perceived by teachers as easy to understand for students, teachers lacked understanding of the long-term dangers of the approach, teachers lacked knowledge of alternative strategies, or that they did not believe that there are other alternatives that can be "helpful" and accessible to students.

Instead of using the “fruit salad algebra” approach to introduce variables to algebra in junior high schools, teachers could consider using the notion of equality of equals sign approach through problems that use students' common knowledge and informal strategies. The following are some examples of approaches that can be used. Ardiansari & Wahyudin (2020) show the steps for expanding the meaning of the equal sign by using the distributive law which can be used to introduce variables as follows. The teacher can explain the steps made by Ardiansari & Wahyudin (2020) through the question-and-answer method in class. Symbols ☐, ■, and △ represent hidden numbers, then one by one these symbols are replaced with letters such as $a, b, c, ...$etc. to represent hidden numbers. The letter that is closely related to the idea of hidden numbers is then referred to as an unknown term.

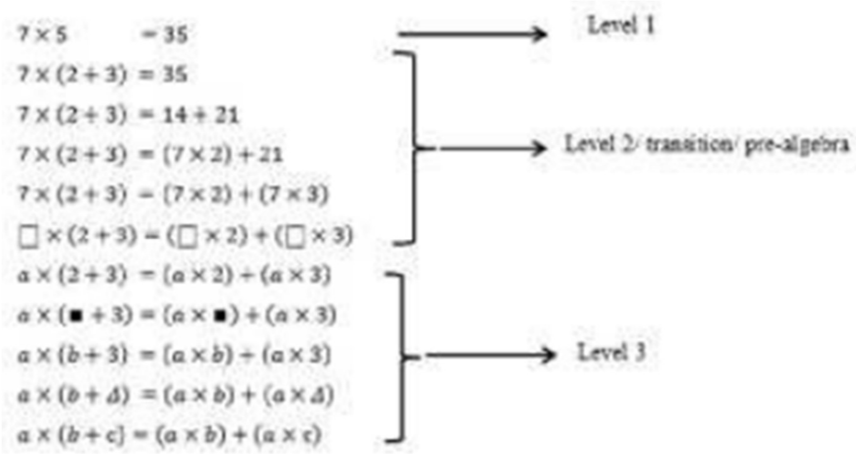


Figure 6: The Steps for Expanding the Meaning of the Equal Sign by using Distributive Law

Another example is through problems in real situations as illustrated by Suryadi (2013) below which can be considered for use in introducing variables. There are three glasses containing Rp. 1000,00 and three other glasses containing Rp. 5000, 00 as shown in Figure 7. Students are asked to find at least three different ways to find the total value of the money in the six glasses.

Figure 7: Problem Illustration

Through class discussion, a number of questions are then asked which encourage students to explain the relationship between the three mathematical representations. Then a further problem is given, namely there are three white glasses where each contains money with the same nominal but it is not yet known how much and three black glasses containing money with the same nominal but it is not known how much it is as shown in Figure 8 to introduce variables.

Figure 8: Further Problem Illustration

Students are asked to find three different ways to determine the total value of the money in the six glasses where the amount of money in the white glass group and the black glass group is not the same.

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