ACTA DIDACTICA NAPOCENSIA

Volume 16, Number 1, 2023 - DOI: 10.24193/adn.16.1.10

RECURRENT AND LINEAR SEQUENCES: A THREE-DIMENSIONAL VISUALIZATION WITH SUPPORT OF GEOGEBRA SOFTWARE

Francisco Régis Vieira ALVES, Carla Patrícia Souza Rodrigues PINHEIRO, Renata Teófilo DE SOUSA, Paula Maria Machado Cruz CATARINO

Abstract: This article is an excerpt from a master's thesis developed in Brazil, in which we approach recurrent and linear sequences, given some intriguing particularities in their definitions and the scarcity of discussion of this topic in the literature of the History of Mathematics, especially with regard to its geometric representation. Thus, we aim to present the identities of Fibonacci, Lucas, Jacobsthal and Padovan in a three-dimensional visualization with the contribution of GeoGebra software. The research methodology chosen was bibliographical, exploratory in nature, where we have theoretical support in works such as Oliveira and Alves (2019), Silva (2017), Souza and Alves (2018), Vieira and Alves (2020). This research brings as results a set of geometric constructions of the identities of the proposed sequences, in three-dimensional perspective, being a support for future works developed around this theme. GeoGebra was essential in the process of constructing and visualizing the sequences, as it provided strategies for understanding the recurrence relations and the properties of the Fibonacci, Lucas, Jacobsthal and Padovan sequences, through the behavior of the visual representations of these identities.

Key words: Recurrent and linear sequences; GeoGebra; Geometric visualization.

1. Introduction

The recurrent and linear sequences are explored in several areas such as Arts, Computing, Science, Mathematics, among others, which denotes a high interest of the scientific community for this subject. The Fibonacci sequence is the most explored by authors of the History of Mathematics, due to its eternalization with the problem of immortal rabbits, which is explored several times in a brief and trivial way. According to Alves (2017), readers cannot trivially understand the evolution of other sequences through the Fibonacci sequence, even though this sequence is the basis for building the others.

The Fibonacci numbers, through unlimited sources, are represented by several algebraic identities, becoming an important sequence for the development of Mathematics (Alves *et al*., 2020). Faced with research such as Oliveira and Alves (2019), Silva (2017), Souza and Alves (2018), Vieira and Alves (2020), which address studies on the generalization of Fibonacci, Lucas, Jacobsthal and Padovan sequences, respectively, there was a need to explore the geometric form of identities related to these sequences, a non-trivial component that was still little explored in previous research. Thus, the objective of this work is to present the identities of Fibonacci, Lucas, Jacobsthal and Padovan in three-dimensional visualization with the contribution of the GeoGebra software.

Alves (2022a) reinforces the importance of some elements for mathematical learning regarding recurrent and linear sequences, which are “the ability to visualize and identify geometric patterns, the adoption of a mathematical notational system consistent with algebraic, arithmetic and geometric invariants involved in each problem and the ability to understand and solve problems” (p. 322).

Other studies such as Alves and Catarino (2017), Mangueira (2022) and Vieira et al. (2019) also point out the lack of exploration of these sequences in the context of the History of Mathematics within undergraduate degrees in Brazil, both with regard to mathematical discussion and their exploration through technological resources. Alves (2022a, 2022b) points out that currently, the History of

Received January 2023.

Cite as: Alves, F. R. V., Pinheiro, C. P. S. R., De Sousa, R. T., & Catarino, P. M. M. C. (2023). Recurrence and linear sequences: a three-dimensional visualization with support of GeoGebra software. *Acta Didactica Napocensia,* 16(1), 140-156, https://doi.org/10.24193/adn.16.1.10

Mathematics is still studied separately from Mathematics itself, disregarding it as the "conductor thread" of an entire epistemological and evolutionary process. Thus, these authors bring research that explore the discussion of these recursive linear sequences, as well as the possibilities of construction and visualization of these sequences in different perspectives, with the contribution of the GeoGebra software.

Generally, the usual approach to study these numerical sequences is algebraic and/or combinatorial representations. The differential of this work in relation to the other research carried out is the exploration of the relation between the 2D and 3D perspectives, their construction, manipulation, and visualization as a contribution to the mathematical understanding and the formation of the undergraduate student in Mathematics.

Starting from the bibliographical study carried out, we bring a research question that supports the construction of these representations: How can the proposed identities be visualized geometrically? From a geometric perspective, considering the three-dimensional plane, we propose the presentation of possible relationships with the Fibonacci, Lucas, Jacobsthal and Padovan sequence through the constructions carried out in the 3D plane with the contribution of the GeoGebra software, as it is a dynamic and easily manipulated tool, which enables the visual component, which is essential to the development of this work.

According to Díaz-Urdaneta, Kalinke and Motta (2019) GeoGebra allows a differentiated approach, enabling the presentation of several Mathematics topics in a single interface, as well as the visualization of these in an algebraic, geometric, and arithmetic way, expanding the possibilities of exploration and development of the student's mathematical thinking.

Based on the above, in the following sections we present the definition of recurrent and linear sequence, as well as a brief historical context about the concepts and identities of the Fibonacci, Lucas, Jacobsthal and Padovan sequences and the respective geometric constructions in a three-dimensional view with the contribution of the GeoGebra, ending with our considerations.

2. Recursive and linear sequences

To start the discussion, let us consider $\mathbb{N} = \{1, 2, 3, 4, ...\}$, where zero does not belong to this set of natural numbers. We use the notation $(x_1, x_2, \dots, x_n, ...)$ to define the sequence, that is, (x_n), $n \in \mathbb{N}$. Thus, $1 \to x_1$, $2 \to x_2$, ..., $n \to x_n$ characterizes the function where each natural number *n* corresponds to a real number x_n.

Matos (2001) reiterates that the sequence is nothing more than an infinite ordered list of *n* real numbers $(x_1, x_2, \dots, x_n, ...)$, in which x_1 is the first term and x_n is the n-th term. With an infinite order, each term x_n has a successor x_{n+1}, and this sequence can be characterized by its general term.

Several sequences are known as recursive, that is, they have a recurrence law. According to Lima (2006) and Pinto (2017), a law of recurrence determines a rule that allows calculating any term x_{n+1}, from the previous term x_n, for all $n \in \mathbb{N}$. Another important definition complements that, for a sequence to be completely defined by a recurrence relation, it is also necessary to inform the first terms from which the others will be obtained.

We realize, through these definitions, that to obtain a sequence of recurrence we must have at least the first term. And from that the sequences can be constructed and classified according to order, linearity, and homogeneity. We define each type of classification in the subsequent items, according to the work of Barros (2021):

- Regarding the *order* of a recurrence equation, we consider the difference between the highest and lowest indices of the terms of its sequence.
- As for *linearity*, when the sequence has a k order linear recurrence equation written in the form: $x_{n+k} = f_1(n)x_{n+k-1} + f_2(n)x_{n+k-2} + \cdots + f_k(n)x_n + f_{k+1}(n)$, in which, $f_i(n)$ is a function n with $i \in \mathbb{N}$ and $1 \leq i \leq k+1$, and $f_k \neq 0$.

- As for *homogeneity,* we have that recurrence is determined when each term depends exclusively on the previous ones. Which implies when a recurrence is classified as non-homogeneous, each term depends on the previous terms and is a function of an independent term.

Based on the above, in the following section we discuss the sequences of Fibonacci, Lucas, Jacobsthal and Padovan from their epistemic-mathematical field, to later present a visual model with their geometric constructions, in a three-dimensional perspective, with the contribution of GeoGebra.

3. Epistemic-mathematical field of recursive and linear sequences

In this topic, the sequences of Fibonacci, Lucas, Jacobsthal and Padovan are presented, in a historical context, establishing a relationship between the definitions and essential identities for a geometric construction through a computational tool.

3.1 Fibonacci sequence

The Fibonacci sequence was developed from the studies of Leonardo Pisano (1180-1250), a mathematician better known as Fibonacci, or "son of Bonaccio". Born in Italy, Leonardo acquired mathematical knowledge in the Arab world, in the areas of Algebra and Arithmetic, however, he was immortalized in history by the problem of reproduction of immortal rabbits, which conceives the sequence that bears his name, the Fibonacci sequence (Santos, 2017).

According to Koshy (2001), the Fibonacci numbers fascinated everyone for centuries, and continue to enchant with their beauty and applications, which can occur in the most varied fields. The magazine *The Fibonacci Quarterly*, published for the first time in 1963, is dedicated to the study of the properties of this sequence. Its definition is:

Definition 1: The Fibonacci sequence F_n has as fundamental recurrence, for any n ∈ ℤ and initial values $F_0 = 0, F_1 = 1, F_2 = 1$ indicated by: $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$.

From the generalization of definition 1, we can also rewrite it as follows: $F_n = xF_{n-1} + yF_{n-2}$, para $n \geq 2,$ for all n ∈ ℤ. Accordingly, according to arithmetic and linear recurrence, further numerical sequences can be explored, which bear a resemblance to this mathematical formula. Performing elementary algebraic operations, it is possible to present one of the properties related to this sequence:

Identity 1: $F_{n+1}^3 = F_n^3 + F_{n-1}^3 + 3F_{n-1}F_nF_{n+1}$, for $n \geq 1$.

The choice of this particular identity was due to the possibility of its three-dimensional representation for a visual exploration through the GeoGebra software, making it possible to reach the proposed objective. With the same purpose, we chose and explained the other sequences and their identities in the subsequent paragraphs.

3.2 Lucas sequence

The Lucas sequence is closely related to the Fibonacci sequence. We have that the first numbers of Lucas are {1, 3, 4, 7, 11, 18, 29, ...} and which can be constructed by what is presented in Definition 2:

Definition 2: $L_n = L_{n-1} + L_{n-2}$, para $n \geq 3$.

When necessary, we use $L_0 = 2$. This sequence is named after its avant-garde François Eduard Anatole Lucas (1842-1891), a French mathematician who made many contributions such as the creation of the Tower of Hanoi, the proof of Fermat's Theorem and several tests for prime numbers, based on sequences. recurrences and linearity.

Lucas discovered the twelfth Mersenne prime, consisting of 39 digits, a number that remained for many years as the largest prime number found. This prime number was considered the largest number found without the aid of technological or computational resources (Eves, 1969).

We observe that the Lucas sequence has recursion similarity with the Fibonacci sequence, in which many identities are similar. Therefore, we chose the identity 2 for the construction of the geometric representation using the Lucas numbers:

Identity 2: $L_{n+1}^3 = L_n^3 + L_{n-1}^3 + 3L_{n-1}L_nL_{n+1}$, for $n \geq 1$.

This property allows the geometric construction through prisms, in which the Lucas numbers are associated with the edges of the lateral faces that are parallelograms.

3.3 Jacobsthal sequence

The Jacobsthal sequence is a second-order sequence with recurrence that resembles the Fibonacci sequence. Its name is related to the German mathematician Ernest Erich Jacobsthal (1882-1965). According to Siegmund-Schultze (2009), Jacobsthal was a former student of Ferdinand G. Frobenius, also a specialist in Number Theory and who was one of the first to study Fibonacci polynomials.

For geometric constructions we propose definitions and theorems that are explored throughout this topic. Thus, we have Definition 3 to support the Jacobsthal sequence:

Definition 3: The Jacobsthal sequence J_n has as fundamental that, for all $n \in \mathbb{Z}$, the initial values are $J_0 = 0,\ J_1 = 1$, indicated by: $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$.

This sequence $J_n =$ {1, 1, 3, 5, 11, 21, 43, ...} linear and recursive is considered a particular feature of Lucas sequence. There are mathematical properties being developed around this sequence. We can highlight, in turn, its vast usefulness to solve mathematical problems related to combinatorial analysis.

According to Koshy (2019), there is one more definition that helps to understand the proposed identities for the geometric construction based on the Jacobsthal numbers, which is:

Definition 4: $J_n(x) = J_{n-1}(x) + x.J_{n-2}(x)$ for the initial values $J_0(x) = 0$ and $J_1(x) = 1$.

Definition 4 is a polynomial representation of the Jacobsthal sequence, which makes a preliminary association with a board to a one-dimensional tile of the type $1\ x\ n$, with the following representations: a square $1\ x\ 1$ has weight 1, a rectangle $1\ x\ 2$, called domino has weight x. These tilings can be represented by the following Theorems 1 and 2:

Theorem 1. Suppose the weight of a square is 1 and that of a domino is x. Then the sum of the weights of the boards of length n is $J_{n+1}(x)$, in which $n \geq 0$.

Based on a particular examination, let's generalize the count of tiles that fill a board $2\ x\ n$, with the Theorem 2:

Theorem 2. Be a board $2\ x\ n$, and the horizontal dominoes $1\ x\ 2$ and vertical $2\ x\ 1$, both with weight 1, black squares $2\ x\ 2$, with weight x. So, the sum of the weights of lenghts n, will be $S_n^2 = J_{n+1}$, for $n \geq 0$ (Koshy, 2019).

According to Craveiro's thesis (2004), there are also some possibilities for the representation of tiling as $3\ x\ 1, 3\ x\ 2, 3\ x\ 3, 3\ x\ 4$, and others, using squares $1\ x\ 1$ and $2\ x\ 2$ to define the total number of tiles possible for a board $3\ x\ n$. These possibilities are supported by Theorem 3:

Theorem 3: Considering a board $3\ x\ n$, with only two types of tiles: a tile $1\ x\ 1$ in orange color and a tile $2\ x\ 2$ in brown color. So, J_n for the number of possible tiles for the board is determined by $j_n = J_n$ for $n \geq 1$.

Thus, we can consider this a mathematical problem proposed and solved by Craveiro (2004) in relation to tiles, in which Jacobsthal numbers are used: the number of pieces q_n that can be calculated for a rectangle $(3\ x\ n)$ using two types of tiles, one of dimensions $(1\ x\ 1)$, in the orange color and the other $(2\ x\ 2)$ in the brown color, relating to the Jacobsthal numbers.

Identity 3: Defining q_0= 0 for a rectangle $(3\ x\ 1)$, we have for a rectangle $(3\ x\ 2)$, $q_2 = 3$, that is, three types of possible tiles; for a rectangle $(3\ x\ 3)$, $q_3 = 5$, following the Jacobsthal sequence.

Based on this identity, we can use the GeoGebra software to visualize this property in a three-dimensional perspective.

3.4 Padovan sequence

Conceived by the Italian Richard Padovan (1935-?), the Padovan sequence was named after the discovery by the architect Hans van Der Laan (Voet & Schoonjans, 2012), and was later analyzed by the mathematician Gérard Cordonnier (1907-1977) who contributed to the knowledge these numbers (Voet & Schoonjans, 2012). This sequence is considered as a "cousin" of the Fibonacci sequence (Alsina & Nelsen, 2015).

Vieira and Alves (2019) state that the Padovan sequence is a third-order, recurrent and linear sequence, that is, previous terms are needed to calculate the next term, in which the initial terms are represented by $P_0 = 0, P_1 = P_2 = P_3 = 1$. Therefore, the sequence can be represented by Definition 5:

Definition 5. $P_n = P_{n-2} + P_{n-3}\,, n \geq 3.$

The Padovan sequence $P_n = \{1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \ldots\}$, brings numerous properties that are currently being studied. However, for its geometric representation in GeoGebra we will focus specifically on Theorem 4:

Theorem 4. $\sum_{i=0}^{n} P_i^2 P_{i+1} = P_n . P_{n+1} . P_{n+2}$

Theorem 4 brings the perspective of a geometric construction through prisms, associated with the Padovan numbers. This identity was chosen due to the ease of observing the geometric form to develop some algebraic conjectures. Thus, initially thinking that this property is related to volume, we built its 3D visualization using GeoGebra.

In the next section, we present the step-by-step construction of geometric shape identities, core of this study.

4. The three-dimensional visualization of identities

For the development of the geometric form of the identities proposed in this research, we used constructions that bring visual approaches, being this model of approach important for the increment of the understanding of these identities, considering that the researches around this theme occur in an approach that advocates the algebraic development.

4.1 Fibonacci sequence

Initially, we adopted $n = 4$, for better figure building. With the substitution of the value for n we have the formula $F_5^3 = F_4^3 + F_3^3 + 3F_3F_4F_5$. In this step, we relate each term of identity 1 to the term of the Fibonacci sequence. The visualization can be constructed from the construction of an edge cube measuring 2 units – this measure relates to F_3 in Fibonacci sequence. For this construction, we select the Cube tool and, on the plane, we select the points that define the edge, in this case it will be 0 to 2 on the x axis, as shown in Figure 1:

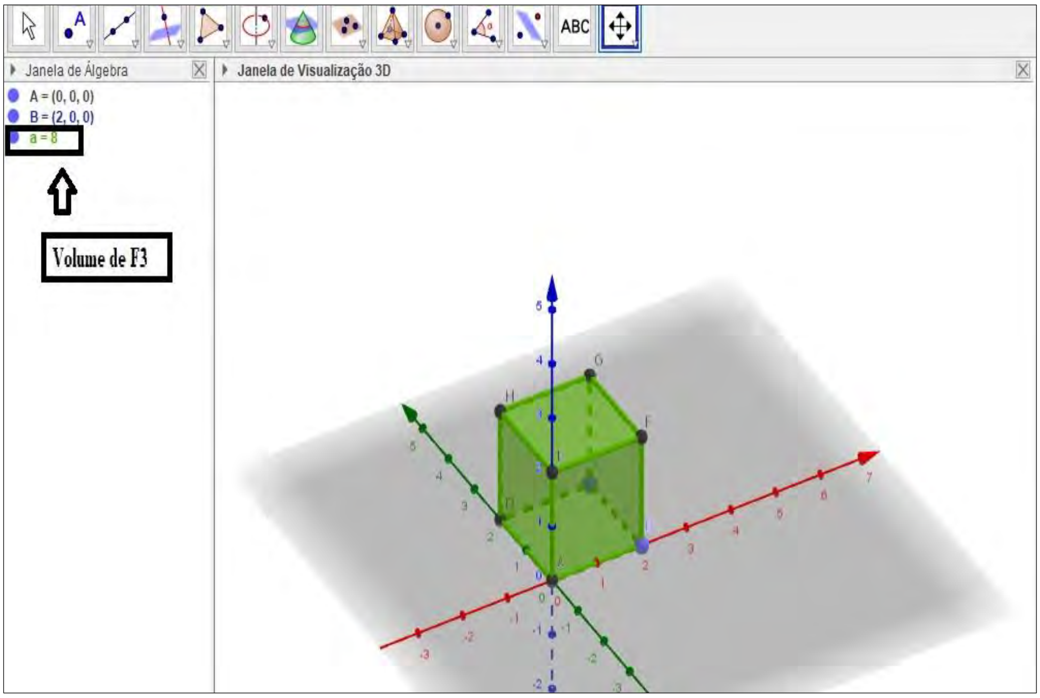

Figure 1. *Construction for* F_{n-1}^3

In Figure 1, notice that the volume value for F_3 is equal 8. In the second step we must construct a rectangular prism, with measures equal to the three consecutive terms of the sequence $F_3F_4F_5$. Therefore, the values for height, width and length will be 2, 3 and 5 respectively (Figure 2):

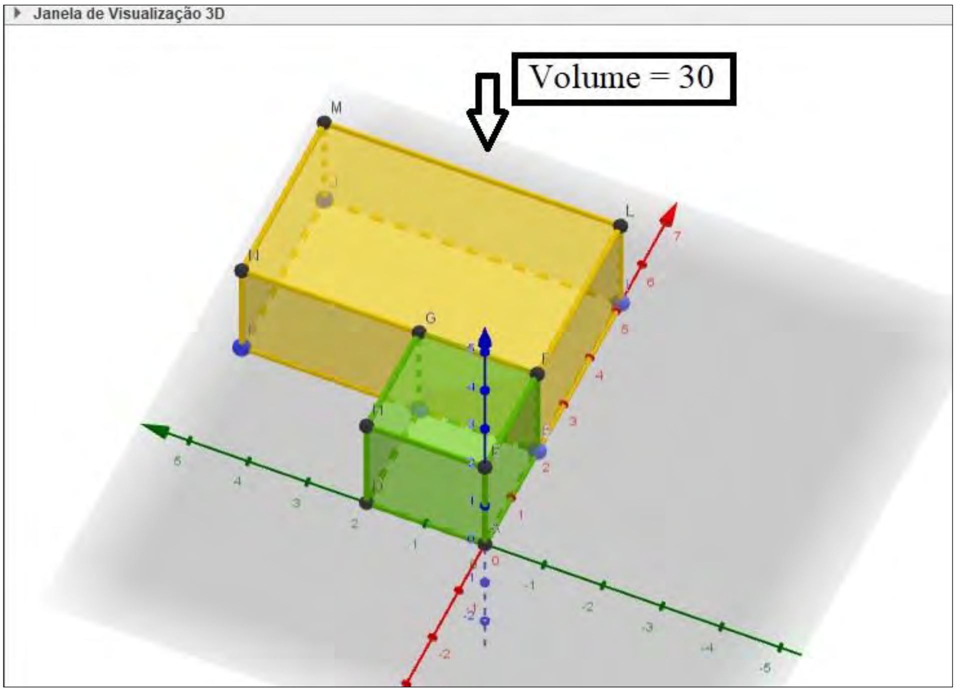

Figure 2. *Construction for* $F_{n-1}F_nF_{n+1}$

For the construction of the rectangular prism, we choose the prism tool and build its base with a rectangle of dimensions 3 and 5 and click on the point that should be its height, which in the case of this construction has a value equal 2.

Analogously to the first step, the construction of the cube corresponding to the F_4. According to Figure 3, this has an edge value equal to 3, which in turn is related to the terms of the Fibonacci sequence:

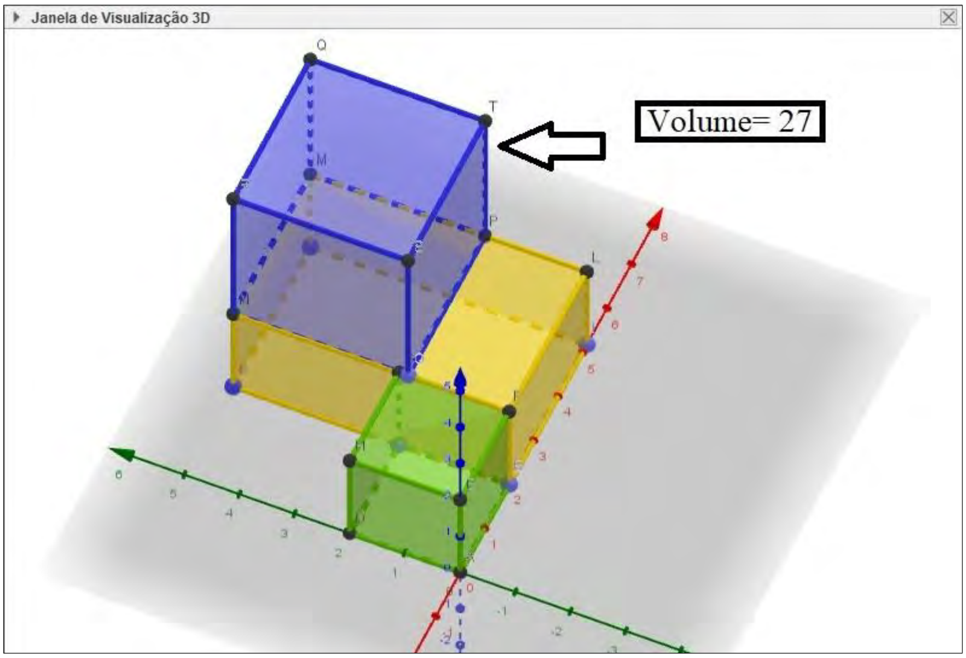

Figure 3. *Construction for* F_n^3

In the same way, following the second step, we build two rectangular prisms of the same volume, with the same measurements used previously, to continue the image, as shown in Figure 4:

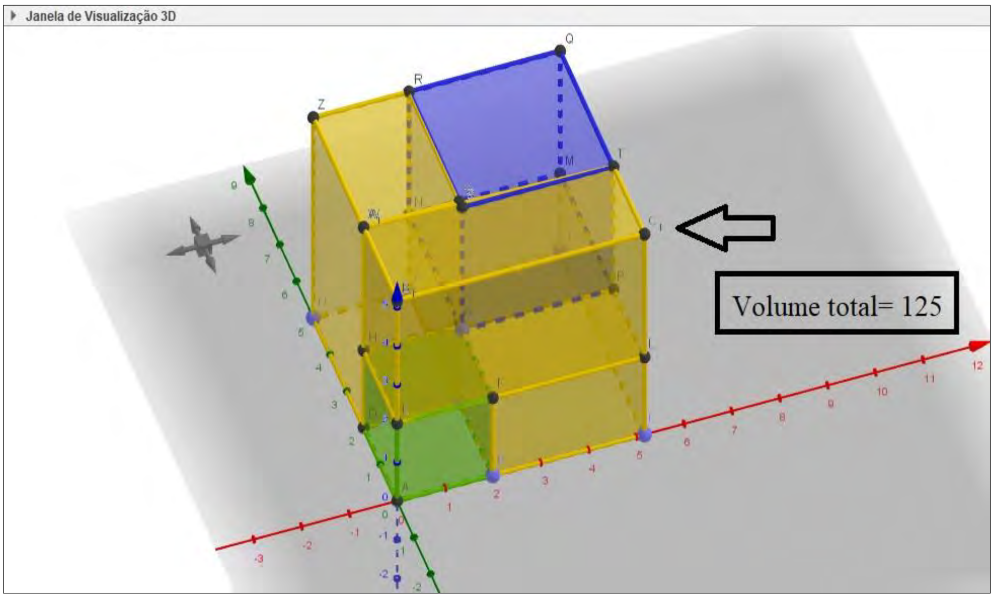

Figure 4. *Construction of identity 1*

Note in Figure 4 that the total volume of the built prism corresponds to the F_5^3, which is relative to the volume of the fifth term of the Fibonacci sequence. If the sequence is followed, we can have a visualization of other terms, geometrically representing this property in a three-dimensional perspective.

4.2 Lucas sequence

For the presentation of identity 2, we start using the value $n = 1$, for better visualization of the construction. Then, with the assignment of the value to n, we have the formula $L_2^3 = L_1^3 + L_0^3 + 3L_0L_1L_2$. In this step, we can relate an element of identity 2 to a term of the Lucas sequence. Carrying out the construction in GeoGebra and using the same tools and steps mentioned above, we obtain Figure 5:

Figure 5. *Description of the volumes of each term of the sum*

Figure 5 shows the volumes that correspond to the elements of identity 2 built. In another perspective, in the 3D window of GeoGebra, we can observe the value of the edge of the purple cube, corresponding to the term L_1; the edge of the yellow cube, associated with the term L_0; and the results that are in green color, which correspond to the volume of the three prisms formed with the height, width and length measurements, relative to the three consecutive terms $L_0L_1L_2$. Figure 6 shows the construction of identity 2 with its total volume:

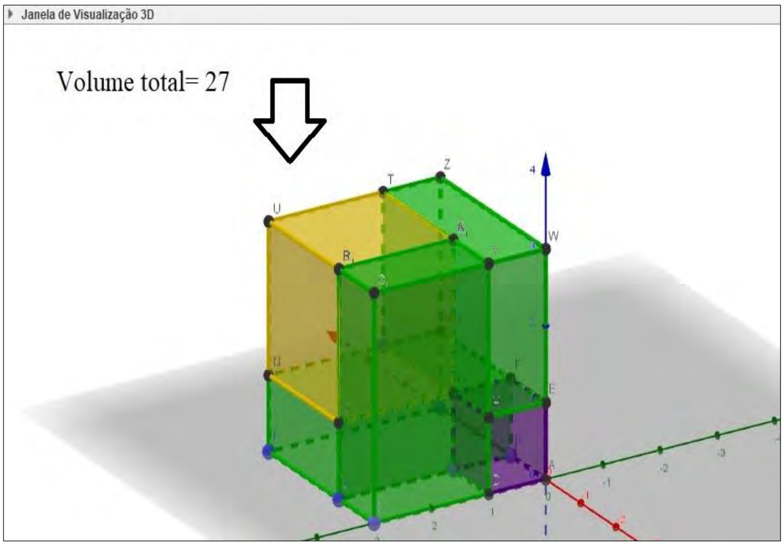

Figure 6. *Construction of identity 2*

In Figure 6, identity 2 can be seen and we have that the total volume is equal to 27, which corresponds to the term L_2^3. Thus, we present a three-dimensional view of Lucas' identity proposed as a support for GeoGebra.

4.3 Jacobsthal sequence

For the construction of identity 3, we must observe the steps for construction, from Theorems 1, 2 and 3.

First, we must search for the "regular polygon" tool in GeoGebra and construct a square with side length 1, and then build the other squares using the proposed theorems to create the tiling. That is, we must insert two squares of side with length 1. Therefore, the next square to be constructed will be the sum of the two values of the two previous edges, which implies a third figure being a square with side 2. We can see this in Figure 7:

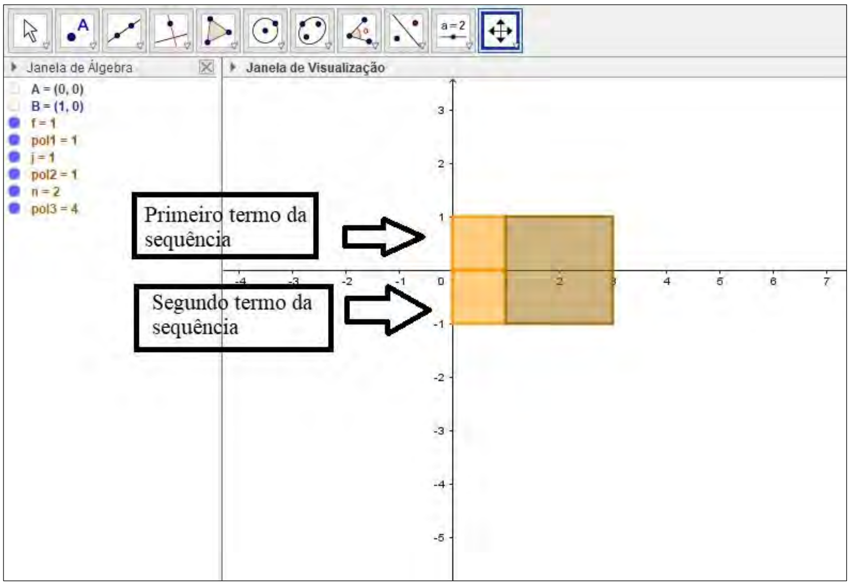

Figure 7. *The initial terms of the Jacobsthal sequence*

In Figure 7, in a two-dimensional view, according to Theorem 2, one thinks of the number of ways to tile a rectangle $3\,x\,n$ with squares of sides $1\,x\,1$ e $2\,x\,2$. Continuing the construction, we have the polynomial representation of the Jacobsthal sequence, as shown in Figure 8:

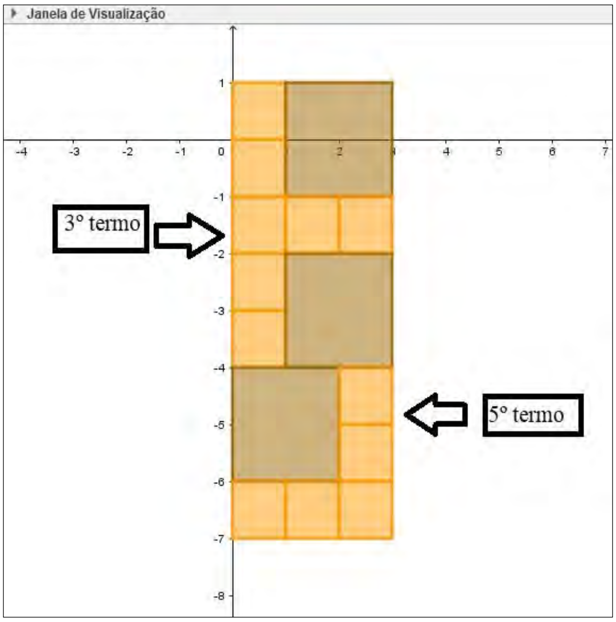

Figure 8. *Next terms of the Jacobsthal sequence*

We observe that one of the sides that corresponds to the Jacobsthal sequence defined in Figure 8 demands even more a construction in a two-dimensional view for the three-dimensional construction. It takes the construction of three terms of the sequence and, thinking of the number of ways we can tile a rectangle $2\ x\ n$, with rectangles $1\ x\ 2$ and squares $2\ x\ 2$.

Therefore, when executing the rectangle construction process using the "polygon" tool, we must select its vertices until completing the figure, finding its initial vertex again. This process must be executed until we find other terms to fill in, as illustrated in Figure 9:

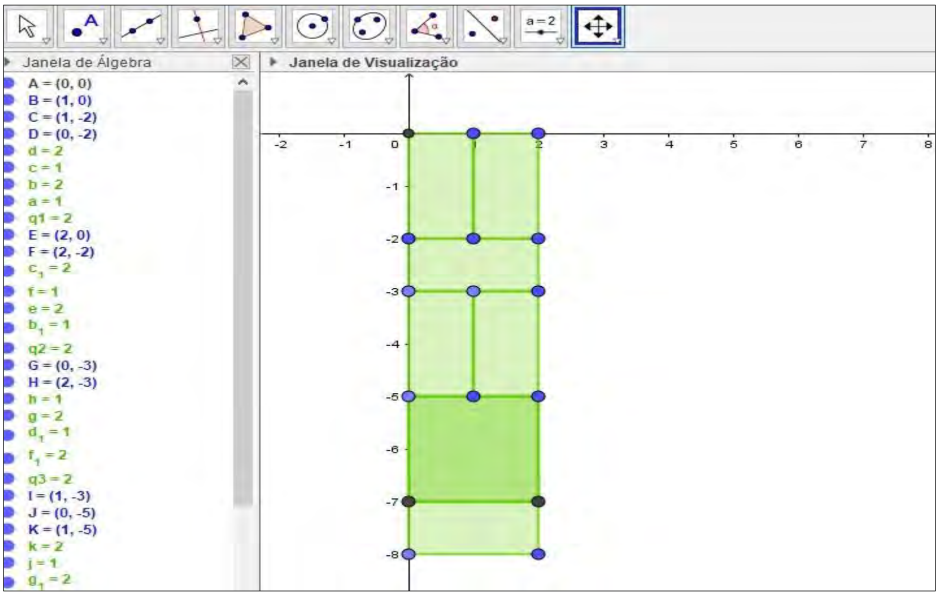

Figure 9. *Construction of the rectangles*

Based on this information and the union of Figures 8 and 9, we created a construction that considers the three-dimensional aspect of these tiles, in which we envision a number of ways to fill a hole with the measurements $2\ x\ 2\ x\ n$ with a figure of measurements $1\ x\ 2\ x\ 2$, according to Figure 10:

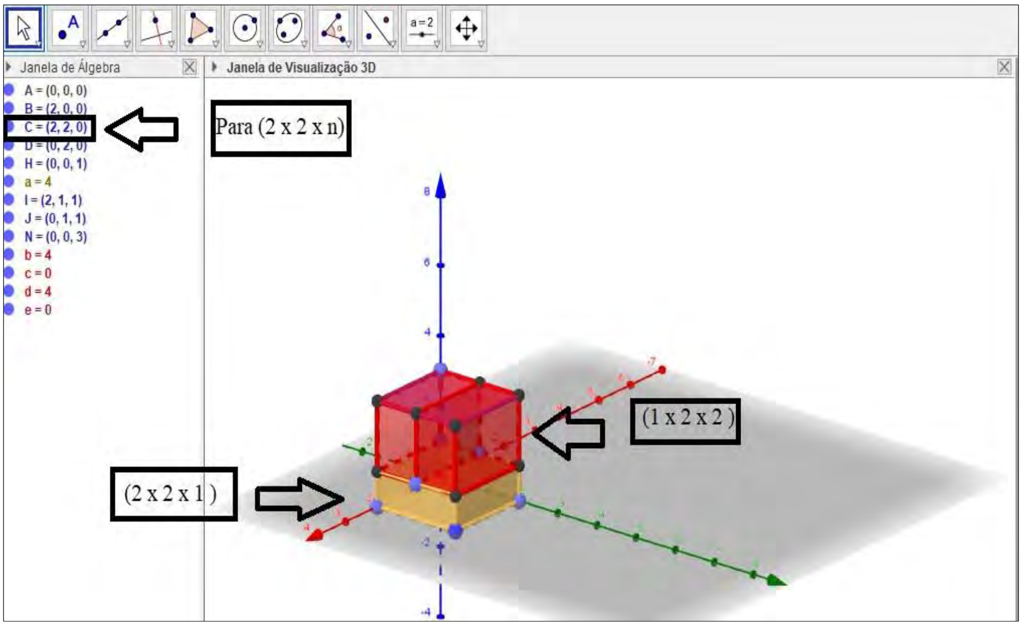

Figure 10. *Initial terms for a three-dimensional visualization*

To build Figure 10, we must select the "prism" tool, as this tool shows us a creation that includes three dimensions (height, width, and length), in which we first delimit the figure from the base. Choosing the points on the x axis, on the yellow prism, we build a square with side 2 and, right after that, move the mouse cursor until we reach the height on the z axis, with measure 1. Continuing the construction process, we build more prisms, as in Figure 11:

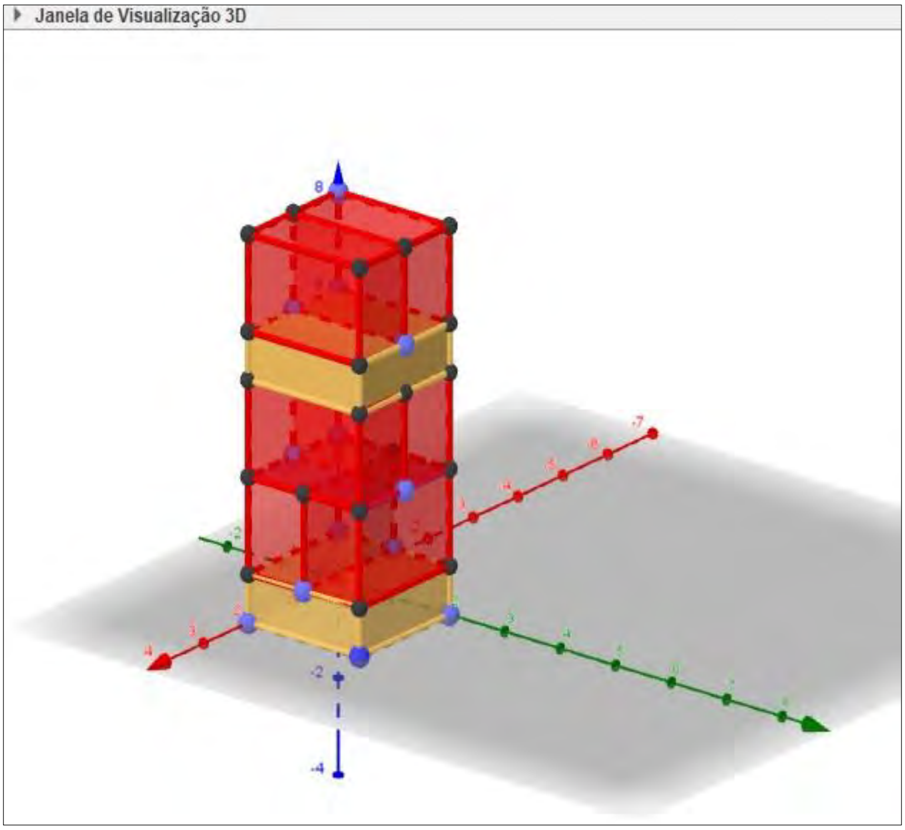

Figure 11. *Jacobsthal sequence in a three-dimensional view*

In Figure 11, we geometrically construct identity 3. By observing the association established between Figures 8 and 9, we can prove that there is an if and only if condition, proving the construction of the Jacobsthal sequence from tiling. Here, it is observed that the computational tool directly helps the construction of mathematical knowledge, being essential for the understanding of these proposed properties.

4.4 Padovan sequence

We start building identity 4 by choosing the "cube" option on the toolbar and defining the value of the edge, which refers to the first term of the Padovan sequence, that is, equal to 1. Then with the same tool we can build the second term, with measures also equal to 1. For the third term, we can choose the "prism" option and, from there, create the image corresponding to the third term, in which the height, width and length measurements are, respectively, 1, 1 and 2, as shown in Figure 12:

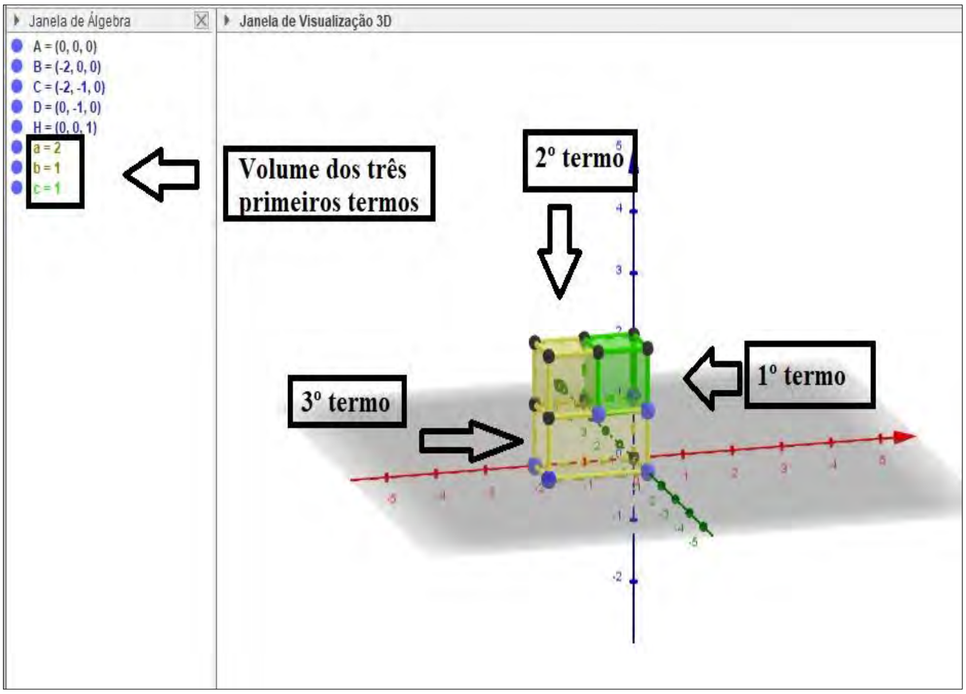

Figure 12. *The first three terms in a 3D visualization.*

Note in Figure 12 that the constructed prisms have volume measurements corresponding to the initial terms and which are also respectively 1, 1 and 2. Using the two tool options mentioned above, analogously, we build two more terms, as shown in Figure 13:

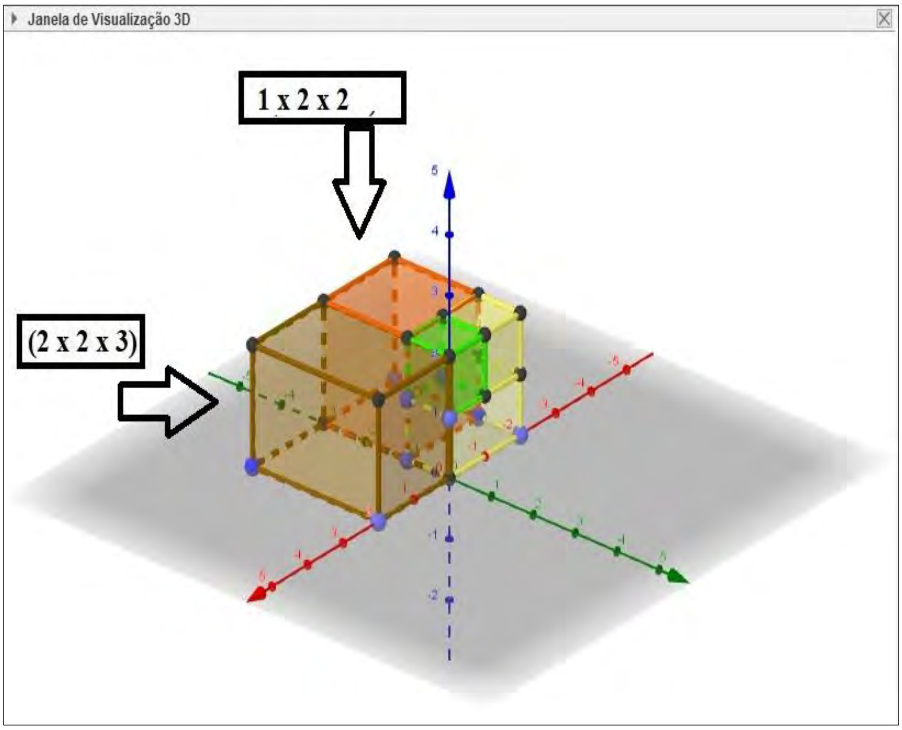

Figure 13. *The first five terms of the Padovan sequence in 3D*

Continuing the construction process and using the same reasoning and tools, we can build the other two subsequent terms, totaling seven prisms, as shown in Figure 14:

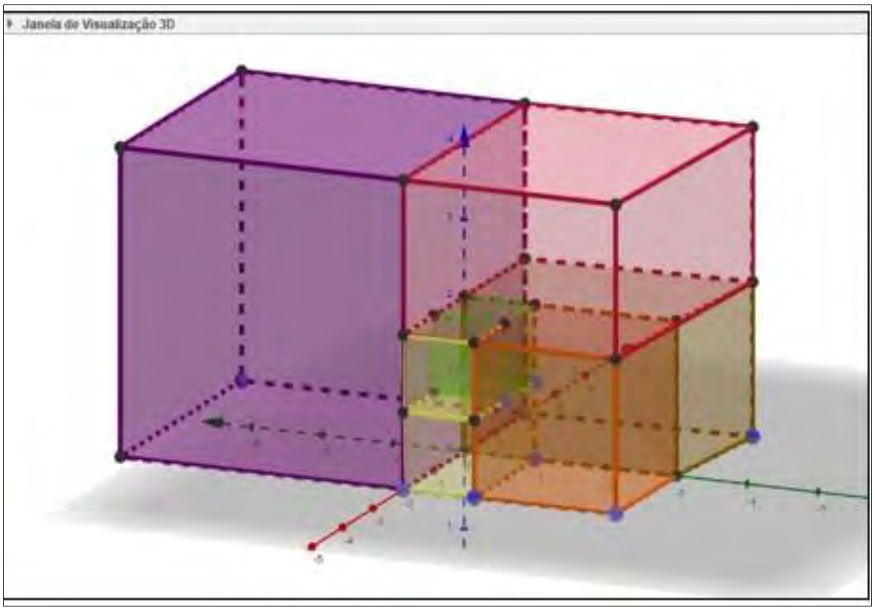

Figure 14. *The first seven terms in the three-dimensional visualization*

In Figure 14 we have steps that point to a geometric representation of identity 4, needing to define the value of n. For all $n \geq 1$, it is understood that this identity results in adding prisms with measures P_i x P_i x P_{i+1} so that your result consists of a larger prism, which explores the component of three-dimensional visualization.

We emphasize that it is extremely important to use GeoGebra for a better understanding of these identities in a bi- and three-dimensional perspective, because by using a visual approach, we can point out important elements that were not observed in the algebraic field. This may allow the teacher or student to trace other paths, in addition to exploring the possibility of demonstrating these properties by other mathematical models.

5. Final considerations

This research aimed to present the identities of Fibonacci, Lucas, Jacobsthal and Padovan in their three-dimensional visualization with the contribution of GeoGebra software. At first, we carried out a bibliographical study on the Fibonacci, Lucas, Jacobsthal and Padovan sequences, where we observed through this survey that these sequences have been little explored during the academic course of Mathematics students in the Brazilian context. In this way, we seek to treat recurrent and linear sequences through a historical context and their mathematical definitions, as a basis for a proposal of geometric constructions that would be a contribution to the study of this subject.

For the construction of the three-dimensional presentations associated with the recursive and linear sequences, we elaborated a three-dimensional proposal of the identities of the Fibonacci, Lucas, Jacobsthal and Padovan sequences, where we observed their characteristics in a geometric view with support from GeoGebra. We emphasize that this software was essential in the process, as the constructions were only possible due to the use of this resource, which enabled a strategy for understanding the proposed mathematical knowledge. In the proposed situation, it is possible to understand the recurrence relation and the properties of the Fibonacci, Lucas, Jacobsthal and Padovan sequences from the behavior of the visual representations of the identities.

Finally, the difficulties and limitations found in this study were the lack of exploration of this subject in initial training in the Brazilian context and the scarcity of constructions and/or geometric representations of the identities of these sequences. Some work has been developed with the aim of contributing to teacher training, with regard to the mathematical basis from its historical context, such as Alves et al.

(2019, 2020) and Alves (2022a, 2022b). In this way, we hope that this research can encourage other works related to the study of different recurrent and linear sequences in Mathematics.

References

Alves, F. R. V. (2017). Didactic Engineering for Jacobsthal's Generalized s-Sequence and Jacobsthal's (s,t)-Generalized Sequence: preliminary and a priori analyzes [Engenharia Didática para a s-Sequência Generalizada de Jacobsthal e a (s,t)-Sequência Generalizada de Jacobsthal: análises preliminares e a priori]. *Revista Iberoamericana de Educación Matemática*, *51*, 83-106. http://funes.uniandes.edu.co/17151/1/Vieira2017Engenharia.pdf

Alves, F. R. V. (2022a). Fibonacci, Tribonacci, etc. sequences and the boards [A Sequência de Fibonacci, Tribonacci, etc. e tabuleiros]. *Boletim GEPEM (Online)*, *80*, 311-323. DOI: 10.4322/gepem.2022.055

Alves, F. R. V. (2022b). Combinatory properties about the Jacobsthal sequence, the notion of the board and some historical notes [Propriedades Combinatórias sobre a sequência de Jacobsthal, a noção de tabuleiro e alguns apontamentos históricos]. *Revista Cearense de Educação Matemática*, *1*, 1-13. DOI: 10.56938/rceem.v1i1.3146

Alves, F. R. V., Vieira, R. P. M., Silva, J. G. & Mangueira, M. C. S. (2019). Didactic Engineering for teaching the Padovan sequence: a study of the extension to the field of integers [Engenharia Didática para o ensino da sequência de Padovan: um estudo da extensão para o campo dos números inteiros]. In: Gonçalves, F. A. M. (Ed.). *Science Teaching and Mathematics Education* [Ensino de Ciências e Educação Matemática]. chapter 2. Atena Editora.

Alves, F. R., Catarino, P. M., Vieira, R. P. & Mangueira, M. C. (2020). Teaching recurring sequences in Brazil using historical facts and graphical illustrations. *Acta Didactica Napocensia*, *13*(1), 87-104, https://doi.org/10.24193/adn.13.1.9

Alves, F. R. V. & Catarino, P. M. M. C. (2017). The Bivariate Fibonacci Polynomals' class (BFP): recent elements on the evolution of a model [A classe dos polinômios bivariados de Fibonacci (PBF): elementos recentes sobre a evolução de um modelo]. *Revista Thema*, *14*(1), 112-136. https://doi.org/10.15536/thema.14.2017.112-136.425

Barros, F. E. (2021). *Didactic Engineering on K-dimensional figurative numbers: recurrent relations, general term, generating function and representations through computational resources* [Engenharia Didática sobre os números figurados K-dimensionais: relações recorrentes, termo geral, função geradora e representações por meio de recursos computacionais]. Master's dissertation (Mestrado em Ensino de Ciências e Matemática). Instituto Federal de Educação, Ciência e Tecnologia do Ceará, Fortaleza, Brasil.

Craveiro, I. M. (2004). *Combinatorial Extensions and Interpretations for Fibonacci, Pell, and Jacobsthal Numbers* [*Extensões e Interpretações Combinatórias para os Números de Fibonacci, Pell e Jacobsthal*]. 117 f. Doctoral thesis. Instituto de Matemática, Estatística e Computação Científica, UNICAMP, Campinas, São Paulo.

Díaz-Urdaneta, S., Kalinke, M. A., & Motta, M**.** (2019). The Didactic Transposition in the elaboration of a learning object in GeoGebra [A Transposição Didática na elaboração de um objeto de aprendizagem no GeoGebra]. *#Tear - Revista de Educação, Ciência e Tecnologia*, *8*(2), 1-12, https://doi.org/10.35819/tear.v8.n2.a3503

Eves, H. (1969). *An introduction to the History of Mathematics*. Third edition. Holt, Hinehart and Winston.

Gil, A. C. (2008). *Methods and techniques of social research* [Métodos e técnicas de pesquisa social]. 6th edition. Atlas.

Koshy, T. (2001). *Fibonacci and Lucas numbers with applications*. Wiley.

Koshy, T. (2019). *Pell and Pell-Lucas numbers with applications II.* Wiley.

Lima, E. L., Carvalho, P. C. P., Wagner, E. & Morgado, A. C. (2006). The High school Mathematics 2 [*A matemática do Ensino Médio 2*]. Coleção do Professor de Matemática, SBM.

Mangueira, M. C. S. (2022). *Didactic Engineering: A Hybridization and Hypercomplexification process of recursive linear sequences* [Engenharia Didática: Um processo de Hibridização e Hiper complexificação de sequências lineares recursivas]. Master's dissertation (Mestrado em Ensino de Ciências e Matemática). Instituto Federal de Educação, Ciência e Tecnologia do Ceará, Fortaleza, Brasil.

Matos, P. M. (2001). *Series, E*. Equações Diferenciais. Prentice Hall.

Oliveira, R. R. & Alves, F. R. V. (2019). Gaussian Fibonacci numbers and two-dimensional recurrent relationships [Os números Gaussianos de Fibonacci e relações recorrentes bidimensionais]. *Revista Thema*, *16*(4), 745-754. https://doi.org/10.15536/thema.V16.2019.745-754.1333

Pinto, F. L. (2017). *The use of recurrences and recursive reasoning in high school* [O uso das recorrências e do raciocínio recursivo no Ensino Médio]. Instituto de Matemática: Departamento de Matemática. Master's dissertation (Mestrado Profissional em Matemática em Rede Nacional). Universidade Federal da Bahia, Salvador, Brasil. http://repositorio.ufba.br/ri/handle/ri/22971

Santos, A. A. (2017). *Didactic Engineering on the Study and Teaching of the Binet Formula as a model for generalization and extension of the Fibonacci Sequence* [Engenharia Didática sobre o Estudo e Ensino da Fórmula de Binet como modelo de generalização e extensão da Sequência de Fibonacci]. Master's dissertation (Mestrado em Ensino de Ciências e Matemática). Instituto Federal de Educação, Ciência e Tecnologia do Ceará, Fortaleza, Brasil.

Siegmund-Schultze, R. (2009). *Mathematicians fleeing from Nazi Germany*: individual fates and global impact. Princeton University Press.

Silva, B. A. E. (2017). *Fibonacci Numbers and Lucas Numbers* [Números de Fibonacci e números de Lucas]. Master's dissertation (Mestrado Profissional em Matemática em Rede Nacional). Universidade de São Paulo, São Carlos, Brasil.

Souza, T. S. A., & Alves, F. R. V. (2018). Didactic Engineering as a methodological instrument in the study and teaching of the Jacobsthal sequence [Engenharia Didática como instrumento metodológico no estudo e no ensino da sequência de Jacobsthal]. *#Tear: Revista de Educação, Ciências e Tecnologia*, *7*(2), 1-19. https://doi.org/10.35819/tear.v7.n2.a3119

Vieira, R. P. M, & Alves, F. R. V. (2020). Didactic Engineering and the sequence of Padovan and Tridovan: a preliminary and a priori analysis [Engenharia Didática e sequência de Padovan e Tridovan: uma análise preliminar e a priori]. *Unión - Revista Iberoamericana de Educación Matemática*, *59*, 227-251. https://union.fespm.es/index.php/UNION/article/view/63

Vieira, R. P. M., Catarino, P. M. M. C. & Alves, F. R. V. (2019). Two-dimensional relations and identities of Leonardo's sequence [Relações bidimensionais e identidades da sequência de Leonardo]. *Revista Sergipana de Matemática e Educação Matemática*, *4*(2), 156-173. https://doi.org/10.34179/revisem.v4i2.11863

Voet, C., & Schoonjans, Y. (2012). Benidictine thought as a catalist for 20th century liturgical space: the motivation behind dom hans van der laan s aesthetic church arquitectury, *Proceedings of the 2nd International Conference of the Europa Architetural History of Network*, 255–261.

Authors

Francisco Régis Vieira Alves, Federal Institute of Education, Science and Technology of Ceará, Fortateza (Brazil). E-mail: fregis@ifce.edu.br

Carla Patrícia Souza Rodrigues Pinheiro, Secretary of Education of the State of Ceará, Fortateza (Brazil). E-mail: carla.patricia62@aluno.ifce.edu.br

Renata Teófilo de Sousa, Secretary of Education of the State of Ceará, Fortaleza (Brazil). E-mail: rtsnaty@gmail.com

Paula Maria Machado Cruz Catarino, Department of Mathematics, University of Trás-os-Montes (Portugal). E-mail: pcatarin@utad.pt

Acknowledgement

We would like to thank the National Council for Scientific and Technological Development - CNPq for the financial support for the development of this research in Brazil. This work was financially supported by National Funds through FCT—Fundação para a Ciência e a Tecnologia, I.P. in Portugal, under the project UIDB/00194/2020.