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Transforming Students' Pseudo-Thinking Into Real Thinking in Mathematical Problem Solving

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Abstract: This exploratory and descriptive study aims to theoretically promote the schema of pseudo-thinking processes in mathematical problem-solving by students. The participants in this study were 36 eighth graders and one math teacher. The researchers collected the data using tests and interviews. The results showed that the structure of pseudo-thinking based on the processes of assimilation and accommodation is theoretically composed of five hierarchical components, namely (a) the structure of the problem, (b) the structure of the subject's thinking, (c) the analytic process, (d) the integration of structures or substructures, and (e) the complete integration of structures. When the subject integrates incomplete substructures into existing thinking schemes, assimilation or accommodation becomes imperfect, resulting in cognitive disequilibrium. The results of such a thought process are called pseudo-thinking. Pseudo-thinking processes can be refined and improved into actual thinking processes through reflection and scaffolding. Assimilation and accommodation occur through defragmentation or organization to rearrange the internal schema so that full structural integration occurs. In the end, the subject experiences cognitive equilibrium so that it becomes an actual student thought process.

Keywords: *Assimilation and accommodation, mathematical thinking, pseudo-thinking, structured thinking.*

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Introduction

Learning mathematics is studying to memorize mathematical formulas and procedures to solve problems in a test and placing more emphasis on thinking processes. The expert opinion states that mathematical thinking is a process involving mathematical knowledge to broaden understanding and problem-solving through reasoning, abstraction, guessing, connecting, and communicating ideas, generalizations, and evidence (Basir et al., 2022; Nepal, 2016; Schoenfeld, 2016). Therefore, the skill to think mathematically is placed as a goal and simultaneously a way of learning mathematics (Schoenfeld, 2016).

According to Fernández et al. (2018), the stages of the thinking process that occur in students include three stages, namely (a) defining the meaning of incoming information, (b) forming new arguments by utilizing existing knowledge, and (c) concluding. This condition will occur if students are given a stimulus as a non-routine math problem so that they will experience cognitive processes in their minds to think of problem-solving strategies (Basir et al., 2022). In learning mathematics, students' thinking processes can be observed from the way students do things through the representations (behaviors) of mathematical thinking (Nepal, 2016).

In solving a problem, students often think they are following a reasoning process when their thinking does not follow a process known as pseudo-thinking (Vinner, 1997). Pseudo-thinking processes are thinking behaviors that often arise and are experienced by students in solving mathematical problems. Pseudo-thinking processes are interesting to discuss because they are thought processes that are not real but "real" experienced by students (Subanji & Nusantara, 2016; Vinner, 1997). This circumstance can occur if students are given questions that have usually not been worked on before or are not routine (Gavaz et al., 2021).

Student responses to math problems have two possibilities: the correct answer or the incorrect answer. If analyzed carefully, student answers (right or wrong) can be assumed that students, when solving problems, experience behaviors of pseudo-thinking processes. Student mistakes in solving math problems need attention because these

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mistakes impact students' understanding of subsequent mathematical concepts. Therefore, teachers need to know the sources of errors students make and how the thinking process occurs.

In the last two decades, researchers have found quite a bit of literature that has discussed pseudo-thinking, namely Vinner (1997), Subanji and Nusantara (2016), Hurst and Hurrell (2020). Vinner (1997) is the first researcher to use the term pseudo-thinking. The results of Vinner's research analyzed student errors in solving mathematical problems, referred to as pseudo-thinking. Subanji and Nusantara (2016) explained that there were student errors in the form of pseudo constructions in covariant reasoning. Meanwhile, Hurst and Hurrell (2020) explained that there was a pseudo-procedural type as a barrier to conceptual understanding. The weakness of the three research results is still partially per case, and researchers have not revealed how the pseudo-thinking takes place, so there is a need to examine the process of pseudo-thinking in further research.

The contribution of this research is to analyze in depth the process of pseudo-thinking in which students solve math problems based on assimilation and accommodation processes. Thus, this research aims to describe the scheme of the structure of the pseudo-thoughts that are formed and how to change the pseudo-thoughts into actual thoughts. The results are linked with those of previous research by experts in an integrated and comprehensive manner so that the results of this study can fill the research gaps in the new literature.

This study aims to analyze and promote an artificial thinking structure based on the processes of assimilation and accommodation that occur when students solve mathematical problems. The results of this study can contribute to a broader field of education than just justifying pseudo-thoughts. Thus, educators and teachers can learn about the pseudo-thinking process and how teachers help students change pseudo-thinking into actual thinking.

Literature Review

To be cognitive refers to mental processes involving thinking and reasoning (Cowan, 2014). Cognitive processes are practices or procedures that combine existing knowledge with new knowledge, generate new knowledge, and make decisions based on that knowledge (Evans & Stanovich, 2013; Newen, 2015). Cognitive functions that play a role in cognitive processes include perception, attention, memory, language, learning, thinking, and so on (Cowan, 2014; Kiryak et al., 2021; Yilmaz, 2019). These cognitive functions work together to integrate new knowledge and create interpretations of the world around us (Cowan, 2014).

The cognitive structure is a mental process or individual mindset to process, understand information, and create meaning (Garner, 2012; Kiryak et al., 2021; Yilmaz, 2019). Each student has different cognitive structures and units, some simple, some complex, depending on their level of cognitive development. Cognitive structures can be developed into rich cognitive structures by repetition or reflection (Garner, 2012; Ifenthaler et al., 2011). The characteristics of cognitive structure or patterns of thinking are unique characteristics that appear as a person's thinking behavior in using cognitive structures to process information and create meaning through the process of (a) making connections, (b) finding patterns of thought, (c) formulating principles or rules, and (d) making principle abstractions (Garner, 2012).

Knowledge construction is the mental process of an individual (student) in finding or changing the information obtained to form a comprehensive understanding or interpretation of the knowledge (Kuldas et al., 2013; Taber, 2011). Piaget clearly stated that knowledge construction is an active, not a passive, process (Piaget, 1964). The development of thinking and the active construction of knowledge is made possible by schemata, adaptation, equilibration, and organization (Joubish & Khurram, 2011; Simatwa, 2010). There is evidence that a process of knowledge construction has taken place, that is, that there is a process of adaptation in the form of assimilation and accommodation, characterized by the stages of thinking (knowledge construction) carried out by the students during the learning process (Bormanaki & Khoshhal, 2017; Zhiqing, 2015).

Assimilation is a cognitive process in which a person collects and classifies new stimuli or information into schemas that have already been formed (Bormanaki & Khoshhal, 2017; Hanfstingl et al., 2022). Assimilation works by using pre-existing schemes to deal with new objects or situations. During assimilation, the incoming stimulus must be by the existing scheme (Zhiqing, 2015). Sometimes the schema does not match the incoming information, so in this situation, the process changes to accommodate the new information. Accommodation is integrating a stimulus by forming a new schema to match the incoming stimulus (Bormanaki & Khoshhal, 2017; Hanfstingl et al., 2022). Accommodation refers to an internal process of changing the new knowledge structure to be consistent with the external reality (Bormanaki & Khoshhal, 2017).

Shlomo Vinner first introduced the term pseudo-thinking in 1997. According to Vinner (1997), many students think they have done a thought process when solving problems; in fact, these students only imitate the procedures carried out by the teacher. Circumstances, where students do not understand what they are thinking can be called pseudo-thinking. Students do not use the actual thinking method to solve problems. However, there is a possibility that students need to think correctly to get answers to the questions given. The right answer is not necessarily the result of the proper thought process, and the wrong answer is not necessarily the result of a wrong thought process. Students who experience a "pseudo" thought process tend to associate with problems they consider the same (Subanji & Nusantara, 2016).

Based on the form of error in constructing a knowledge concept, pseudo-thought processes include pseudo "true" and pseudo "wrong." Pseudo "true" occurs when students get the correct final answer but through wrong reasoning. Pseudo "wrong" occurs when students get the wrong final answer through correct reasoning (Subanji & Nusantara, 2016; Wibawa et al., 2018). Meanwhile, pseudo-thinking includes analytical pseudo-thinking and conceptual pseudo-thinking based on a student's understanding of a concept (Vinner, 1997). Pseudo-analytic thinking is a mental activity not based on the controls and procedures chosen and used (Kusmaryono et al., 2020; Vinner, 1997; Wibawa et al., 2018). Pseudo-conceptual thinking is a mental activity that occurs when a person cannot understand the concepts used and make connections between concepts (Vinner, 1997; Wibawa et al., 2018). The indicators of pseudo-thinking behavior are presented in Table 1.

Table 1. Indicators of Pseudo-Thinking Behavior

Indicator	Description
Loss of individual control stages	a. Responding to an idea in a hurry without thinking it through. b. Did not check the correctness of the information obtained. c. Ignoring one of the components that must be known in the information or ideas obtained.
Learn by rote	a. Work on problems by memorizing formulas. b. Absorb new information by rote without connecting to previous information or knowledge.
Study habits	a. Solve the test questions using the usual procedure used in the previous questions. b. Feel confident using certain procedures even if they are not required to.
Lack of understanding of concepts	a. Lack of proper understanding of prerequisite concepts. b. Less able to connect between mathematical concepts in accordance with the problems faced.

Methodology

Research Design

This research used an exploratory descriptive approach (Creswell, 2014). Exploratory-descriptive research describes the state of a phenomenon. This study does not aim to test a specific hypothesis but describes what a variable, symptom, or condition is. This study aims to describe the schema (structure) of pseudo-thinking that students experience when solving mathematical problems and to investigate the process by which pseudo-thinking phenomena occur.

Participants

The participants in this study were 36 eighth graders students and a math teacher with more than five years of teaching experience. The students were male and female, aged between 11 and 13 years. Participants were only students who allegedly experienced "true" or "wrong" pseudo-thinking when completing math tests. The teacher participated in six learning meetings as a mathematics teaching staff during the research process.

Collection of Data and Instruments

Researchers collected data through tests and interviews. The test instrument was three math questions, and the interview instrument was an interview guide sheet. The development of the test items included the stages of (a) identifying the construct of problem-solving abilities, (b) developing test items, (c) validating by mathematics curriculum experts, (d) testing the test instrument, and (e) testing the validity of the instrument. The three math problem items had the same difficulty level: extended abstract (the fourth level in the taxonomic structure of the observed learning outcomes) (Kusmaryono, 2018). The three items of the mathematics test were tested for the validity of the test instrument using the Pearson correlation product moment statistic. Based on the results of the statistical test, the Pearson correlation was (.666) for item 1, (.809) for item 2, and (.774) for item 3. The three test items were valid because the Pearson correlation value was higher than > .444 (Suresh & Raju, 2022; Tsang et al., 2017).

Researchers conducted interviews with students. The students interviewed were selected through purposive sampling techniques (Taherdoost, 2016). Researchers prepared interview questions in a semi-structured form. The interview questions included the stages of (a) determining the topic and purpose of the interview; (b) formulating questions (conversations) to explore the subject's thought processes in solving problems; (c) the interview questions validated by a team of validators namely two cognitive development theorists, and they provided feedback; and (d) the researcher revises the interview questions (Yeong et al., 2018). Examples of interview questions can be seen in Table 2.

Table 2. Examples of Interview Questions

No.	Questions
Q-1	What strategy do you think of to solve this question item?
Q-2	Where did you get the idea to solve this question item?
Q-3	What are your difficulties when solving this question item?
Q-4	Did you check the answers carefully?
Q-5	Why did you do an analytical process to solve this calculation?
Q-6	Did you notice any procedural errors at this step?
Q-7	Are you sure and satisfied your answer is correct?

Researchers invited participants to this interview using a purposive sampling technique. The interviews ended in 120 minutes and were conducted after students had completed the formative tests. The interview was recorded with a Lavalier microphone.

Material

The math questions used as the test consisted of three items. The test items focused on solving reasoning problems about the volume of triangular prisms, cylinders, and pyramids. The following is an example of a math problem about the volume of a triangular prism (item number 3).

Look at the picture of the ABC.DEF right-angled prism vessel. The vessel contains water as high as CH with a length ratio of CH: HF = 3: 1 . The base ABC is right-angled at point C, length AC = 8 dm and length AB = 10 dm, and height AD = 16 dm.

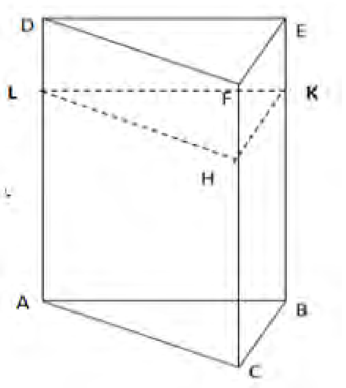


Figure 1. Mathematics Test Materials

Data Analysis

The student's math test results were analyzed using descriptive statistics to classify correct and incorrect answers. The researcher identified the answers of students suspected of experiencing "true" or "wrong" pseudo-thinking when completing math tests. Researchers interviewed students who experienced pseudo-thinking.

The researcher analyzed the interview data by reading the transcripts, coding, categorizing, and interpreting the interviews (Lester et al., 2020). To test the credibility of the data, the researcher expanded observations, increased persistence, triangulated data sources and theories (findings) from previous experts, data sources and theories and opinions (findings) from previous experts (Miles et al., 2019), analyzed negative cases, and member checks (Nowell et al., 2017; Stahl & King, 2020). The researcher also conducted an objectivity test using a confirmability test to ensure the reliability of qualitative data analysis in a discussion forum attended by researchers, a team of experts, and teachers (Adler, 2022).

Research Procedure

This research lasted for six weeks, starting from (a) permits for research sites and preparation of research instruments (1 week), (b) data collection in the field (3 weeks), and (c) conducting data analysis and discussion of research results (2 weeks). The collection of research data in the school for six meetings took three weeks because the school had set up a schedule for completing the subject matter in which there were two math learning meetings each week. The teacher's duties included preparing learning scenarios, implementing mathematics learning, and compiling test questions. The students' assignments took part in learning in mathematics class, completed formative tests, and attended interviews with researchers. At the beginning of the program, the teacher taught mathematics to 36 students. Students attended six face-to-face meetings in mathematics class. At the end of the program, the teacher gave a math problem-solving test. Students completed the math test, and the researcher checked the results of the test answers. Then the researchers analyzed the test results by classifying the correct and wrong answers and identifying students who experienced "true" or "wrong" pseudo-thoughts. Researchers conducted interviews with students. In the final stage, the researcher conducted, reduced and tabulated the data made, coding and interpreting the interview results according to the pseudo-thinking process. The researcher constructed the pseudo-thinking structures experienced by the students based on the appropriate theory.

Results

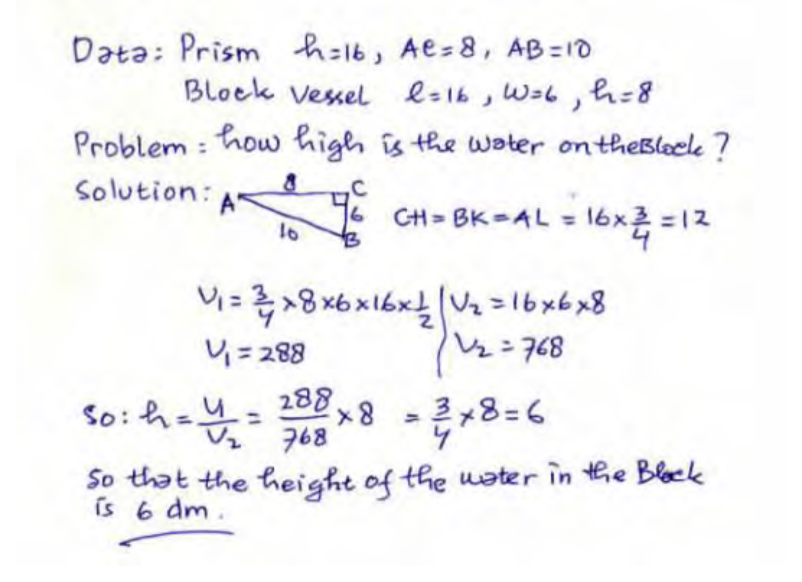
The math test results of 36 students were carefully corrected and analyzed. Analysis of the quality of student answers was grouped into correct, wrong, and no (Table 3).

Table 3. Recap of the Quality of Student Answers

Math Problems	**N**	**Answer Quality**		
		Incorrect	**Correct**	**No Answers**
Problem 1	36	3	26	7
Problem 2	36	5	28	3
Problem 3	36	12	24	-

The results of searching student answer sheets and interviewing researchers with 36 students revealed that 16 subjects were suspected of experiencing pseudo-thinking processes. Based on the data in Table 3, the researcher took an example of the answers from question 3, where all subjects responded (true or wrong). Then the subject representatives being interviewed are the subject (S.07; S.21), which represents students who answer "wrong" in question 3, and the subject (S.19; S.35), which represents students who answer "correctly" in question 3. The interview aimed to confirm the problem-solving process and get in-depth information about students' problem-solving thinking processes. For example, if the selected subject did not experience pseudo-thinking processes, the researcher moved on to other subjects until the researcher found a subject with pseudo-thinking cases.

The example of the answer to the math test taken from question 3 has the same error location as most test takers. The following is an example of wrong answers and correct answers assuming students are indicated to have pseudo-thinking processes.



Data: Prism $h=16$, $AC=8$, $AB=10$

Block Vessel $l=16$, $w=6$, $h=8$

Problem : how high is the water ontheBlock ?

Solution:

$CH = BK = AL = 16 \times \frac{3}{4} = 12$

$V_1 = \frac{3}{4} \times 8 \times 6 \times 16 \times \frac{1}{2}$ | $V_2 = 16 \times 6 \times 8$

$V_1 = 288$ | $V_2 = 768$

So: $h = \frac{V_1}{V_2} = \frac{288}{768} \times 8 = \frac{3}{4} \times 8 = 6$

So that the height of the water in the Block is 6 dm.

Figure 2. Response to the "Incorrect" Answer From the Subjects (S.21)

Based on the subject's answer (S.21) in Figure 2, it can be explained that the subject can handle or solve several problem topics, namely finding the height of the water on the prism, which is 12, finding the height of the triangle (base of the prism) which is 8 through the use of Pythagorean theorem, and using the results of calculations to calculate the volume of an upright triangular prism that is 288 (V1). Subject (S.21) can also determine the volume of the beam, which is 768 (V2). Then the subject (S.21) estimated to solve the problem, namely the comparison of the volume of V1 and V2 (V1:V2), then multiplied by the height of the beam. Subject (S.21) can establish relationships between one topic and another. However, the subject (S.21) did not properly take advantage of the relationship between structures, so the calculation results are obtained (the water level in the beam vessel of 6 dm is not the right solution). From the subject's response in Figure 2, the researcher (R) conducted interviews to determine the thinking process.

Researcher : *What do you think about this issue?*

(S.21) : *This problem is quite heavy and I have to think hard.*

Researcher : *Why would you compare the volume of V1 to V2 and then multiply by the height?*

(S.21) : *I understand there is a comparison of the volumes of two different containers to determine the water level.*

Researcher : *Where did you get the idea to calculate ¾ x 8 = 6?*

(S.21) : *I guess it's the same as the water level in the new container.*

Researcher : *Did you check the answers carefully?*

(S.21) : *I didn't double-check this answer.*

Researcher : *Are you sure your answer is correct?*

(S.21) : *I am not sure.*

Researcher : *Please check your answer.*

(S.21) : *The volume of water in the beam is 16 x 6 x 6 = 576.*

Based on the information from the interview, it was known that the subject (S.21) realized his mistake, so the answer was incorrect, so the researcher allowed the subject (S.21) to reflect. The following is a snippet of the interview during the reflection process.

Researcher : *What do you know about the volume of water when it is moved?*

(S.21) : *The volume of water remains the same even though the container is different, which is 288 liters.*

Researcher : *Do you know where this troubleshooting error lies?*

(S.21) : *Yes, the error is at (V1:V2) = 3/4*

Researcher : *What do you do with the reflection process?*

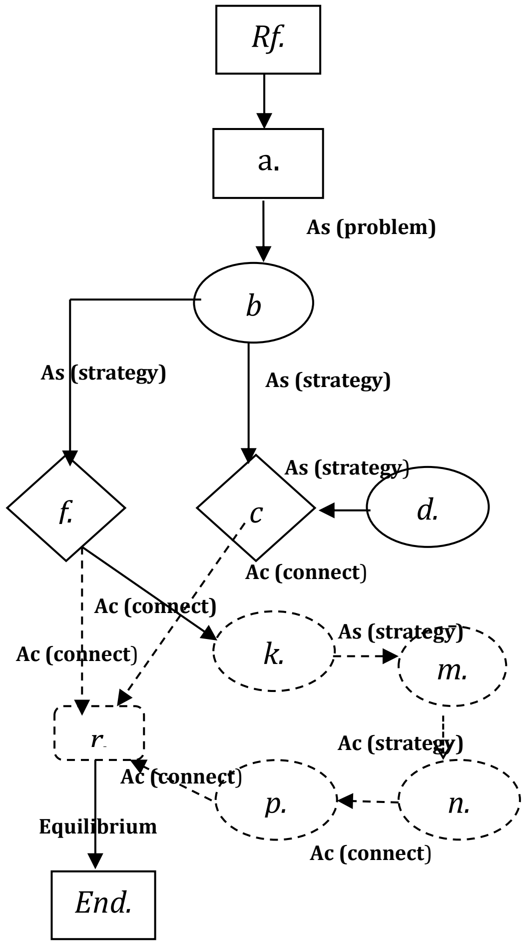
(S.21) : *Comparing the ratio (V1:V2) x height V1, it is obtained (288:768) x 8 = 3*

Researcher : *Are you sure the result is correct?*

(S.21) : *Sure, sir, the water level on the beam is 3 dm.*

Because the volume of water is 16 x 6 x 3 = 288 (equal to the volume of water in the prism).

Based on the analysis of the subject's answers (S.21) in Figure 2 and the results of the interview data reduction before and after reflection, the schema of the subject's thinking flow (S.21) can be described. The schema of the subject's thinking flow (S.21) in solving mathematical problems before and after the reflection process is shown in Figure 3.

Schema	Code	Description
	*Rf.*	Reflection
	a.	Problem to find: The height of the water in the beam container
	b.	Presenting data information: volume formula of a triangular prism
	c.	Calculating the volume of water in a triangular prism
	d.	Apply the Pythagorean theorem to calculate the area of the base of a triangular prism
	k.	Checking the water volume of the beam is not the same as the volume of the triangular prism
	m.	Formulate the volume equation for a triangular prism and a cuboid
	n.	Solve the equation to get the final result
	p.	The final result was found the water level in the beam
	r.	Checking the volume of water on the beam is equal to the volume of water in a rectangular prism
	End.	Work completed: the subject is confident in his work

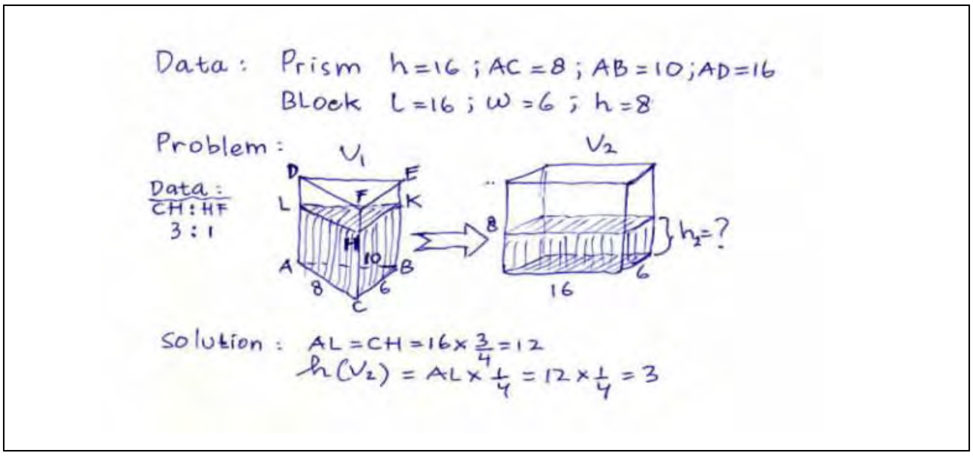
Description :

——→ = Subject line of thought before reflection

- - - - → = Subject line of thought after reflection

Figure 3. The Flow of the Subject's Thought Process (S.21)

Furthermore, the results of the subject's work (S.35) are in Figure 4. The results of the subject's work (S.35) are examples of answers with correct final results, but students are indicated to experience pseudo-thinking processes.


Data : Prism h=16 ; AC=8 ; AB=10 ; AD=16
Block L=16 ; ω=6 ; h=8

Problem :

Data :
CH : HF
3 : 1

Solution : $AL = CH = 16 \times \frac{3}{4} = 12$
$h(V_2) = AL \times \frac{1}{4} = 12 \times \frac{1}{4} = 3$

Figure 4 Response to the "Correct" Answer From the Subject (S.35)

Paying attention to the subject's response (S.35), Figure 4 illustrates that the subject (S.35) carried out problem-solving steps briefly and efficiently. Subject (S.35) described the problem through image representation. However, the subject (S.35) failed to understand how to establish the relationship between the structures that comprise the whole. Although the final answer obtained by the subject (S.35) is correct, this finding still needs to be further confirmed through the following interview.

Researcher : *Are you thinking of answering this problem?*

(S.35) : *Wow..., I have to rack my brain to find the right strategy*

Researcher : *Are you sure your answer is correct?*

(S.35) : *I'm not sure, because I didn't double-check.*

Researcher : *Why don't you check the completion steps from start to finish?*

(S.35) : *No time to double-check.*

Researcher : *Where can you write the multiplication of 12 by?*

(S.35) : *I assume the volume of the prism is the volume of the beam, and the length of AC = 8 is ½ x 16. So the height of the water on the beam is 12 x ½ x ½ = 12 x ¼ = 3.*

Researcher : *Sorry, the explanation doesn't have a solid foundation. Please double-check the steps for the solution.*

Based on the interviews, it is known that the subject (S.35) could not explain convincingly that the answer was correct. Next, the researcher allowed the subject (S.35) to reflect. The following is a snippet of the interview during the reflection process.

Researcher : *What do you think to explain your answer?*

(S.35) : *Shall I build the equation V1 = V2?*

Researcher : *Why did you choose the equation V1 = V2?*

(S.35) : *I assume the volume of water (V1) remains the same even though it is transferred to a different container (V2).*

Researcher : *What's different about the two containers?*

(S.35) *The difference is the water level in the container.*

Researcher : *What is the next step in solving it?*

(S.35) : *V1 = V2*

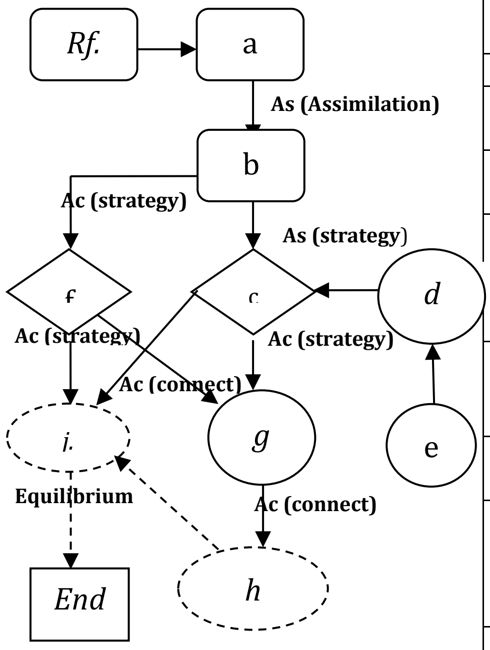
288 = 16 x 6 x h

288 = 96 x h → h = 288 : 96 → h = 3

Researcher : *Are you sure about the conclusion you get from this solution?*

(S.35) : *I believe. The water level in the beam is 3 dm. It turned out to be lower than the water level in the prism because the beam is larger (area) than the prism.*

Based on the analysis of the subject's answers (S.35) in Figure 4 and the results of interview data reduction before and after reflection, the subject's thinking flow scheme (S.35) can be described. Figure 5 shows the schema of the subject's thinking flow (S.35) in solving mathematical problems before and after the reflection process.

Schema	Code	Description
	*a.*	Problem to find: The height of the water in the beam container
	b.	*Presenting* data
	c	Questioning the volume of triangular prism
	d.	Find the area of the base of a triangular prism
	e.	Applying the Pythagorean Theorem
	f.	Questioning the volume of the beam
	g.	Compose the ratio of the volume of water in a triangular prism to the volume of the beam
	h.	Obtained the water level on the beam
	j.	Checking: formulate an equation for the volume of water in a triangular prism equal to the volume of water in the beam
	End.	Completed work: the subject is sure that his work is correct

Description :

——▶ = Subject line of thought before reflection

- - - -▶ = Subject line of thought after reflection

Figure 5. The Flow of the Subject's Thought Process (S.35)

Discussion

Students' problem-solving behaviors were analyzed to find out what students thought and how they related concepts to the given problems. The case of pseudo-thinking in the assimilation and accommodation process was traced based on the researcher's suspicion of the student's response in problem-solving, namely (a) the answer was "wrong," but the subject (S.21) was able to solve it correctly, and (b) the answer was "true" but the subject (S.35) cannot justify the answer. This finding means that the subject (S.21) experiences a "wrong" pseudo-thinking process, and the subject (S.35) experiences a "true" pseudo-thinking process.

Some researchers suggest that pseudo-thinking processes can be improved or eliminated through a process of reflection (with or without scaffolding); this idea is in line with Vygotsky's theory of the existence of a zone of proximal development (ZPD) (Kusmaryono et al., 2021). Starting from this ZPD, subjects (S.26 and S.35) were allowed to reflect because students have the skill (potential) to be in a real state of thinking (Kusmaryono et al., 2021; Suranata et al., 2018).

The Process of Changing Pseudo-Thinking Into Actual Thinking on the Subject (S.21)

The subject's thinking behavior (S.21) in solving problems produced wrong answers. The subject's thinking structure (S.21) resulted from a pseudo-thought process or what is referred to as a "wrong" pseudo-analytic thinking process. Errors in thinking classified as pseudo-analytic were errors related to problem-solving strategies or processes (Vinner, 1997). According to Vinner (1997), the 'pseudo-analytic' thought process begins with the incompleteness of the thinking substructure in the accommodation process.

One of the causes of "wrong" pseudo-thinking in solving math problems is that students only learn by rote and lack understanding of prerequisite concepts. The subject (S.21) experiences a thought process system that is fast, automatic, effortless, unconscious, and inflexible (Kusmaryono et al., 2020). However, according to Vinner (1997), these students experience a fuzzy memory thought process or remember vaguely. Considering that the 'pseudo-analytical' thinking process is artificial and not a real thinking structure, the subject (S.21) can reflect.

Based on the results of the interviews, there is an imperfect understanding of the subject's concept (S.21) regarding the properties of liquids (the volume of water in a vessel). Researchers identified the emotions of impatience, anger, and annoyance when the subject failed. Solid cognitive commitment also does not support this situation, so subjects easily give up on solving math problems. In the case of the subject (S.21), it is illustrated that potential negative emotions in students can accompany a lack of conceptual understanding of mathematics. We argue that students learning mathematics should not only understand the heuristic steps of problem-solving but also be able to neutralize the potential negative emotions associated when students fail to find the correct answer (solution).

When the subject (S.21) was allowed to reflect, the teacher provided a little scaffolding so that the subject could continue and tried again to work on the problem until it produced the correct answer. At the time of reflection, the subject (S.21) defragmented the problem (Kusmaryono et al., 2020; Vinner, 1997; Wibawa et al., 2018). The subject (S.21) rearranged the structure of his thinking through defragmentation. It helped to change his thinking process by making a relationship, namely compiling an equation between the volume of water in a triangular prism container and the water level in a new container (block). Once confirmed, the subject (S.21) can explain the solution to the problem and prove the answer is correct with good reasons. Subject (S.21) felt confident about his answer because he had gone through re-checking (reflection).

Based on the reflection process, the subject (S.21) felt satisfied and was in a state of balance (equilibration) where there was a match between the structure of thinking and the structure of the problem (Bormanaki & Khoshhal, 2017). From a practical perspective on learning mathematics, it is recognized that students feel satisfied when they complete assignments and can understand the mathematics material being studied. This satisfaction will continue to motivate and challenge students to learn mathematics (Schukajlow et al., 2017). Thus, the defragmentation process carried out by the subject (S.21) has succeeded in rearranging (restructuring) the "wrong" pseudo-thinking process to become the subject's actual thinking process (S.21). This finding implies that the process of setting reflection by scaffolding from the teacher results in students being able to form actual thinking structures and change students' positive emotional states. This emotional state strongly impacts students' attitudes and beliefs in subsequent mathematics learning.

The Process of Changing Pseudo-Thinking Into Actual Thinking on the Subject (S.35)

The subject's thinking behavior (S.32) in solving problems yielded correct answers. However, when the answer was confirmed, the subject (S.35) could not provide a correct explanation and justification or reason. The subject's thinking structure (S.35) stemmed from a "true" pseudo-thinking process. At the primary and secondary school level, consciously or not, many teachers teach mathematics through pseudo-procedural or pseudo-conceptual methods. This method can be seen from the results of students' work when solving problems by following procedures that do not make sense even though the answers are correct (Hurst & Hurrell, 2020). This pseudo-procedural method also occurs in the subject (S.35) when solving mathematical reasoning problems. In this case, the subject (S.35) could not apply concepts to his cognitive structure. Even if the correct answer was obtained, the justification is not well-founded.

During the reflection process, the subject (S.35) reviewed the steps for solving problems and checking calculations. Subject (S.35) proved his answer with the equation for the volume of water in a block and the volume of water in a triangular prism so that the correct answer was obtained. Subject (S.35) improved his thinking structure and formed a new one. The subject's thinking process (S.35) is called an organizational process in this reflection process. The organization is a person's tendency to regulate mental processes (thinking) by rearranging internal schemas and exploring relationships and associations between schemas (Piaget, 1964). Through the process of organizing this scheme, the subject (S.35) felt confident and satisfied with the results of his work, so in this last process, the cognitive balance occurred (see Figure 5) (Bormanaki & Khoshhal, 2017; Hanfstingl et al., 2022). According to Piaget's view, children (individuals) also change their schemas according to the organization. Organizational processes aim to develop interconnected cognitive systems to be more effective than before. Piaget named this organization a high-level cognitive system (Piaget, 1964).

Pseudo-Thinking Structure Based on Assimilation and Accommodation Processes

The results of an in-depth and thorough analysis of the subject's thinking process (S.21 and S.35) revealed that the pseudo-thinking structure based on the assimilation and accommodation process consists of five components, namely (a) problem structure, (b) student structure, (c) analytical processes, (d) integration of structures or substructures, and (e) integration of complete structures. The schema of the pseudo-thinking structure based on the process of assimilation and accommodation of the subject is presented in Figure 6.

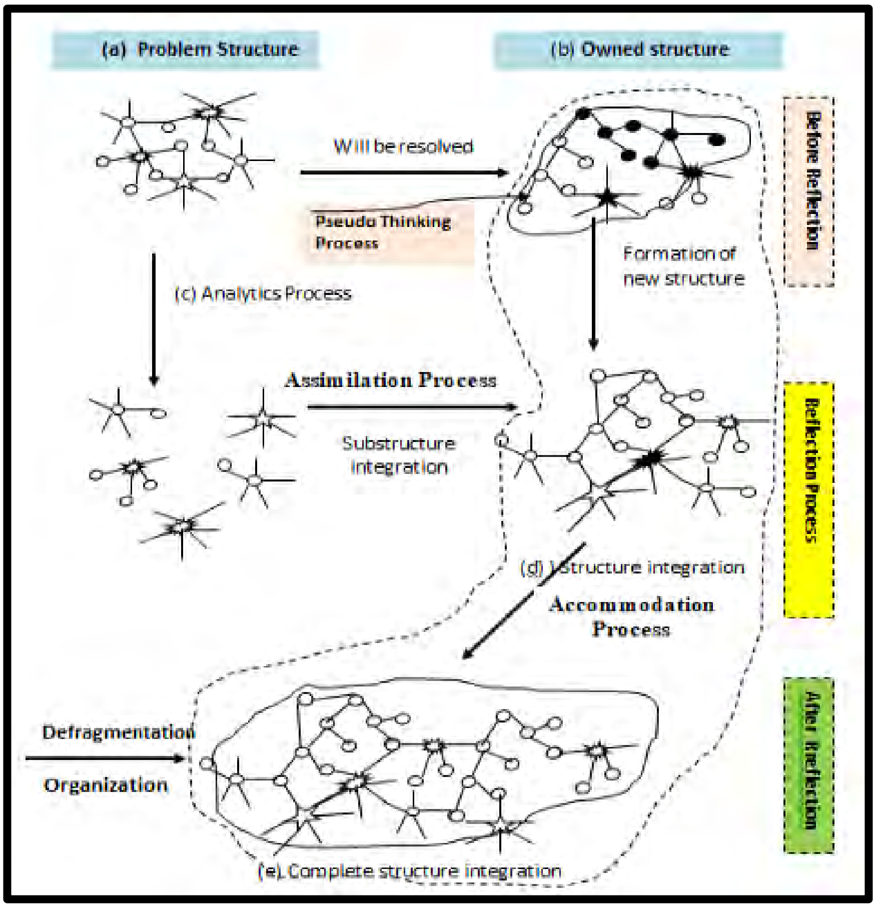


Figure 6. The Structure of Pseudo-thinking Based on the Process of Assimilation and Accommodation

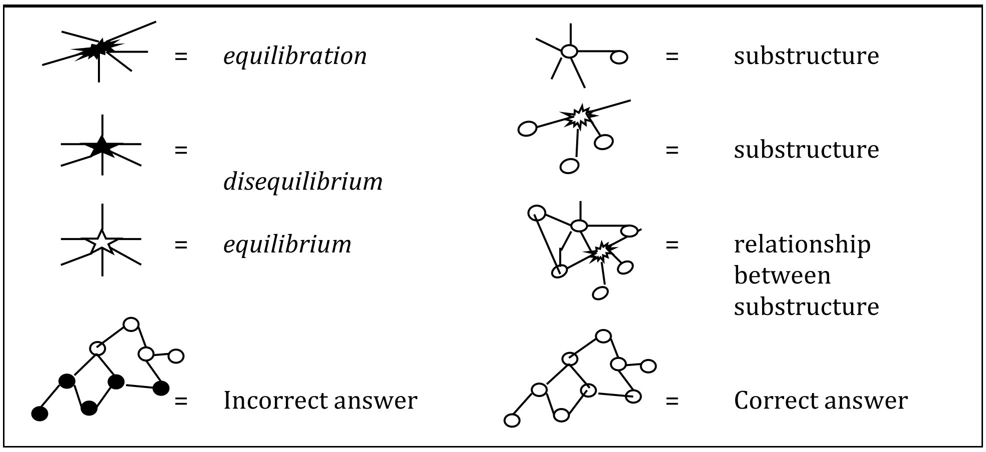


Figure 7. Pseudo-Thinking Substructure Symbols

Figure 7 is an abstract substructure symbol forming a pseudo-thinking pattern (scheme). Theoretically, the scheme of the pseudo-thinking structure based on the processes of assimilation and accommodation presented in Figure 6 can be explained through the following description.

A math problem has a predetermined problem structure (see Figure 6; code: a), then the problem will be solved by the subject (students). The results of each student's problem-solving vary depending on the structure of their thinking (see Figure 6; code: b). The subject's thought process when solving problems begins with disequilibrium and continues with adaptation (assimilation and accommodation) (Zhiqing, 2015). In the adaptation process, incomplete accommodation occurs so that when integrated into the problem structure, it produces wrong or correct answers that need to be obtained correctly. In this case, the subject experiences pseudo-thought processes, namely pseudo-analytical ones that are "wrong" and pseudo-conceptual ones that are "true." This pseudo-thinking process occurs because in the problem-solving process, the subject experiences an accommodation process that is imperfect (incomplete), and the subject does not reflect or re-check.

The pseudo-thinking process can still be improved through the reflection process. At the time of reflection, there is a defragmentation process of the thought process to correct errors through (a) an analytical process on the substructure (see Figure 6; code: c), (b) incomplete structural changes (assimilation process), (see 6; Figure code: d) or (c) establishment of a new structure (accommodation) (see Code figure: e). During reflection, defragmentation and organization can also rearrange the internal schema to integrate the complete structure (see Figure 6; code: e). After the subject experiences a process of deconstruction and organization, the subject is in the process of cognitive equilibrium, or subjective knowledge of mathematics is the same as objective knowledge (Ernest, 1991, as cited in Marsigit et al., 2020). So that there is no pseudo-thinking, it will become a student's accurate thinking process. This process will continue when a person learns or receives a new stimulus so that a person's thinking process will become more complex over time (Zhiqing, 2015).

In line with the teacher's task in helping the development of students' cognitive structures, it is suggested that learning always emphasizes "learning how to learn" (Hasanah et al., 2022). The purpose of "learning how to learn" is for students to understand what is being learned and that learning has deep meaning. Since pseudo-thinking is not an actual thinking process, students can still improve it through reflection. During reflection, it is suggested that the teacher provide scaffolding to help students improve their thinking structure. If students learn without assistance, they will remain in their true territory and cannot progress to a higher level of potential development (Breive, 2020). Improvement of this thinking structure is based on the belief that when students are in the zone of proximal development (ZPD) or the zone of potential construction (ZPC), their thinking structures have the potential to develop optimally (Kusmaryono et al., 2021).

The results of this study have similarities with the results of previous studies (Hurst & Hurrell, 2020; Subanji & Nusantara, 2016), namely, we used pseudo-thinking as a rationale, where subjects in solving problems can experience pseudo-thinking, and we agree that pseudo-thinking can be transformed into real thinking. Our difference with their research is that we have theoretically presented the process of pseudo-thinking based on cognitive theory (processes of assimilation, accommodation, equilibrium, and organization of knowledge) and pseudo-thinking processes in the form (of schemes) of pseudo-thinking so that researchers and readers can easily understand pseudo-thinking processes. Meanwhile, the thinking and the subject's knowledge construction are mental (active) processes in acquiring and using knowledge (Yilmaz, 2011).

In cognitive learning theory, learning is an interaction between stimulus and response and involves various factors within the individual. Therefore, cognitive learning theory emphasizes that learning involves active mental activities to acquire, remember, and use knowledge (Holland, 2008). This cognitive learning theory is more concerned with the learning process than learning outcomes because it involves very complex thinking processes (Yilmaz, 2019).

Conclusion

Based on the research findings, it can be concluded that if every student solving math problems feels unsure, dissatisfied, and unable to justify their performance results, then it is indicated that they are experiencing a pseudo-thinking process. Theoretically, the structure of pseudo-thinking based on the processes of assimilation and accommodation consists of five components, namely (a) the structure of the problem, (b) the structure of the subject's thinking, (c) the analytical process, (d) the integration of structure or substructure, and (e) the integration of the complete structure. When the subject integrates incomplete substructures into existing thinking schemes, assimilation or accommodation becomes imperfect, resulting in cognitive disequilibrium. The results of thinking in this process are referred to as pseudo-thinking. This process can be corrected and increased to become an actual thinking process through the teacher's reflection and scaffolding process. During reflection, assimilation and accommodation occur through defragmenting or organization to rearrange the internal schema so that a complete structural integration occurs. In the end, the subject experiences a cognitive equilibrium so that it becomes an actual student thinking process. The improvement of this thinking structure is based on the belief that when students are in the zone of proximal development (ZPD) or the zone of potential construction (ZPC), their thinking structure can develop optimally. The results add new insights to the literature and contribute to previous studies, as they have not only revealed the existence of analytic pseudo-thinking and conceptual pseudo-thinking but also presented a way of thinking (a schema) of pseudo-thinking. Thus, there is a pseudo-thinking (schema) structure, so researchers or teachers can help students (subjects) avoid these pseudo-thinking processes.

Recommendations

Based on the results of this study, the researchers suggest that (a) the teacher needs to provide students with scaffolding tailored to their needs when learning mathematics and (b) students need to be involved in a review step to check the correctness of answers to avoid pseudo-thinking and improve their understanding of mathematics. In the future, other researchers can conduct studies with a more comprehensive and broader set of participants, ranging from elementary school students to high school students, so that the results can be more representative and the best formula can be found to help students get out of pseudo-thinking.

Limitations

The limitation of this study is the small number of participants and the restriction to junior high school students. The results of this study are preliminary and refer to students in grade eight. Thus, there is still the possibility of conducting further research to determine changes in the structure of pseudo-thinking depending on the characteristics of the participants by the researchers.

Authorship Contribution Statement

Nizaruddin: Research method, preparation of instruments, data analysis, interpretation of research results, and final approval, Kusmaryono: Basic ideas of research, collection data and analysis of data, critical revision of manuscripts, grammatical corrections of manuscripts, and verification of literature.

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