# Problem-solving: Growth of Students' Mathematical Understanding in Producing Original Solutions

Octavina Rizky Utami Putri[1,2], Susiswo[1], Erry Hidayanto[1], Slamet[1]

[1]Universitas Negeri Malang, Indonesia,

[2] Universitas Muhammadiyah Malang, Indonesia

octavina@umm.ac.id, susiswo.fmipa@um.ac.id*, erry.hidayanto.fmipa@um.ac.id,

slamet.fmipa@um.ac.id

*Abstract: Students' deep understanding of problem-solving can stimulate the presence of original solutions. This statement led this present study to explore the growth of students' mathematical understanding in producing original solutions through problem-solving. Fifty-five students, from two different junior high schools, are solving a two-dimensional figures problem. Students who produce original solutions are interviewed to investigate their growth in mathematical understanding when generating their solutions. The original solution indicates that the answer fulfills the three aspects: different, unique, and correct. The study observes students' activities, both 'acting' and 'expressing', which refers to layers of understanding of Pirie and Kieren's Model. 'Acting' and 'expressing' were observed to investigate the movement of students' understanding in solving-problem. Students whose understanding grows to the layers of 'image making' and 'image having' will come up with original ideas. The ideas become more complex as students' understanding grows to the 'property noticing' layer. Besides, the original ideas combined with conceptual and procedural knowledge can formally support the presence of an original solution. Students produce original solutions when their understanding has reached the layers of 'formalizing' and 'observing'.*

Keywords: growth, mathematical understanding, original idea, original solution, problem-solving

## INTRODUCTION

Problem-solving has been a long-standing concern for both learning and research in mathematics education (Hidayah, Sa'dijah, Subanji, & Sudirman, 2020). Previous studies have shown that mathematical problem-solving activities can encourage a deeper and more meaningful understanding (Kotsopoulos & Lee, 2012; Plaxco & Wawro, 2015). Students are given the opportunity to implement their knowledge in problem-solving (Li et al., 2020). In fact, the problem faced by the students requires the implementation of knowledge in new situations as well as high-level thinking skills (Kotsopoulos & Lee, 2012). The various knowledge combinations and the involvement of understanding are crucial in this skill to support students' mathematical competence (Edo & Tasik, 2022; Spencer, Fuchs, & Fuchs, 2019; Stadler, Herborn, Mustafić, & Greiff, 2020; Torbeyns, Schneider, Xin, & Siegler, 2015). Therefore, mathematics problem-solving activities not only encourage students to apply their understanding but also have the potential to form a deep understanding to improve their mathematical competency.

Understanding measures the quality and quantity of the relationship between new knowledge and prior knowledge that is already owned. In other words, the more relationships in the knowledge network, the better the understanding (Walle, Karp, & Bay-Williams, 2019). The level of mathematical understanding is determined by the number and strength of the network of connections between mathematical concepts, procedures, and facts. Understanding can be comprehended thoroughly as long as it is associated with a network of connections that are more numerous or stronger (Hiebert & Carpenter, 1992). Furthermore, this deep mathematical understanding can encourage flexibility of thinking and connection of ideas in problem-solving (Martínez-Planell, Trigueros Gaismán, & McGee, 2017; Musgrave & Carlson, 2017; Weber, 2009). The more ideas linked in problem-solving, the more likely it is to produce original solutions (Agnoli, Franchin, Rubaltelli, & Corazza, 2015).

Original solutions in problem-solving emphasize unique and different ideas (Sidi, Torgovitsky, Soibelman, Miron-Spektor, & Ackerman, 2020). The originality of the solution can be assessed based on objective and subjective perspectives. An original solution objectively refers to various ideas considered from a whole subject in a specific group. Meanwhile, the subject's perspective emphasizes the assessor's point of view. The assessment of originality by the researcher, however, has progressed from subjective to objective, i.e., referring to the only subject in the group which has generated different ideas (Dumas & Dunbar, 2014; Mones & Massonnié, 2022). However, an original solution in solving a problem not only focuses on differences in ideas but also must prioritize the uniqueness and correctness of these ideas (Silver, 1997).

The original solutions do not appear suddenly but through a series of thought processes involving understanding (Munahefi, Kartono, Waluya, & Dwijanto, 2020; Wessels, 2014). Likewise, the results of the preliminary study by the researcher showed that students involved in understanding by linking knowledge to produce original solutions. Students also use various methods in solving problems, and one of these methods is original, resulting in a new, deeper understanding. This initial finding relates to understanding which is not a static learning point but an evolving mental activity (National Research Council, 2002). In other words, mathematical understanding is a dynamic process that shifts from informal actions to more formal abstractions, which can be observed through *acting* and *expressing* (Pirie & Kieren, 1994). It can be communicated through the students' representation in mathematical problem-solving activities (Quintanilla & Gallardo, 2022).

The theory of growth in mathematical understanding was first developed by Pirie and Kieren (1994). This theory explains eight potential layers that describe a person's level of understanding of a particular concept. These layers, from the innermost to the outermost circle, include *primitive knowing, image making, image having, property noticing, formalizing, observing, structuring,* and *inventising*. *Primitive knowing* is the initial point of understanding observed as the whole thing that has been known and done by the students. *Image making* and *image having* are related to creating new knowledge that is different from initial knowledge. However, *image making* involves triggers when creating new knowledge, whereas *image having* does not involve triggers to help understand images. After the image is owned and well understood, students are ready to connect prior and new knowledge at *property noticing* (Gulkilik, Moyer-Packenham, Ugurlu, & Yuruk, 2020; Pirie & Kieren, 1994). *Formalizing* is related to students' ability to use formal mathematical definitions or algorithms (Bobis & Way, 2018). In *observing*, students reflect and coordinate their formalization activities. For example, after formalizing the procedure, then do reasoning (Yao & Manouchehri, 2022). At *structuring*, students realize that a collection of theorems are interrelated and ask for verification through logical thinking. Then, students bring new understanding to create a new concept as an *inventising* achievement (Pirie & Kieren, 1994). The growth of understanding occurs through continuous reciprocating movements through layers of understanding, as illustrated in Figure 1.

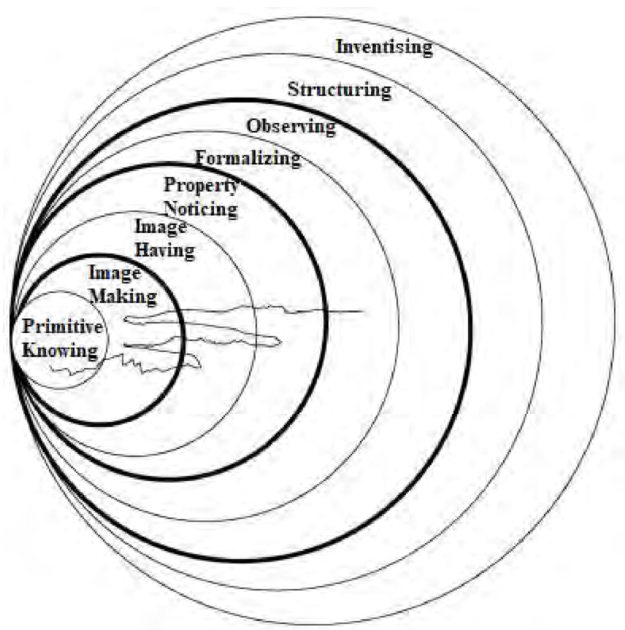


Figure 1: Growth in Mathematical Understanding by Pirie and Kieren (1994)

The eight circles in Figure 1 are depicted in a nested form which represents each layer that contains all the layers in it. Each layer, apart from *primitive knowing* and *inventising*, contains *acting* and *expressing,* which complement each other as activities that can be observed in each layer of understanding. *Acting* contains mental and physical activities, while *expressing* is related to conveying these activities to others and oneself. The pairs of *acting* and expressing can be observed in *image making*, *image having*, *property noticing*, *formalizing*, *observing*, and *structuring,* respectively called *image doing* and *image reviewing*, *image seeing* and *image saying*, *property predicting* and *property recording*, *method applying* and *method justifying*, *featuring identifying* and *featuring prescribing*, and *theorem conjecturing* and *theorem proving*. Pirie and Kieren proposed a model of growth in mathematical understanding with key features: *don't need boundaries, folding back,* and the complementarities of *acting* and *expressing* (Pirie & Kieren, 1994). Moreover, the study focuses on *acting* and *expressing* to observe the movement of the level of understanding.

Mathematical understanding can grow when a person learns new things (Gulkilik et al., 2020). Mathematical understanding also grows when students carry out mathematical problem-solving activities (Patmaniar, Amin, & Sulaiman, 2021). The formation of new understandings and a deeper understanding of mathematics can be facilitated through the assignment of problem-solving that emphasizes the productivity of ideas (Bajwa & Perry, 2021). Hence, the growth of students' mathematical understanding can be triggered through these tasks.

Previous studies have applied Pirie and Kieren's model to investigate the thinking process in solving mathematical problems. These studies focused on *don't need boundaries* (Rahayuningsih, Sa'dijah, Sukoriyanto, & Qohar, 2022), *folding back* (Patmaniar et al., 2021; Risley, Hodkowski, & Tzur, 2015), *primitive knowing* (Putri & Susiswo, 2020), and low-skilled students' understanding growth (Sengul & Yildiz, 2016). However, the studies showed its limitations in dealing with the theory of growth in mathematical understanding in problem-solving that focused on achieving an original solution, while such understanding is closely related to the presence of originality (Paulin, Roquet, Kenett, Savage, & Irish, 2020).

Pirie and Kieren's model has the potential to be the reference for observing mathematical growth and understanding in producing original solutions to problem-solving activities. Some people hold the view that mathematical understanding is a dynamic process that can build connected knowledge and flexible thinking (Martin & Towers, 2016), thus encouraging students to be able to generate new, unique, and useful ideas (Sitorus & Masrayati, 2016). Therefore, this study aims to explore the growth of students' mathematical understanding in producing original solutions to problem-solving activities.

## METHOD

This research is an exploratory study with a qualitative approach. There were fifty-five students from two different Junior High Schools (JHS-A and JHS-B) in Malang-Indonesia as participants. Researchers prepared instruments in the form of worksheets and interview guidelines to obtain data on the growth of students' understanding of producing original solutions. The tasks sheet contained problems in the area of two-dimensional figures adapted from Siswono (2010). Students are asked to design any two-dimensional figures that represent a park that has exactly 1200 $m^2$ as shown in Figure 2.

Mr. Dani, the rural village head of Maju Jaya, plans to design a creative park covering an area of 1200 $m^2$. He needs others to help him make the design. Assist Mr. Dani based on the following conditions:

a. Create as many creative parks design as possible, and the length of the sides according to the known area!
b. Pat attention to the creative park shape that you think is the most unique one! Tell us in detail how you determine the length of the sides!

Figure 2: Problem-Solving Task

The results of problem-solving are analyzed based on originality in an objective perspective, namely the different ideas of a particular group of people (Dumas & Dunbar, 2014; Mones & Massonnié, 2022) and uniqueness and correctness in problem-solving (Silver, 1997). All student answers were classified based on the solution's types, characteristics, and accuracies, as shown in Table 1.

| Types | Characteristics | Accuracies | Code | Category |
|---|---|---|---|---|
| Similar | Ordinary | Incorrect or Correct | SOI or SOC | Unoriginal |
| Different | Ordinary | Incorrect or Correct | DOI or DOC | Unoriginal |
| | Unique | Incorrect | DUI | Unoriginal |
| | | Correct | DUC | Original |

Table 1: Classification of Solutions in Problem-Solving

Based on Table 1, the researcher identified the problem-solving solutions of fifty-five participants shown in Figure 3.

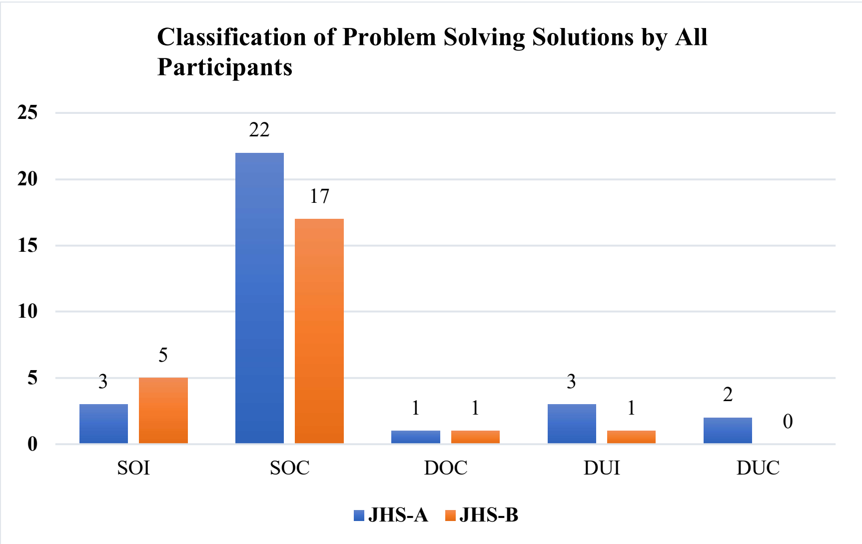


Figure 3: Frequency graph of Problem-Solving Solutions by Fifty-Five Participants

Figure 3 shows that two students provided original solutions from JHS-A. The researcher summarizes the many original solutions for the two selected students presented in Table 2.

| No | School | Name | Gender | Number of solutions | |
|---|---|---|---|---|---|
| | | | | Original | Unoriginal |
| 1 | JHS-A | A[1] | Female | 1 | 4 |
| 2 | JHS-A | A[2] | Male | 1 | 1 |

Table 2: The Number of Solutions Generated by A[1] and A[2]

Table 2 shows that even though A[1] produces more solutions than A[2], both of them can produce one original solution. Furthermore, A[1] and A[2] were interviewed to observe the growth of mathematical understanding in producing original solutions. The interview guideline refers to the descriptors of each layer of mathematical understanding. The descriptor of each layer was developed from Pirie and Kieren's model presented in Table 3.

| Mathematical Understanding Layers | Descriptors by Pirie and Kieren | Descriptors in This Research |
|---|---|---|
| *Primitive knowing* | Preliminary knowledge is needed to build certain concepts. | Initial knowledge is needed to solve the problem. |
| *Image making* | Creating a new image as a differentiator from previous knowledge through mental or physical activity. | Making new knowledge different from prior knowledge by involving triggers such as simple examples. |
| *Image having* | Understanding certain concepts without acting on objects. | Modifying knowledge to acquire new knowledge without involving triggers. |
| *Property noticing* | Combining images to form certain specific properties. | Combining knowledge to form certain specific properties. |
| *Formalizing* | Making generalizations and developing formal mathematical ideas. | Building problem-solving steps in accordance with formal mathematical procedures. |
| *Observing* | Thinking of the latest formal ideas and use them to create algorithms. | Reflecting knowledge to be applied in various problem-solving situations. |
| *Structuring* | Being aware of the interrelationships between theorems. | Linking between theorems by involving logical arguments as a form of verification in solving problems. |
| *Inventising* | Bringing new understanding to create new concepts. | Having a new understanding that can be known through a conclusion statement. |

Table 3: Descriptors for Each Layer of Mathematical Understanding

Data were analyzed using data condensation, data display, and conclusion drawing (Miles, Huberman, & Saldana, 2014). Data condensation was done by providing the codes of *acting* and *expressing* on the achievement of each layer of students' mathematical understanding (Pirie & Kieren, 1994), as shown in Table 4.

| Mathematical Understanding Layers | Observed Aspects | | | |
|---|---|---|---|---|
| | **Acting** | **Acting Code** | **Expressing** | **Expressing Code** |
| *Primitive knowing* | - | - | - | - |
| *Image making* | *Image doing* | *Ac-IM* | *Image reviewing* | *Ex-IM* |
| *Image having* | *Image seeing* | *Ac-IH* | *Image saying* | *Ex-IH* |
| *Property noticing* | *Property predicting* | *Ac-PN* | *Property recording* | *Ex-PN* |
| *Formalizing* | *Method applying* | *Ac-F* | *Method justifying* | *Ex-F* |
| *Observing* | *Featuring identifying* | *Ac-O* | *Featuring prescribing* | *Ex-O* |
| *Structuring* | *Theorem conjecturing* | *Ac-S* | *Theorem proving* | *Ex-S* |
| *Inventising* | - | - | - | - |

Table 4: *Acting* and *Expressing* in The Mathematical Understanding Layers

Furthermore, data display was done by providing examples of the presence of *acting* and *expressing* at each level of student understanding. Data is also displayed visually to illustrate the presence of original solutions in the mathematical understanding layer. Finally, the researcher provides conclusions regarding the growth of students' mathematical understanding in producing original solutions through problem-solving.

During the data collection and analysis process, researchers conducted member checking and peer debriefing to obtain credible data (Nowell, Norris, White, & Moules, 2017). Member checking is done through interviews with students who have produced original solutions to clarify the results of student problem-solving. In-depth interviews are used to explore the growth of students' mathematical understanding in producing original solutions. Peer debriefing is carried out by researchers through discussions with colleagues of doctoral students and mathematics education lecturers other than the research team to get suggestions regarding the data that has been obtained.

## RESULTS

Two students were chosen as research subjects, A[1] and A[2], because both produced original solutions and provided clear information on each *acting* and *expressing,* observed from each mathematical understanding layer achievement. The two subjects have different characteristics in the movement of the growth of mathematical understanding in producing original solutions, but both achieve original solutions at the same level of understanding.

### The growth of A[1]'s understanding of producing original solutions

A[1] produces five solutions, and only one reflects originality. The original solution is a composite two-dimensional figure combining two rectangles, as seen in Figure 4 (E).

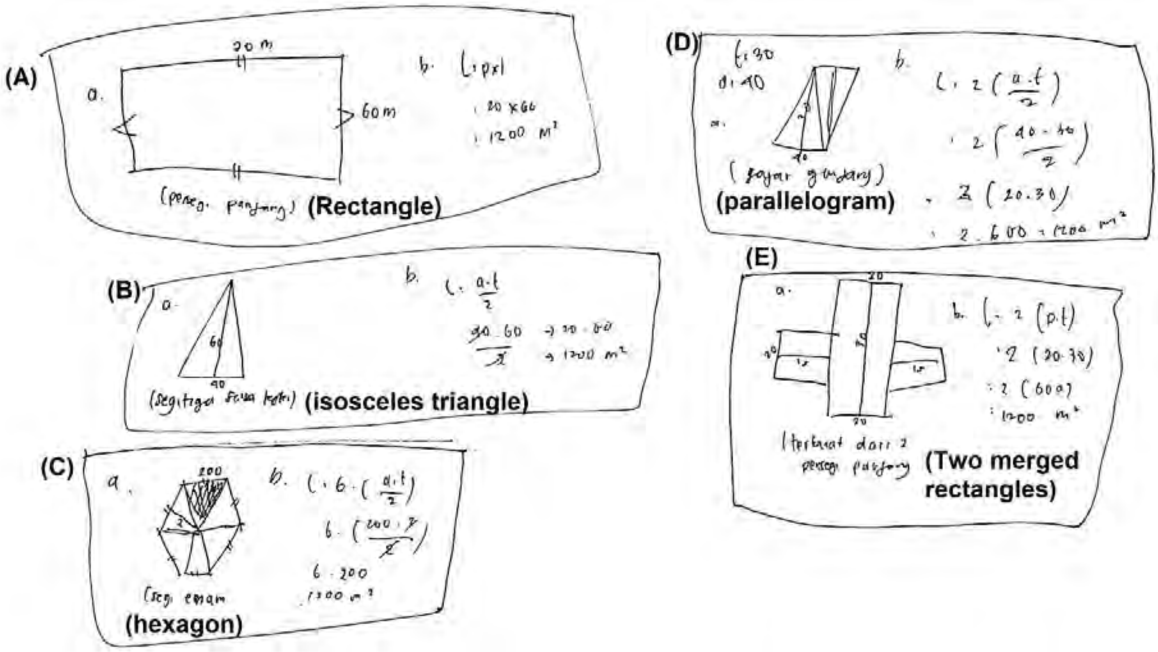


Figure 4: A[1]'s Problem-Solving Task Results

Figure 4 shows that most students in A[1]'s class created similar solutions, as shown in (A) and (B). Meanwhile, solution (C) reflects different solutions from all students and the idea can be said to be unique, but this solution is incorrect. Solution (D) is also different from all of the students' solutions in class A[1], but this idea is quite ordinary because parallelograms have been studied before. Solution (E) is a different solution from all student solutions in class A[1], the idea can be said to be unique, and conceptually the truth can be accepted mathematically. Therefore, this study distinguishes between original ideas and solutions. An original solution is obtained from an original idea equipped with the accuracy of the solution according to formal mathematical concepts.

Furthermore, the researcher conducted interviews with A[1] to explore her *primitive knowing*. A[1] had studied basic two-dimensional figures before and mentioned several examples, such as triangles, rectangles, parallelograms, trapezoids, and circles. However, she can only mention the formula of triangle and rectangle areas. A[1] said that the problem-solving task completed by A[1] was a new learning experience for her. A[1] stated that she never made composite two-dimensional figures of the same area.

In the early stages of solving the problem, A[1] represents a park by drawing a basic two-dimensional figure in the form of a rectangle (A) and a triangle (B) as a trigger for the presence of a composite two-dimensional figure. A[1] also gives the lengths of the sides of rectangles and triangles. Then, A[1] creates a composite two-dimensional figure which is a combination of triangles that form a hexagon, as shown in Figure 2 part (C) ***(Ac-IM)***. A[1] said that this idea came from combining triangles to become a composite two-dimensional figure ***(Ex-IM)***. This *acting* and *expressing* reflect *image making*, namely making new knowledge that is different from *primitive knowing*, so as to produce original ideas in solving problems. A[1] draws a composite two-dimensional figure, which is a hexagon, by combining six triangles ***(Ac-IH)***. A[1] explained that this aims to make it easier for him to divide the area of each forming triangle ***(Ex-IH)***. This *acting* and *expressing* reflects how A[1] 's knowledge grows into *image having*. A[1] also said that he chose as many as six triangles to make it easier to divide 1200 into six parts for each area of the triangle ***(Ac-PN)***. Now A[1] has new knowledge that in creating a composite two-dimensional figure, it is necessary to pay attention to many trigger basic two-dimensional figures to make it easier to determine each area and side, so this is an achievement *property noticing*. A[1] states that if the area of each basic two-dimensional figure has been obtained, then A[1] can easily determine the length of the sides in the form of integers ***(Ex-PN)***.

In *formalizing*, A[1] uses the formula for the area of a triangle, as shown in Figure 2 part (C), to be able to determine the length of the base and height of the triangle. A[1] uses a similar method to generate the lengths of the sides of other two-dimensional figures ***(Ac-F)***. A[1] states that the length of the sides of the composite two-dimensional figure can be found by using the formula of the area of each basic two-dimensional figure ***(Ex-F)***.

A[1] always involves a trigger two-dimensional figure to produce a composite two-dimensional figure. A[1] combines triangles to get a hexagon or parallelogram and also combines rectangles to form other composite two-dimensional figures ***(Ac-O)***. A[1] conveys the steps for determining the sides of a composite two-dimensional figure, namely: drawing a composite two-dimensional figure from the same type of basic two-dimensional figure, dividing the area of the two-dimensional figure by the number of the basic two-dimensional figure, and using the formula for the area of the basic two-dimensional figure to determine the length of the sides of the composite two-dimensional figure ***(Ex-O)***. A[1] has made a schematic in her knowledge that to produce a composite two-dimensional figure, A[1] needs to combine several identic basic two-dimensional figures. This solution step is used to come up with an original solution. It means that A[1] has reached *observing*, even though A[1] always needs triggers to draw a composite two-dimensional figure.

Whenever A[1] needs a trigger to produce an original solution, then A[1] returns to *image making* and expands understanding. This phenomenon of returning to a deeper layer of understanding is referred to as *folding back*. In the end, at *formalizing* and *observing*, A[1] produced an original solution: a composite two-dimensional figure of two identical rectangles with the correct side length. The researcher summarizes the results of observations on *acting* and *expressing* by A[1], which are presented in Table 5.

| Mathematical Understanding Layers | *Acting* | *Expressing* |
|---|---|---|
| *Image Making* | The student draws a triangle and a rectangle as triggers, then draws other composite two-dimensional figures built by combining multiple triangles or rectangles. | The student stated that she could create a composite two-dimensional figure by combining identical basic two-dimensional figures. |
| *Image Having* | The student draws composite two-dimensional figures by combining congruent basic two-dimensional figures to make it easier to divide the area. | The student stated that if all basic two-dimensional figures that form composite two-dimensional figures are the same, then it is easy to determine the area and length of each side. |
| *Property Noticing* | The student pays attention to the many congruent basic two-dimensional figures to make it easier to determine the sides' area and length. | The student stated that each basic two-dimensional figure's area must be considered to produce the length of the sides in the form of integers. |
| *Formalizing* | The student determines the length of the sides of the composite shape by writing the area formula of the basic two-dimensional figures. | The student stated that the sides of a composite two-dimensional figure could be found by paying attention to the area formula of each basic two-dimensional figure. |
| *Observing* | The student uses a similar pattern of the solution steps to create any composite two-dimensional figures and determine the lengths of the sides. | The student stated that the steps for determining the sides of a composite two-dimensional figure, namely: drawing a composite two-dimensional figure from several identical basic two-dimensional figures, dividing the area that is known by the number of basic two-dimensional figures, and using the area formula of the basic two-dimensional figure to determine the length of the sides of the composite two-dimensional figure. |

Table 5: A[1]'s *acting* and *expressing* in producing original solutions

**The growth of A[2]'s understanding of producing original solutions**

A[2] produces two solutions, and only one reflects originality, which creates three rectangles combinations, as shown in Figure 5 (F).

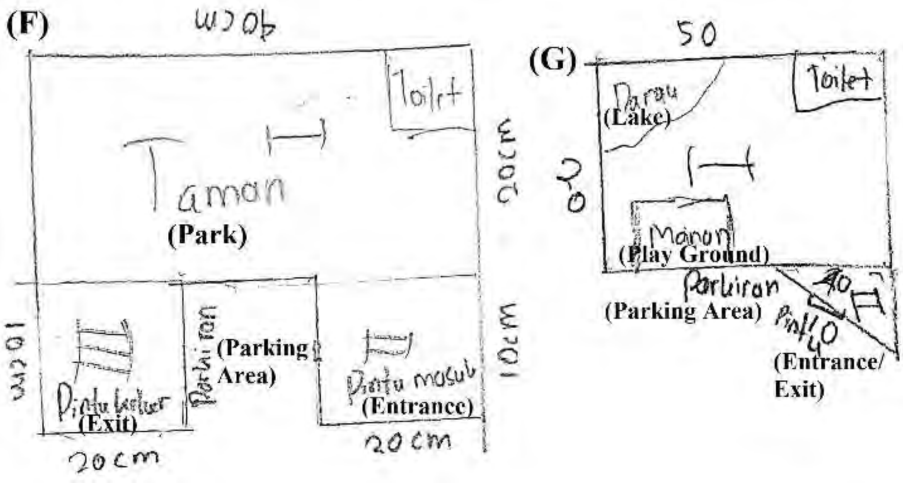


Figure 5: A[2]'s Problem-Solving Task Results

Figure 5 shows that A[2] creates two composite two two-dimensional figures. Figure 5 (F) includes three different rectangles. Figure 5 (G) includes rectangles and triangles, but the length of the base and height of the triangle does not fit the right triangle concept.

Through interviews, the researcher explores A[2] 's understanding of producing original solutions. A[2] already has *primitive knowing,* which is indicated by the statement that A[2] has studied basic two-dimensional figures such as trapezoids, parallelograms, rectangles, squares, triangles, and circles at school. A[2] has also studied pentagons and hexagons with his parents at home. A[2] can state the formula for the area of a basic two-dimensional figure and states that it is difficult to remember the formula for the area of a trapezoid, pentagon, hexagon, and circle. A[2] is used to solve the problem of determining the area of a basic two-dimensional figure by knowing the length of the sides. On the other hand, solving the composite two-dimensional figure problem is a new learning experience for him.

Based on the results of solutions by A[2], as shown in Figure 5, it shows that A[2] has created new knowledge that is different from previous learning experiences. Without using a trigger, A[2] can represent the garden by creating composite two-dimensional figures, as explained in Figure 5 (F) and (G) ***(Ac-IH)***. A[2] also states that the shape of a garden can be a combination of squares, rectangles, or triangles ***(Ex-IH)***. This *acting* and *expressing* indicate that A[2] has reached *image having* in the growth of his understanding. A[2] does not need a trigger to create a composite two-dimensional figure. The researcher conducted further interviews related to A[2] 's understanding of determining the length of the sides of the composite two-dimensional figure he had made. The following is an excerpt from the interview of the researcher (R) and A[2].

Interview Excerpt (1)
R : *Do you depend on formulas when solving problems?*
A[2] : *No, I don't. The problem is that the area of the garden is 1200* $m^2$. *So I determine the area of each basic flat shape (points to the division of three parts of the flat shape area) is 200, 200, and 800.*
Interview Excerpt (2)
R : *How can you determine the side length of the composite plane shape?*
A[2] : *I divide the area into several parts, then I determine the length of each side (points to the length of the sides).*

Interview Excerpt (1) showed that A[2] determines the length of the sides of the composite two-dimensional figure, starting with dividing the area into three parts, namely for rectangles I, II, and III, as shown in Figure 5 part (F). In Figure 5 (F), this division is based on the proportions of the size of basic two-dimensional figures made, namely rectangle I is given an area of 800, while rectangles II and III are each given an area of 200 ***(Ac-PN)***. A[2] states that A[2] must consider the division of this area based on the proportion of the size of rectangle I, which is larger than rectangles II and III ***(Ex-PN)***. This shows that A[2]'s understanding has reached *property noticing*. A[2] combines knowledge regarding image proportions and their area to get each basic two-dimensional figure's area.

The achievement of the *formalizing* layer can be seen in the Interview Excerpt (2). A[2] determines the length of the sides of the composite two-dimensional figure by determining the length of the sides of each rectangle by multiplying two integers. The concept of the area of a rectangle seems to stick to A[2] 's memory, so A[2] can quickly determine the length of the sides of the rectangle without writing down the formula ***(Ac-F)***. A[2] says that the area of a rectangle can be found by multiplying the lengths of the two adjacent sides ***(Ex-F)***. However, it appears that there are disproportionate side lengths, as shown in Figure 5 (F). The researcher conducted further interviews with A[2], as disclosed in the following interview excerpt.

Interview Excerpt (3)
*R : What is the length that you determine from this side? Is it 40? (points to the length of the rectangle I). What about this? (points to rectangles II and III)*
*A[2] : Yes 40. The length of this side is 20 because it is a rectangle, a square, and a square. Oh, sorry, these are all rectangles. If so, I change the length to 10 (while writing). The length of the side of rectangle I is 40 (points to the length of rectangle I), if the length of side 20 plus 20 (points to the length of rectangles II and III), then the length of both is the same as rectangle I. So I change the length.*

Based on the Interview Excerpt (3), A[2] realized that the side lengths that had been made were disproportionate, as shown in Figure 5 (F). Rectangles II and III each have a length of 20 and are separated by some sides of the rectangle I. A[2] then revised it by giving the lengths of the sides of rectangles II and III, respectively, the length and width were originally 20 and 10 to 10 and 20. The revision of the idea that has been carried out by A[2] forms a new understanding by A[2];

namely, A[2] must pay more attention to the details between the given figure and the given side length so that it looks more proportional.

The researcher summarizes the results of observations on *acting* and *expressing* A[2], which are presented in Table 6.

| **Mathematical Understanding Layers** | ***Acting*** | ***Expressing*** |
|---|---|---|
| *Image Having* | The student draws composite two-dimensional figures by combining several types of basic two-dimensional figures without involving triggers. | The student stated that basic two-dimensional figures that are made could be any basic two-dimensional figures. |
| *Property Noticing* | The student pays attention to the proportion of the size of each basic two-dimensional figure to make it easier to determine the area and length of each side. | The student stated that they had to consider the proportion of the size of each of the basic two-dimensional figures and the area determined. |
| *Formalizing* | The student directly determines the length of the composite two-dimensional figures' sides without writing down the formula for the area of the basic two-dimensional figures. | Students stated that they did not depend on the formula, but it had been used to determine the side length of basic two-dimensional figures. |
| *Observing* | The student uses the solution steps in a similar pattern to create any composite two-dimensional figures and determine the lengths of the sides. | The student stated the steps for determining the sides of a composite two-dimensional figure, namely: drawing a composite two-dimensional figure, giving the area of each basic two-dimensional figure based on the proportions of the figure and the known area, using the area formula of the basic two-dimensional figure to determine the length of the sides of the composite two-dimensional figure. |

Table 6: A[2]'s *Acting* and *Expressing* in Producing Original Solutions

Furthermore, the researcher conducted further interviews to explore A[2]'s idea of making a composite two-dimensional figure. A[2] states that A[2] can directly create a composite two-dimensional figure without involving any triggers ***(Ac-O)***. A[2] states that similar solving steps are used to create other composite two-dimensional figures along with determining the length of the sides, namely: drawing the composite two-dimensional figure, giving the area of each basic two-dimensional figure based on the proportions of the drawing and the known area, using the area procedure of the basic two-dimensional figure to determine the lengths of the sides of the composite two-dimensional figure ***(Ex-O)***. The completion steps that have been used by A[2] make it easier for him to solve the problem given by the researcher. However, A[2] does not pay attention to the concept of triangles in this second solution but only emphasizes procedural knowledge in determining the length of a triangle's sides. As a result, the length of the height and base of the triangle in Figure 5 part (G) does not match the right triangle made by A[2]. In accordance with the right triangle in Figure 5 part (G), conceptually, if the base length is 10 and the height is 40, it will form an acute triangle, not a right triangle, as described in A[2]. Thus, A[2]'s understanding can be said to achieve *observing* but not achieving *structuring*. Even so, A[2] stated that he had to be more careful in making a composite two-dimensional figure and the length of their sides by paying attention to the properties of the basic two-dimensional figure.

**The achievement of original solutions in the understanding layer**

This observation, namely *acting* and *expressing*, clarifies A[1] and A[2] achievement at each layer. The movements between layers indicate a growth in understanding from the emergence of original ideas to original solutions. Although their growth movements are different, both produce original ideas and solutions at the same level of understanding. The relationship between achieving the original solution in problem-solving and the understanding layer is presented in Figure 6.

Figure 6 explains that *primitive knowing* provides primitive ideas to support original ideas. The original ideas emerge as the student's understanding grows to *image making*, *image having*, and *property noticing*. When students have reached the *formalizing* layer, they formally involve a mathematical procedure in solving the problem to activate the original solution. Meanwhile, when they reach *observing*, they will become more consistent in using the same procedures to come up with another solution. Therefore, students' growth in understanding begins with primitive ideas that develop into original ones that are different from what they produce at *primitive knowing*. The original ideas grow into original solutions when students' understanding grows to reach *formalizing* and *observing*.

Moreover, A[1] and A[2] understanding does not reach the *structuring*. They do not relate to concepts of the basic two-dimensional figure and the composite two-dimensional figure. A[1] does not pay attention to the triangle conditions that can form a hexagon. A[1] Arranging six triangles into a hexagon is an original idea. However, if A[1] combines six identical triangles with a base length of 200 and a height of 2, it will not form a regular hexagon with a side length of 200. Therefore, A[1] cannot reach *structuring*. Similarly, A[2] only focuses on the broad procedures of the basic two-dimensional figure. A[2] does not notice that a triangle with a base length of 10 and a height of 40 cannot form a right triangle.

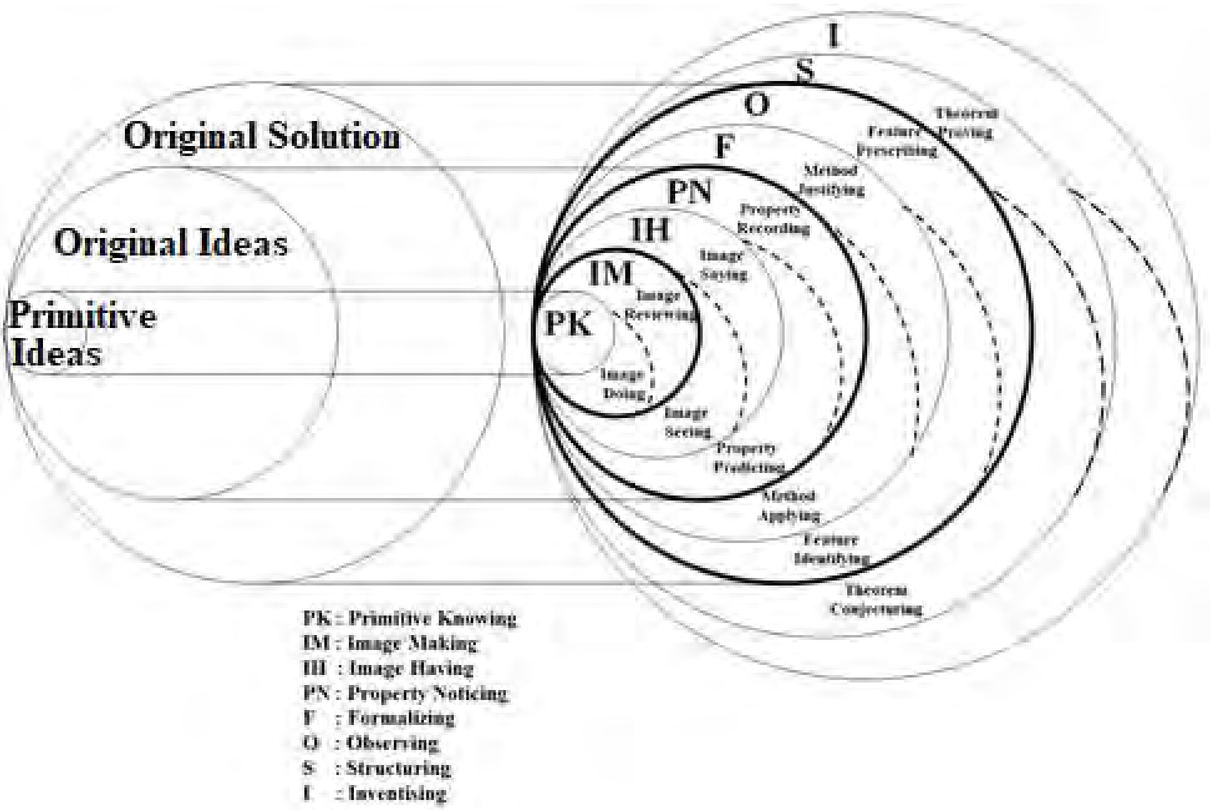


Figure 6: The Relationship between the Achievement of Original Solutions and Understanding Layers in Problem-Solving

## DISCUSSION

In the activity of solving two-dimensional figure problems, students can generate original ideas such as forming new composite two-dimensional figures from a combination of rectangles and triangles. The composite two-dimensional figure formed by students is evidence of the productivity of ideas that are unique and appear to be different from other students. According to Dumas & Dunbar (2014), many ideas generate from problem-solving activities. A different idea from the ideas of all subjects in a particular group can be referred to as the original idea. (Sidi et al., 2020) stated that original ideas must emphasize their uniqueness. However, the original idea does not merely become original solutions in solving problems. It has to be applied equally with the procedural and conceptual understandings to produce one. According to (Silver, 1997), original solutions in problem-solving are not only emphasized original ideas but also must pay attention to the accuracy of the solution. Therefore, an original idea can grow into an original solution balanced by the accuracy of the solution according to the context of the problem being solved.

The solution's accuracy is related to one's understanding capacity in connecting mathematical concepts and procedures in solving problems. The accuracy of the solution is not only assessed based on the uniqueness of the composite two-dimensional figures made by students but also must pay attention to the length of the sides of the composite two-dimensional figure that is made so that the area matches the problem being solved. Likewise, a student creates a regular hexagon from a combination of six isosceles triangles with a base length and height of 200 and 2. It certainly does not allow the area of a regular hexagon with a side length of 200 to have an area equal to the sum of the area of six triangles with a base length of 200 and high of 2. Thus, the appearance of unique ideas has to balance with an excellent mathematical understanding. It is relevant to the opinion of the researchers that without good understanding, one cannot make the right decision (Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Rupalestari, Juandi, & Jupri, 2021). A deep mathematical understanding is necessary to solve problems by connecting mathematical concepts, which aims to support the productivity of new ideas in problem-solving (Beaty et al., 2020; Paulin et al., 2020; Syahrin, Dawud, Suwignyo, & Priyatni, 2019; Xu, Geng, & Wang, 2022). Not only connecting concepts but also relating concepts with the mathematical procedure to gain mathematical understanding. Legesse, Luneta, & Ejigu (2020) mentioned that the involvement of conceptual and procedural understandings in problem-solving is interdependent, which illustrates the linkage between mathematical concepts and problem-solving procedures. Therefore, mathematical understanding plays a vital role in producing the accuracy of solutions to problem-solving.

Moreover, mathematical understanding highly supports problem-solving. Still, each layer-achieved skill in students' mathematical understanding depends on their *primitive knowing*, in which this layer of understanding can significantly impact the original solution achievement. The area formula for the basic two-dimensional figure, such as rectangles and triangles, is attached to the students to produce a composite two-dimensional figure. Previous studies stated that the critical point for this to succeed is *primitive knowing* (Putri & Susiswo, 2020). However, it should be noted that *primitive knowing* is not the lowest level of understanding but rather background knowledge as an initial basis for the growth of mathematical understanding (Pirie & Kieren, 1994). This background knowledge students obtain from their previous learning experience can be used as the basis of growth in mathematical understanding (MacDonald, 2022). It can also benefit the students to solve problems by emphasizing constructive ideas. So, it can deepen their understanding (Husband, 2021), which they can use to produce the original ideas (Auliasari, Sujadi, & Siswanto, 2021; Beaty et al., 2020; Kao, 2022; Lee & Therriault, 2013).

The original ideas are present as the students form a composite two-dimensional figure from identical and different basic two-dimensional figures. It is where *image making* or *image having* achieved, that is, creating different knowledge from *primitive knowing*. New knowledge occurs when one performs mental or physical actions, creating a new image while processing it (Gulkilik et al., 2020). In this state, students engage in activities that help them to develop mathematical ideas through certain representations to get an idea of a concept. This image background is not merely visual but a verbally expressed idea or action (Martin & Towers, 2016). Similarly, students who reach the initial level of abstraction do not need a trigger to understand the image (Bobis & Way, 2018). Original ideas are also present when the students watch several basic two-dimensional figures that are identical or not to make them easier to figure out each area and the length of the side. It relates to a person's ability to manipulate or combine the image's aspects to form specific characteristics (Pirie & Kieren, 1994). As students reach the image, they will be ready to connect and differentiate between previous and present understandings (Bobis & Way, 2018).

Besides, original ideas must balance with excellent conceptual and procedural knowledge to produce original solutions to a problem. Many original ideas appear when the students are asked to design the park. Nevertheless, students have applied the procedure of the basic two-dimensional figure to achieve a *formalizing* layer of understanding. Accordingly, students might use formal mathematical procedures and coordinate formal activities. Similarly, in *observing*, students could predict another solution concerning the procedure being formed (Pirie & Kieren, 1994). It will trigger the conceptual and procedural understandings to solve the problem and produce the right solution. Therefore, the presence of original solutions can result from the presence of original ideas balanced by good procedural and conceptual knowledge.

Students' mathematical understanding to produce original solutions can shift and grow. A[2] does not require a trigger to produce them, but he can directly achieve *image having*. This is due to the involvement of the *don't need boundaries* phenomenon, which grows the students' understanding not to fasten to the previous layer. Rahayuningsih et al. (2022) mention that this phenomenon can be between *image making* and *image having*. As students reach *image making,* they can go through it and arrive at *image having*. Similarly, A[1]'s mathematical understanding while producing original solutions involved *folding back* in. *Folding back* occurs each time A[1] makes a composite two-dimensional figure, and he needs a trigger to form it. Previous studies found that the *folding back* phenomenon is regarded as the way students expand their understanding and connect it conceptually. Since their background knowledge is insufficient to solve new problems, so they must return to the deeper layer to expand it (Martin & Towers, 2016). *Folding back* is not only to remember but also to view the former understanding from a new perspective (Palha, Dekker, Gravemeijer, & van Hout-Wolters, 2013).

## CONCLUSIONS

Mathematical understanding is crucial to producing an original solution in problem-solving activities. Besides, observing students' understanding by *acting* and *expressing* themselves can help clarify the students' growth movement. The presence of original solutions while solving a problem is not sufficiently observed based on the occurrence of the original idea. It should emphasize the accuracy of ideas as well. The accuracy has to come from a person's understanding capacity to link between conceptual and procedural knowledge so thus the original solutions in the problem-solving activities appear.

Pirie and Kieren's theory can be relied upon as a good reference for investigating students' mathematical understanding in producing original solutions to problem-solving activities. In comparison, *primitive knowing* provides a great primitive idea that can trigger students to think of original ideas while solving problems. Originality ideas arise when they start to understand a new concept that differs from the previous ones. It grows an initial understanding of the layers of *image making* and *image having*. And it grows more complex as it reaches *property noticing*, where students make the details of original ideas by noticing specific characteristics. Furthermore, original solutions are shown when the students formally connect their original ideas with the procedure – this skill achieved grows at the *formalizing* layer. A similar procedure the students used is to produce another solution that makes them grow to the level of *observing.* Nevertheless, these students are unaware of any relationship between the theorems that can be applied in solving the problem, so their understanding does not reach the *structuring* level. Therefore, further researchers are expected to investigate the growth of students' understanding of problem-solving activities by considering another characteristic of the research subject and setting.

**References**

[1] Agnoli, S., Franchin, L., Rubaltelli, E., & Corazza, G. E. (2015). An Eye-Tracking Analysis of Irrelevance Processing as Moderator of Openness and Creative Performance. Creativity Research Journal, 27(2), 125–132. https://doi.org/10.1080/10400419.2015.1030304

[2] Auliasari, R., Sujadi, I., & Siswanto. (2021). Junior High School Students' Creative Thinking Process on Equation of Straight Lines. IOP Conference Series: Earth and Environmental Science, 1808(1). https://doi.org/10.1088/1742-6596/1808/1/012061

[3] Bajwa, N. P., & Perry, M. (2021). Features of a pan balance that may support students' developing understanding of mathematical equivalence. Mathematical Thinking and Learning, 23(1), 1–27. https://doi.org/10.1080/10986065.2020.1700587

[4] Beaty, R. E., Chen, Q., Christensen, A. P., Kenett, Y. N., Silvia, P. J., Benedek, M., & Schacter, D. L. (2020). Default Network Contributions to Episodic and Semantic Processing during Divergent Creative Thinking: A Representational Similarity Analysis. NeuroImage, 209(December 2019), 116499. https://doi.org/10.1016/j.neuroimage.2019.116499

[5] Bobis, J., & Way, J. (2018). Building Connections Between Children's Representations and Their Conceptual Development in Mathematics. 55–72. https://doi.org/10.1007/978-981-10-7153-9_4

[6] Dumas, D., & Dunbar, K. N. (2014). Understanding Fluency and Originality: A latent variable perspective. Thinking Skills and Creativity, 14, 56–67. https://doi.org/10.1016/j.tsc.2014.09.003

[7] Edo, S. I., & Tasik, W. F. (2022). Investigation of Students' Algebraic Conceptual Understanding and the Ability to Solve PISA-Like Mathematics Problems in a Modeling Task. Mathematics Teaching-Research Journal, 14(2), 44–60

[8] Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An Exponential Growth Learning Trajectory: Students' Emerging Understanding of Exponential Growth Through

Covariation. Mathematical Thinking and Learning, 18(3), 151–181. https://doi.org/10.1080/10986065.2016.1183090

[9] Gulkilik, H., Moyer-Packenham, P. S., Ugurlu, H. H., & Yuruk, N. (2020). Characterizing the growth of one student's mathematical understanding in a multi-representational learning environment. Journal of Mathematical Behavior, 58(December 2019), 100756. https://doi.org/10.1016/j.jmathb.2020.100756

[10] Hidayah, I. N., Sa'dijah, C., Subanji, & Sudirman. (2020). Characteristics of Students' Abductive Reasoning in Solving Algebra Problems. Journal on Mathematics Education, 11(3), 347–362. https://doi.org/10.22342/JME.11.3.11869.347-362

[11] Hiebert, J., & Carpenter, T. P. (1992). Learning and Teaching with Understanding. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 65–97). https://doi.org/10.5860/choice.30-2122

[12] Husband, M. (2021). Prospective Teachers Working to Connect Their Own Understanding of Subtraction. Canadian Journal of Science, Mathematics and Technology Education, 21(4), 840–855. https://doi.org/10.1007/s42330-021-00179-x

[13] Kao, C.-Y. (2022). How broad cognitive abilities contribute to traditional analogies, creative analogies, and general creativity. Thinking Skills and Creativity, 45, 101068. https://doi.org/https://doi.org/10.1016/j.tsc.2022.101068

[14] Kotsopoulos, D., & Lee, J. (2012). A naturalistic study of executive function and mathematical problem-solving. Journal of Mathematical Behavior, 31(2), 196–208. https://doi.org/10.1016/j.jmathb.2011.12.005

[15] Lee, C. S., & Therriault, D. J. (2013). The cognitive underpinnings of creative thought: A latent variable analysis exploring the roles of intelligence and working memory in three creative thinking processes. Intelligence, 41(5), 306–320. https://doi.org/10.1016/j.intell.2013.04.008

[16] Legesse, M., Luneta, K., & Ejigu, T. (2020). Analyzing the effects of mathematical discourse-based instruction on eleventh-grade students' procedural and conceptual understanding of probability and statistics. Studies in Educational Evaluation, 67(July), 100918. https://doi.org/10.1016/j.stueduc.2020.100918

[17] Li, L., Zhou, X., Huang, J., Tu, D., Gao, X., Yang, Z., & Li, M. (2020). Assessing kindergarteners' mathematics problem solving: The development of a cognitive diagnostic test. Studies in Educational Evaluation, 66(March). https://doi.org/10.1016/j.stueduc.2020.100879

[18] MacDonald, A. (2022). Contextual influences on children's mathematical understandings. Learning, Culture and Social Interaction, 35, 100632. https://doi.org/https://doi.org/10.1016/j.lcsi.2022.100632

[19] Martin, L. C., & Towers, J. (2016). Folding back, thickening and mathematical metbefores. Journal of Mathematical Behavior, 43, 89–97. https://doi.org/10.1016/j.jmathb.2016.07.002

[20] Martínez-Planell, R., Trigueros Gaismán, M., & McGee, D. (2017). Students' understanding of the relation between tangent plane and directional derivatives of functions of two variables. Journal of Mathematical Behavior, 46, 13–41. https://doi.org/10.1016/j.jmathb.2017.02.001

[21] Miles, M. B., Huberman, A. M., & Saldana, J. (2014). Qualitative Data Analysis: A Methods Sourcebook. Los Angles: SAGE Publications, Inc.

[22] Mones, P., & Massonnié, J. (2022). What can you do with a bottle and a hanger? Students with high cognitive flexibility give more ideas in the presence of ambient noise. In Thinking Skills and Creativity. https://doi.org/10.1016/j.tsc.2022.101116

[23] Munahefi, D. N., Kartono, Waluya, B., & Dwijanto. (2020). Analysis of Creative Thinking Process Based on Metacognitive with Project Work Models. International Conference on Science and Education and Technology (ISET 2019) Analysis, 443(Iset 2019), 180–185. https://doi.org/10.2991/assehr.k.200620.035

[24] Musgrave, S., & Carlson, M. P. (2017). Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching. Journal of Mathematical Behavior, 45, 137–149. https://doi.org/10.1016/j.jmathb.2016.12.011

[25] National Research Council. (2002). Learning and understanding: Improving advanced study of mathematics and science in U.S. high schools. In J. P. Gollub, M. W. Bertenthal, J. B. Labov, & P. C. Curtis (Eds.), Quarterly Journal of Experimental Psychology (Vol. 3). https://doi.org/10.1080/17470215108416768

[26] Nowell, L. S., Norris, J. M., White, D. E., & Moules, N. J. (2017). Thematic Analysis: Striving to Meet the Trustworthiness Criteria. International Journal of Qualitative Methods, 16(1), 1–13. https://doi.org/10.1177/1609406917733847

[27] Palha, S., Dekker, R., Gravemeijer, K., & van Hout-Wolters, B. (2013). Developing shift problems to foster geometrical proof and understanding. Journal of Mathematical Behavior, 32(2), 142–159. https://doi.org/10.1016/j.jmathb.2012.12.002

[28] Patmaniar, Amin, S. M., & Sulaiman, R. (2021). Students' Growing Understanding in Solving Mathematics Problems Based on Gender: Elaborating Folding Back. Journal on Mathematics Education, 12(3), 507–530. https://doi.org/10.22342/JME.12.3.14267.507-530

[29] Paulin, T., Roquet, D., Kenett, Y. N., Savage, G., & Irish, M. (2020). The effect of semantic memory degeneration on creative thinking: A voxel-based morphometry analysis. NeuroImage, 220(August 2019), 117073. https://doi.org/10.1016/j.neuroimage.2020.117073

[30] Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? Educational Studies in Mathematics, 26(2–3), 165–190. https://doi.org/10.1007/BF01273662

[31] Plaxco, D., & Wawro, M. (2015). Analyzing student understanding in linear algebra through mathematical activity. Journal of Mathematical Behavior, 38, 87–100. https://doi.org/10.1016/j.jmathb.2015.03.002

[32] Putri, R. A., & Susiswo. (2020). Analysis of layer of primitive knowing of high school students in linear function material: A study of application of student activity sheets based on Pirie Kieren theory. The 3rd International Conference on Mathematics and Sciences Education (ICoMSE) 2019, 2215. https://doi.org/10.1063/5.0000525

[33] Quintanilla, A., & Gallardo, J. (2022). Interpreting the relationship between emotions and understanding in mathematics : An operational approach applied to measurement with preservice elementary teachers. 68(October).

[34] Rahayuningsih, S., Sa'dijah, C., Sukoriyanto, & Qohar, A. (2022). Exploring students' understanding layers in solving arithmagon problems. Cakrawala Pendidikan, 41(1), 170–185. https://doi.org/10.21831/cp.v41i1.33837

[35] Risley, R., Hodkowski, N. M., & Tzur, R. (2015). Jake'S Conceptual Operations in Multiplicative Tasks: Focus on Number Choice. Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 181–188.

[36] Rupalestari, D., Juandi, D., & Jupri, A. (2021). The creative thinking process of junior high school students in solving mathematical problems. Journal of Physics: Conference Series, 1806(1). https://doi.org/10.1088/1742-6596/1806/1/012116

[37] Sengul, S., & Yildiz, S. G. (2016). An Examination of the Domain of Multivariable Functions Using the Pirie-Kieren Model. Universal Journal of Educational Research, 4(7), 1533–1544. https://doi.org/10.13189/ujer.2016.040706

[38] Sidi, Y., Torgovitsky, I., Soibelman, D., Miron-Spektor, E., & Ackerman, R. (2020). You may be more original than you think: Predictable biases in self-assessment of originality. Acta Psychologica, 203(December 2019), 103002. https://doi.org/10.1016/j.actpsy.2019.103002

[39] Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. Zentralblatt Für Didaktik Der Mathematik, 29(3), 75–80. https://doi.org/10.1007/s11858-997-0003-x

[40] Siswono, T. Y. E. (2010). Leveling Student's Creativity in Solving and Posing Mathematical Problem. Journal on Mathematics Education, 1(1), 17–40.

[41] Sitorus, J., & Masrayati. (2016). Students' creative thinking process stages: Implementation of realistic mathematics education. Thinking Skills and Creativity, 22, 111–120. https://doi.org/10.1016/j.tsc.2016.09.007

[42] Spencer, M., Fuchs, L. S., & Fuchs, D. (2019). Language-related longitudinal predictors of arithmetic word problem solving: A structural equation modeling approach. Contemporary Educational Psychology, 60, 101825. https://doi.org/10.1016/j.cedpsych.2019.101825

[43] Stadler, M., Herborn, K., Mustafić, M., & Greiff, S. (2020). The assessment of collaborative problem solving in PISA 2015: An investigation of the validity of the PISA 2015 CPS tasks. Computers and Education, 157(June). https://doi.org/10.1016/j.compedu.2020.103964

[44] Syahrin, A., Dawud, Suwignyo, H., & Priyatni, E. T. (2019). Creative thinking patterns in student's scientific works. Eurasian Journal of Educational Research, 2019(81), 21–36. https://doi.org/10.14689/ejer.2019.81.2

[45] Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. Learning and Instruction, 37, 5–13. https://doi.org/10.1016/j.learninstruc.2014.03.002

[46] Walle, J. A. Van de, Karp, K. S., & Bay-Williams, J. M. (2019). Elementary and Middle School Mathematics Tenth Edition: Teaching Developmentally.

[47] Weber, K. (2009). How syntactic reasoners can develop understanding, evaluate conjectures, and generate counterexamples in advanced mathematics. Journal of Mathematical Behavior, 28(2–3), 200–208. https://doi.org/10.1016/j.jmathb.2009.08.001

[48] Wessels, H. (2014). Levels of mathematical creativity in model-eliciting activities. Journal of Mathematical Modelling and Application, 1(9), 22–40

[49] Xu, W., Geng, F., & Wang, L. (2022). Relations of computational thinking to reasoning ability and creative thinking in young children: Mediating role of arithmetic fluency. Thinking Skills and Creativity, 44, 101041. https://doi.org/https://doi.org/10.1016/j.tsc.2022.101041

[50] Yao, X., & Manouchehri, A. (2022). Folding back in students' construction of mathematical generalizations within a dynamic geometry environment. Mathematics Education Research Journal, 34(2), 241–268. https://doi.org/10.1007/s13394-020-00343-w