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Causes of proof construction failure in proof by contradiction

Deni Hamdani[1], Purwanto[2], Sukoriyanto[2], Lathiful Anwar[2]

[1]Department of Mathematics Education, University of Mataram, West Nusa Tenggara, Indonesia
[2]Department of Mathematics, Universitas Negeri Malang, East Java, Indonesia
*Correspondence: purwanto.fmipa@um.ac.id

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Abstract

Failure to deduce false suppositions in proof by contradiction is still considered "more difficult" than proving the conditional p to q in proof by contraposition. This study aims to identify the types of proof construction failures based on the action steps of proof by contradiction, then offer a framework of construction failure hypothesis specifically used in proof by contradiction. The research data were collected and analyzed from the work of students who have agreed to be research participants, a total of 83 students. The results of the analysis of student work successfully identified four types of failures, namely formulating suppositions, constructing and manipulating suppositions, identifying contradictions, and disproving suppositions. These four types of failures then became the material for the development of the hypothesis framework of a failure to construct proof by contradiction, which consists of 17 hypothesis nodes divided into three main hypotheses, namely: operational (action), affective (emotional), and foundational (logical reasoning). The failure hypothesis framework justifies that the sources of the failure of proof construction in proof by contradiction are understanding of the act of producing a proof by contradiction, emotionality towards the coherence of the construction steps, disproving suppositions, beliefs, use of appropriate definitions-theorems and axioms, and cognitive tension in proof by contradiction; and formal logic of the act of producing a proof by contradiction, as well as differences in the underlying logic with other acts.

Keywords: Causation, Failure, Proof by Contradiction, Proof Construction, The Hypothesis of Contradiction

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Constructing proofs is one of the important competencies that mathematics students must have to follow courses that require logical thinking and analytical skills. Constructing proofs has become a major goal of university mathematics courses (Alcock & Weber, 2010). However, proof construction is not a competency that is easily mastered by every student, many students experience difficulties in constructing proofs and eventually fail to produce valid proofs. This is because constructing proofs is a problem-solving activity (Netti & Herawati, 2019), which demands the ability to use appropriate proof approaches, axioms, definitions, lemmas, and theorems (Selden & Selden, 2008; 2009).

Difficulties that fail to construct proofs have received attention in mathematics education research (Harel & Sowder, 2007; Alcock & Weber, 2010). Among them are difficulties with the underlying mathematical logic (Stavrou, 2014; Knipping, 2008; Moore, 1994), difficulties understanding mathematical concepts in proofs (Mejía-Ramos et al., 2015; Stavrou, 2014; Moore, 1994), and difficulties using certain proof methods, such as proof by mathematical induction (Relaford-Doyle, 2020; Andrew, 2007; Baker, 1996; Dubinsky, 1986; Harel, 2001), proof by contraposition (Doruk, 2019; Brown, 2018;

Ko, 2010; Stylianides et al., 2004; Epp, 2003) and proof by contradiction (Rabin & Quarfoot, 2021; Antonini & Mariotti, 2006; Reid & Dobbin, 1998; Harel & Sowder, 1998; Lin et al., 1998; Barnard & Tall, 1997), difficulty knowing how to use a proof approach to prove a statement (Hanna, 2000; Hoyles, 1997; Moore, 1994; Weber, 2004), and difficulty constructing proofs with statements containing quantifiers (Dubinsky & Yiparaki, 2000; Piatek-Jimenez, 2010; Sellers et al., 2021; Schüler-Meyer, 2022).

The five research focuses on constructing proofs above show that in addition to the ability to use appropriate axioms, definitions, or theorems, it turns out that the ability to determine certain proof methods also greatly determines the success of constructing valid proofs of a mathematical statement, for example, a statement that requires the ability to construct proofs in proof by contradiction. Failure to construct proofs in proof by contradiction has not been widely studied. Though proof by contradiction is an important form of proof content in mathematics, it is still thought to be "more difficult" to use than direct proof (Reid & Dobbin, 1998; Antonini & Mariotti, 2006; Tall et al., 2012; Brown, 2018) failing to identify contradictions (Barnard & Tall, 1997; Antonini & Mariotti, 2009; Antonini, 2019). This failure to identify contradictions may be triggered by the fact that proofs by contradiction require debates with true negation suppositions, to arrive at the contradiction and thus prove the statement to be true indirectly (Quarfoot & Rabin, 2021), the information students have is limited and conflicting (Rabin & Quarfoot, 2021), and perhaps their prior knowledge/understanding cannot be used to construct proofs by contradiction.

The difficulty of constructing proofs by contradiction has been identified through several studies, such as negating intuitive claims (Dubinsky et al., 1988; Reid & Dobbin, 1998; Sellers, 2018), constructing and manipulating contradictory mathematical objects (Antonini & Mariotti, 2008; Leron, 1985), and identifying contradictions (Antonini & Mariotti, 2009; Barnard & Tall, 1997; Rabin & Quarfoot, 2021; Antonini, 2019; Brown, 2018). These three research focuses appear to be the steps of constructing proofs by contradiction and were identified to be the main basis for the development of the difficulty hypothesis framework for constructing proofs in proof by contradiction (Quarfoot & Rabin, 2021; Rabin & Quarfoot, 2021), namely the framework: operational hypothesis, affective hypothesis, and foundational hypothesis. The difficulty hypothesis framework does not address the occurrence of construction failure when students engage in proof by contradiction. However, the nodes in the main hypotheses, such as the operational hypothesis, contain the idea that this hypothesis classification provides support and opportunities for students to achieve success when dealing with proof by contradiction.

Between difficulty and failure, the term "failure", in our opinion, is more appropriate to describe the process of producing proofs (proof construction), not "difficulty", because "difficult" does not always mean "failure". The phenomenon of failing to construct mathematical proofs, in general, can be identified when learners construct the statement "if n^2 is even then n is even" in proof by contraposition on the problem of proving irrationality 2 (Brown, 2018), realize and can apply the necessary facts but still fail to construct the proof (Weber, 2001), rely the belief in the validity of a statement on the failure to find a counterexample or a carefully selected example (Weber, 2008), deviate from the proof construction process (discrepancy of proof construction results) based on the established scheme (Netti & Herawati, 2019), and develop inappropriate justifications to explain why the conjecture of a statement must be true (Ozgur et al., 2019).

This paper then tries to identify the failure of constructing proofs by contradiction method through an initial research study, which was conducted in a class of Mathematics Education study program students at the University of Mataram totaling 83 students, spread over 20 students in semester V and 63 students in semester VII. In addition to the researcher himself, the instrument of this initial study used a proof problem that required students to construct proofs in proof by contradiction, namely: "There is no rational number x that satisfies the equation $\mathrm{x}^2 = 3$". This initial study succeeded in identifying a failure hypothesis that had not been accommodated in the development of the hypothesis framework for difficulty constructing proofs as conducted by David Quarfoot and Jeffrey M. Rabin (Quarfoot & Rabin, 2021), namely the hypothesis of disproving suppositions. In addition to adding operational hypothesis nodes, the addition of one type of hypothesis to the main hypothesis increases the failure hypothesis to four, namely formulating the negation supposition of the proof problem, constructing and manipulating suppositions, identifying contradictions, and disproving suppositions. The addition of the hypothesis "disproving the supposition" to be the fourth hypothesis in constructing proofs by contradiction, and this addition is also based on research that shows that common ways of reasoning in proof by contradiction are less numerous, less accessible to students, or less trained by teachers, and it seems that not many researchers have empirically explored them (Antonini & Mariotti, 2008; Brown, 2018; Hanna & de Villiers, 2021; Tall, 1980; Thompson, 1966).

Thus, these four failure hypotheses will be empirically explored and justified to be the cause of the failure of the act of producing proof by contradiction (operational hypothesis) and can add to the development of emotional responses or emotional views (affective hypothesis) and add to the complexity of logical foundations that can support construction results when engaging in proof by contradiction (foundational hypothesis). The emergence of failure hypotheses that have not been accommodated according to the hypothesis framework of David Quarfoot and Jeffrey M. Rabin (Quarfoot & Rabin, 2021) motivates to empirically examine how prior knowledge/understanding is used (activated) to construct proof by contradiction and describe failures in activating sources of knowledge/understanding into a hypothesis framework of proof construction failures in proof by contradiction that contains types of construction failures and their sources of causes.

METHODS

This study uses an exploratory descriptive research design with a qualitative approach, to explore the types of proof construction failures and their sources of causes, through the phenomenon of students' failure to activate their prior knowledge/understanding to construct proofs, especially when students are involved with indirect proofs (proof by contraposition and proof by contradiction). Students who became participants in this study were taken from several student populations of the Mathematics Education Study Programme, Faculty of Teacher Training and Science Education, University of Mataram. This sampling was carried out by recruiting students who were willing to become participants during the study (voluntary sampling), and the total number of willing participants was 83 students spread across students in semester V and semester VII. The recruitment of participants in this study adopted the research sampling method according to Josephine Relaford-Doyle & Rafael Núñez (Relaford-Doyle & Núñez, 2021), except that this study requires that prospective participants have programmed courses that contain proof and proving content, such as logic and set courses, calculus, number theory, and real analysis.

Research Instruments

The identification of the types of proof construction failures and the exploration of the sources of the failures were explored using two research instruments. Firstly, the first instrument was validated by each research member and validated by the validator, before it was finally approved as an instrument to answer the research formulation. This instrument aimed to identify the types of failures in constructing proofs, which consisted of the following two proof problems.

1. Using the proof by contraposition. Prove that if a is an integer and a^2 is a multiple of 3, then a is a multiple of 3.
2. Using proof by contradiction. Prove that there is no rational number x that satisfies the equation $x^2 = 3$.

Second, the second instrument is the main instrument of data collection or in the case of the researcher himself, who observes, asks, hears, asks, and takes research data. Based on the types of construction failures, the researcher explores the sources that cause proof construction failures using semi-structured interview techniques. This semi-structured interview is intended to explore the ideas (prior knowledge/understanding) used as material to construct proofs for the problems posed (proof problems), and then how these ideas are activated to show the success or failure of construction.

Data Collection

Previously, the interviewed research subjects were selected based on the following selection mechanism, to obtain uniqueness in each failure, such as classical test, prospective subjects constructed proofs using the proof method instructed, correction/identification of construction success criteria on each item of the proof problem, if after identifying the success criteria a participant was found who successfully constructed proof problem-1 (PP1) and failed to construct proof problem-2 (PP2); or failed to construct both proof problems then the participant can be a prospective interview subject, and identify four or more types of failures in each prospective subject's construction results, if less than four then the prospective subject does not become a research subject interviewed or otherwise, and) the process of getting subjects is carried out several times until getting the same pattern (data saturation).

Data Analysis

Analyze proof answers in the form of written construction sheets to identify types of construction failures, based on the action steps, such as formulating supposition, manipulating and constructing supposition, identifying contradiction, and disproving supposition. The results of this analysis were then followed by semi-structured interviews. The analysis of the recorded interviews was based on the act of producing proofs and aimed at exploring/justifying the source of the causes of each type of construction failure, then used to explore the complexity of the logical basis that led to the failure of proof construction in proof by contradiction. After identifying the types of failures and exploring their causal sources, the next step is to code the types and causal sources of construction failures into the proof construction failure hypothesis framework. The failure hypothesis framework has three main hypotheses, namely operational hypotheses, affective hypotheses, and foundational hypotheses that are scattered in each type of failure. The results of this analysis are then intended to provide an overview of the sources of causes of failure in each type of student failure in constructing proof, as well as to develop a hypothesis framework for proof construction failure in proof by contradiction.

RESULTS AND DISCUSSION

Types of Failure in Constructing Proof

Identification of the types of proof construction failures begins with determining the frequency of failure criteria (success and failure) in constructing PP1 and PP2, where the frequency of students who succeed in constructing PP1 is 43 students, and the remaining 40 students fail. While the frequency of students who failed to construct PP2 was 65 students, and the rest were declared a success as many as 18 students. Not until determining the frequency of failure criteria, the next step is grouping the frequency of failure in constructing PP1 and PP2. This grouping of failure frequency groups students into 4 groups, namely: 1) failed to construct PP1 and PP2, 35 people, 2) failed to construct PP1 but successful in constructing PP2, 30 people, 3) success in constructing PP1 but failed to construct PP2, 5 people, and 4) success in constructing PP1 and PP2, 13 people.

The grouping of failure frequencies aims to facilitate the sampling of subjects who have types of proof construction failures in proof by contradiction. This grouping then decided to choose one subject representing each group, except for subjects who came from the group failing to construct PP1 but success in constructing PP2, and subjects who came from the group successful in constructing PP1 and PP2. Taking one subject based on failure in constructing PP2, is based on the ability to construct proofs in proof by contradiction contained in PP2. Some other reasons are based on some research results that show that "working with direct proofs is easier than working with contradiction proofs" (Tall, 1980; Reid & Dobbin, 1998; Lin et al., 1998; Brown, 2018; Chamberlain & Vidakovic, 2021), proof by contradiction is important in mathematical content (Rabin & Quarfoot, 2021), and "proof by contraposition can be material in constructing proof by contradiction" (Lin et al., 1998; Stylianides et al., 2004; Antonini & Mariotti, 2008; Jourdan & Yevdokimov, 2016; Brown, 2018; Doruk, 2019).

Taking one subject from each group decided that subject FTR and subject LSW as the focus subject of analysis. Subjects FTR and LSW, both failed in constructing PP2. Failure to construct PP2 is seen based on failure in the steps of formulating supposition, constructing and manipulating supposition, identifying contradictions, and disproving supposition. The failure to construct proof in these steps of proof by contradiction was then classified into four types of failure. The first three types of failure are based on the results of the study which show that students still have difficulties in (1) negating intuitive claims (Dubinsky et al., 1988; Reid & Dobbin, 1998; Sellers, 2018), (2) constructing and manipulating contradictory mathematical objects (Leron, 1985; Antonini & Mariotti, 2008), and (3) identifying contradictions (Barnard & Tall, 1997; Antonini & Mariotti, 2009; Brown, 2018; Antonini, 2019; Rabin & Quarfoot, 2021).

The identification of previous papers by David Quarfoot and Jeffrey M. Rabin shows that the "disproving supposition" step does not appear to be the main basis for developing the difficulty hypothesis framework in proof by contradiction proof, because disproving supposition is not the same as the lack of target hypothesis node (Quarfoot & Rabin, 2021; Rabin & Quarfoot, 2021), coupled with two papers by Darryl Chamberlain and Draga Vidakovic showing that disproving suppositions do not appear to be a focus in developing understanding in constructing proofs in proof by contradiction (Chamberlain & Vidakovic, 2016; 2021), and three other papers showing that the importance of disproving suppositions to conclude the supposition is false and is an important part of proof by contradiction (Balacheff, 1991; Reid et al., 2008; Lew & Zazkis, 2019). The last reason is like the previous introduction "disproving supposition" being the fourth step in constructing a proof in proof by contradiction, and based on the results of identifying the types of failures in this study which show that the steps of proof by contradiction have not been widely presented so that they are less understandable, and this result is also supported by research that shows common ways of reasoning in proof by contradiction are less numerous, less accessible to students, or less trained by teachers than ways of reasoning in direct proof (Antonini & Mariotti, 2008; Brown, 2018; Hanna & de Villiers, 2021; Tall, 1980; Thompson, 1966).

Thus, the results of this study, state that disproving supposition becomes the last step (the fourth step) which is very important to complete the types of failure (the fourth type of failure) in this study, as well as being material for the hypothesis framework for the failure of proof construction in proof by the following contradiction (see Figure 1).

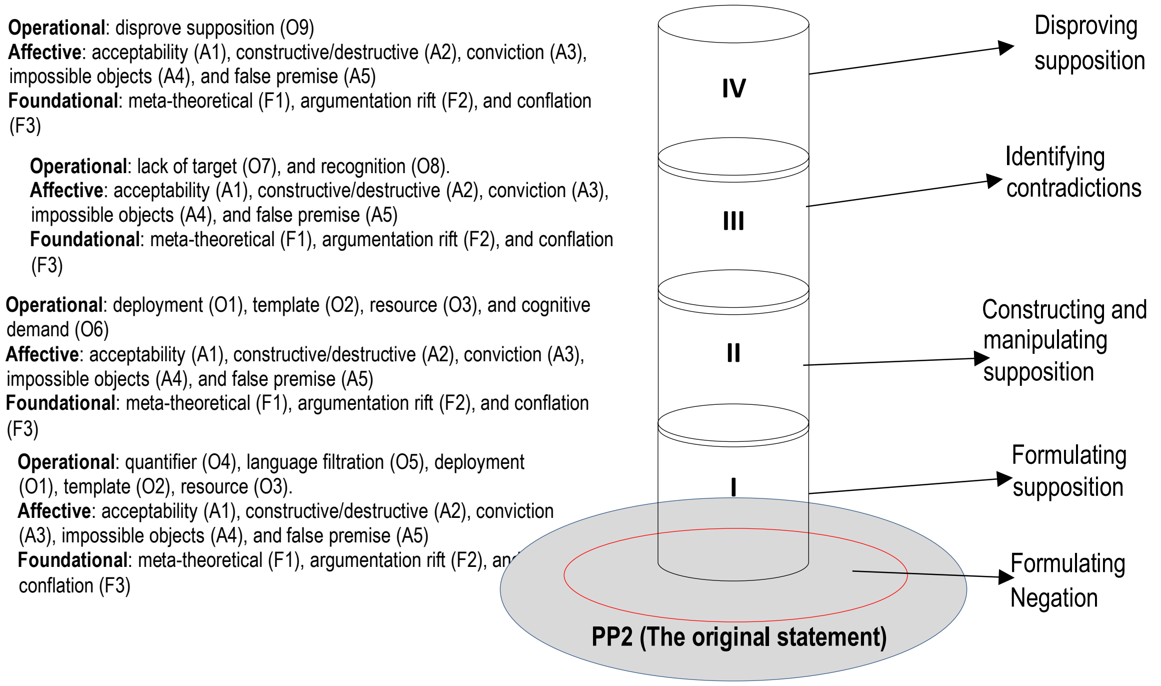


Figure 1. The Hypothesis Framework for (students' failure with) proof by contradiction

This failure hypothesis framework is composed of the four types of failure as described earlier. The nodes of the main hypotheses (operational, affective, and foundational) are distributed within each failure type. The 9 operational hypothesis nodes are divided into each failure type with different numbers, while the affective and foundational hypothesis nodes do not change, always being inside each failure type. The operational hypothesis has quantifier and language filtration hypothesis nodes which are two nodes connected by the negation hypothesis edge, and the negation hypothesis edge is one of the nodes in the training hypothesis. Not only that, the recognition node and lack of target hypothesis are two nodes connected by the contradiction hypothesis edge. The affective hypothesis has the acceptability node and constructive/destructive hypothesis which are two nodes connected by the socio-mathematical hypothesis edge, and the impossible object node and false premise hypothesis are two nodes connected by the false world hypothesis edge. The foundational hypothesis has three hypothesis nodes, namely: meta-theoretical, argumentation rift, and conflation.

Failure of Proof Construction, Emotional Response, and Logical Foundation

Based on the results of identifying the types of proof construction failures in proof by contradiction. Two subjects, each representing a selected group, will be traced for their construction action step failures based on the following three main hypotheses.

Subject FTR

The results of the analysis of the subject's proof answer sheet based on the action step of constructing proof, subject FTR was declared: 1) success in formulating the negation supposition of the original statement, 2) failure in constructing and manipulating suppositions, 3) failure in identifying contradictions, and 4) failure in disproving suppositions. The causes of success and failure as well as the logical basis for constructing this proof are then explained based on the following operational, affective, and foundational hypotheses.

Success in Formulating Supposition

The first step in formulating the supposition begins by ensuring that the original statement is true, then negating it so that the original statement is false, and finally formulating the supposition of the negation of the original statement. In this step, the FTR subject succeeded in formulating the supposition of the negation of the original statement, but the mathematical language structure of the supposition formulation was still considered inaccurate (see Figure 2).

Andaikan Ada bilangan rasional x yang memenuhi persamaan $x^2=3$. Karena x adalah bilangan rasional, maka $x=\frac{a}{b}$, dimana $a,b \in \mathbb{Z}$ dan $b \neq 0$ (a dan b relatif prima atau $\boxed{GCD(a,b)=1}$). //	**Translation** Suppose there is a rational number x that satisfies the equation $x^2 = 3$. Since x is a rational number, then $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. (a and b are relatively prime or $GCD(a, b) = 1$).

Figure 2. Subject FTR success in formulating supposition

The operational hypothesis indicated by this step shows that the supposition formulation is still not suitable to represent the correct supposition formulation (language filtration hypothesis). The use of proof by contradiction as PP2's proof method is only known based on the question clues (deployment hypothesis). This is because similar experiences related to the representation of proof problems are still lacking (template hypothesis). Therefore, the success in formulating supposition still needs to be improved by emphasizing the understanding of the negation of quaternary statements, the definition of rational and irrational numbers (resource hypothesis), and routinely activating knowledge through the activity of solving related proof problems (training hypotheses). The following interview results show the subject's emotional response in formulating a supposition.

Researcher	:	*Is the statement "there is no rational number x that satisfies the equation $x^2 = 3$" True or false?*
Subject FTR	:	*The statement is true because the values of x that satisfy the equation $x^2 = 3$ are $x = -\sqrt{3}$ and $x = \sqrt{3}$; and x is an irrational number.*
Researcher	:	*What about the negation of this statement?*
Subject FTR	:	*The negation of this statement is "There is a rational number x that satisfies the equation $x^2 = 3$".*
Researcher	:	*Then what method of proof do you use to show that this statement is true?*
Subject FTR	:	*Like the question instructions. The statement "there is no rational number x satisfies the equation $x^2 = 3$" is asked to be proved by the proof by contradiction.*
Researcher	:	*If the statement does not have a clue. How will you prove that this statement is true?*
Subject FTR	:	*Other than proof by contradiction. I still don't have much experience with this proofs problem.*
Researcher	:	*Okay, let's mention the steps of proving proof by contradiction.*
Subject FTR	:	*First, supposing the negation of an original statement is true, then proving the supposition using the contradiction proof method to get a contradiction, and finally stating the supposition is false, so the statement is true.*
Researcher	:	*Now try to formulate the supposition of the negation of this statement!*
Subject FTR	:	*Suppose there is a rational number x that satisfies the equation $x^2 = 3$.* *Since x is a rational number, then $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$, and $b \neq 0$.*

		a and b are relatively prime or $GCD(a, b) = 1$
Researcher	:	*Then, what does "a and b are relatively prime or $GCD(a, b) = 1$" mean in the formulation of the supposition.*
Subject FTR	:	*The fraction $\frac{a}{b}$ is the simplest form.*

The question in this interview shows that the success in formulating the supposition is influenced by the understanding of formulating the proof problem as a true value statement, then formulating the negation based on the quantifier, and formulating the supposition of the negation of the statement requires the subject to activate his knowledge. The activation of this knowledge leads the subject to inhabit a logical world that is false (false world hypothesis). The false world containing the formulation of the supposition of negation (false premise hypothesis) will require the subject to dare to work with the existential quantifier "there exists", the definition of a rational number, and the greatest common factor as the representational effect of the formulation of the true negation supposition (impossible object hypothesis).

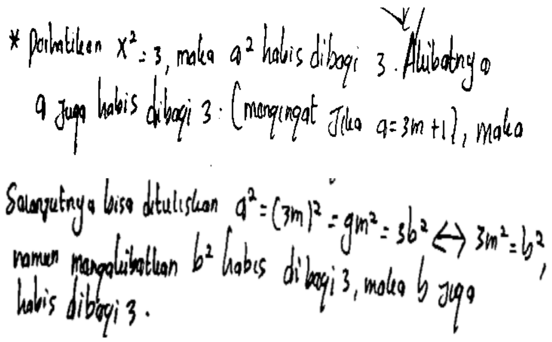
The demand of constructing the effect of this supposition, demands an understanding of the negation quantifier "there is no", and an understanding of the definition of rational numbers and the content of the greatest common factor that builds it (socio-mathematical hypothesis). This demand can be a construction medium to the stage of constructing and manipulating supposition (acceptability hypothesis), as well as a constructive means of activating knowledge (constructive/destructive hypothesis). Finally, understanding the steps of formulating a supposition, such as a statement must be true, understanding the quantifier contained, the negation of the quantifier, and understanding the effect of the negation of the quantifier can provide confidence in the deductive logic that is present (conviction hypothesis).

The logical foundation for success in formulating supposition is characterized in three foundational hypothesis activities. The first activity, characterized by the formulation of supposition which is the object of focus to be constructed, contains elements of statement, proof, and theory. In this case, the irrationality theorem 3 is represented in the mathematical sentence "There is no rational number x that satisfies the equation $x^2 = 3$" is considered successfully understood, so the subject succeeds in formulating the meta-theoretical hypothesis. The success in understanding the representation of the proof problem contributed to the cause of success in minimalizing the rift of action steps (argumentation rift hypothesis), and the success in formulating the supposition shows the association of ideas manifested in the formulation of the correct supposition (conflation hypothesis).

Failure in Constructing and Manipulating Supposition

The formulation of the assumption of negation then requires the subject to construct and manipulate the supposition based on the action step. The results in Figure 3 show that the subject failed to construct and manipulate the supposition.

The results of the construction and manipulation of the supposition show that the subject made a mistake in formulating the statement a^2 is divisible by 3. The next mistake is $a^2 = \cdots = 3b^2 \leftrightarrow 3m^2 = b^2$, or $a^2 = b^2$. These two errors seem to say a^2 is divisible by 3 based on $x^2 = 3$ without being based on a strong logical reason (deployment hypothesis), as a result, a is also divisible by 3, then choosing $a = 3m + 1$ or $a = 3m + 2$ to get $a^2 = 3n + 1$ (cognitive demand hypothesis). This is due to the lack of experience in constructing and manipulating similar/same ideas related to problem representation (template hypothesis), so it becomes a source of failure in constructing and manipulating supposition (resource hypothesis).



* perhatikan $x^2 = 3$, maka a^2 habis dibagi 3. Akibatnya a juga habis dibagi 3. (mengingat Jika $a = 3m + 1$), maka

Selanjutnya bisa dituliskan $a^2 = (3m)^2 = 9m^2 = 3b^2 \leftrightarrow 3m^2 = b^2$, namun mengakibatkan b^2 habis dibagi 3, maka b juga habis dibagi 3.

Translation
Consider $x^2 = 3$, then a^2 is divisible by 3. Hence a is also divisible by 3 (remembering that if $a = 3m + 1$), then … $a^2 = 3n + 1$
or if $a = 3m + 2$, then … $a^2 = 3n + 1$
Furthermore, we can write
$a^2 = (3m)^2 = 9m^2 = 3b^2 \leftrightarrow 3m^2 = b^2$, but as a result b^2 is divisible by 3, then b is also divisible by 3.

Figure 3. Subject FTR failed in constructing and manipulating supposition

The following interview results show the subject's emotional response in constructing and manipulating hypotheses.

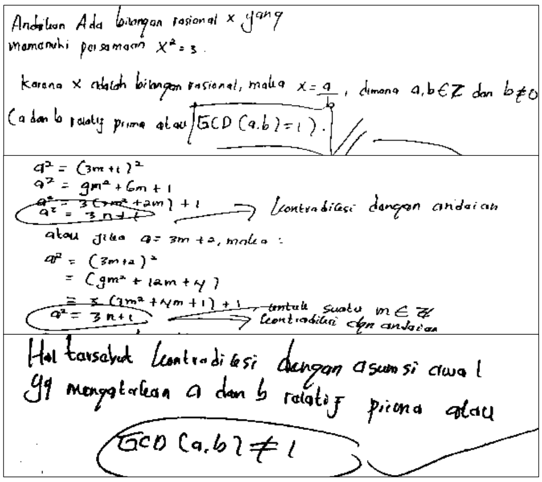
Researcher	:	*What made you write $x^2 = 3$, then a^2 is divisible by 3 and consequently a is also divisible by 3?*
Subject FTR	:	*Since x and a have the same degree of 2, $x^2 = a^2$, and since $x^2 = 3, a^2$ is divisible by 3, consequently, a is also divisible by 3.*
Researcher	:	*What do you understand by a number divisible by 3?*
Subject FTR	:	*A number divisible by 3 is a number whose sum of digits is divisible by 3.*
Researcher	:	*What made you choose $a = 3m + 1$ or $a = 3m + 2$?*
Subject FTR	:	*Well sir, since a^2 is divisible by 3, then a is also divisible by 3, to get a contradiction I chose $a = 3m + 1$ or $a = 3m + 2$, where $a = 3m + 1$ or $a = 3m + 2$ are both numbers not divisible by 3.*
Researcher	:	*Why did you choose $a = 3m$ in $a^2 = (3m)^2 = \cdots = 3b^2 \leftrightarrow 3m^2 = b^2$?*
Subject FTR	:	*Since a is divisible by 3, then $a = 3m$*
Researcher	:	*What is your reasoning for saying that $3m^2 = b^2$ results in b^2 being divisible by 3, so b is also divisible by 3?*
Subject FTR	:	*Because $b^2 = 3m^2$ is the definition of a number divisible by 3.*

The subject's success in formulating suppositions shows that the subject has not realized that he is entering a false world, so at this step, he decides to show that the world he is entering is indeed false in a different way and fails (constructive/destructive hypothesis), so this failure cannot be used as a medium to identify contradictions (acceptability hypothesis), and cannot provide confidence that the construction results are correct (conviction hypothesis). Awareness of the constructed and manipulated supposition formula can require the use of all available cognition to show that the subject is in a contradictory area (false world hypothesis). Constructing a false world in this area is manipulating a rational number x that satisfied the equation $x^2 = 3$ (false premise hypothesis), so the subject must dare to work to get results that contradict the supposition formulation (impossible object hypothesis).

The logical basis for the failure to construct and manipulate supposition is scattered in three foundational failure hypotheses. The first logical foundation, understanding working with suppositions to obtain contradictory results, then choosing seemingly to correct but erroneous steps will produce unacceptable construction results, such as $x^2 = a^2$ (meta-theoretical hypothesis). Basing the idea on $x^2 = a^2$ may lead to a^2 divisible by 3, so that a is divisible by 3. This would certainly have a rift with choosing $a = 3m$ to get $b^2 = 3m^2$ (argumentation rift hypothesis). So, if $x^2 = 3$ is used to say a^2 is divisible by 3 then using $a = 3m$ to get b^2 is divisible by 3, it will eventually contradict the conclusion that a and b are both divisible by 3 (conflation hypothesis).

Failure in Identifying Contradictions

The results of the identification of the subject's contradictions based on the results of the construction and manipulation of the supposition in Figure 4 produce two contradictions that show the results of the construction that contradict the initial supposition. These two contradictions then show that the subject did not work with the supposition, but instead chose to use something else that looked right but was wrong.



Translation

Suppose there is a rational number x that satisfies the equation $x^2 = 3$.
Since x is a rational number, then $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$.
(a and b are relatively prime or $GCD(a, b) = 1$).
....
if $a = 3m + 1$, then ... $a^2 = 3n + 1$, contradiction with the supposition
if $a = 3m + 2$, then ... $a^2 = 3n + 1$ contradiction with the supposition
....
This is a contradiction the earlier assumption that a and b are relatively prime or $GCD(a, b) = 1$.

Figure 4. Subject FTR failed in identifying contradiction

The two constructions that contradict the supposition are: 1) the construction result $a^2 = 3n + 1$ contradicts the supposition (unknown), and 2) the construction result $GCD(a, b) \neq 1$ is a contradiction $GCD(a, b) = 1$. Both contradictions fail, because one of the supposition references is unknown, and the construction result $GCD(a, b) \neq 1$ cannot be justified because it comes from an illogical idea (contradiction hypotheses). The constructed and manipulated hypothesis formulation cannot produce results that contradict the hypothesis, because the results of constructing $a^2 = 3n + 1$ and $b^2 = 3m^2$ both come from illogical ideas (lack of target hypothesis). Both contradictions also show that working with hypotheses requires an understanding of the hypotheses that are constructed and manipulated, to find a contradictory situation (recognition hypothesis). The following interview results show the subject's emotional response in identifying contradictions.

Researcher	:	*Other than $GCD(a, b) = 1$. Explain why $a^2 = 3m$ is also a supposition?*
Subject FTR	:	*The supposition $a^2 = 3m$ is based on the supposition that $x^2 = 3$ is the definition of a number divisible by 3.*
Researcher	:	*How do you find a construction that contradicts $a^2 = 3m$?*
Subject FTR	:	*Since $a = 3m$ is a number divisible by 3, as opposed to a number not divisible by 3, $a = 3m + 1$, or $a = 3m + 2$ is chosen.* *Substitute the value of $a = 3m + 1$, or $a = 3m + 2$ into the value of a^2, and we get $a^2 = 3n + 1$, where $n = 3m^2 + 2m$.* *Or $a^2 = 3n + 1$, where $n = 3m^2 + 4m + 1$.* *So $a^2 = 3n + 1$ contradicts the supposition $a^2 = 3m$.*
Researcher	:	*Can we base a contradiction (the result of construction and manipulation) on two suppositions?*
Subject FTR	:	*Because in my proofs answer two suppositions appear, namely $a^2 = 3m$ and*

		$GCD(a, b) = 1$. So I am still not sure if it is allowed, because I don't have an example. Please give me some guidance, sir.
Researcher	:	*Based on the statement "if a^2 is divisible by 3 then a is also divisible by 3" and "if b^2 is divisible by 3 then b is also divisible by 3". How do you determine the values of a and b?*
Subject FTR	:	*According to the definition of divisibility of integers. Firstly, since a is divisible by 3, there is an integer m such that $a = 3m$. Secondly, since b is divisible by 3, there is an integer n such that $b = 3n$.*
Researcher	:	*How do you determine that there is a contradiction?*
Subject FTR	:	*Since $GCD(a, b) = (3m, 3n) \neq 1$, it is a contradiction with $GCD(a, b) = 1$.*

This interview shows one of the contradictions: $a^2 = 3n + 1$ is said to contradict the supposition $a^2 = 3m$. This contradiction cannot be used to infer the existence of a contradiction, because the contradiction material $a^2 = 3n + 1$ and the refuted material $a^2 = 3m$ are both not based on appropriate logical ideas, while the second contradiction "$GCD(a, b) \neq 1$ contradicts $GCD(a, b) = 1$" is true, but the material of the contradiction comes from an inconsistent argument/idea, i.e. a is obtained from analogizing $x^2 = 3$ (x^2 is divisible by 3) with a^2, so a^2 is divisible by 3, and consequently a is divisible by 3, and so is b (constructive/destructive hypothesis). So, these two contradictions cannot be used as a medium to conclude that the supposition is false (acceptability hypothesis), and therefore they do not provide confidence in identifying contradictions based on logical constructions (conviction hypothesis). The identification of contradictory construction results, giving the subject experience that two contradictions cannot be a measure of success in finding contradictions, especially the contradiction material used to refute comes from the construction and manipulation of false ideas (false world hypothesis). Premise material derived from false ideas, of course, cannot be used to obtain the values of a and b (false premise hypothesis), so it takes effort to argue with cognition so that the premise used to obtain contradiction material must come from a logical construction idea (impossible object hypothesis).

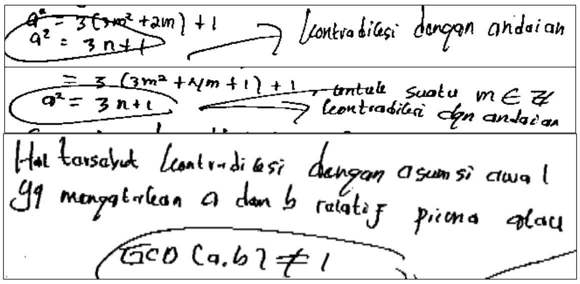
The failure to identify contradictions also has logical foundations that are spread over three foundational failure hypotheses. First logical foundation. The activity underlying this hypothesis is characterized by the appearance of $a^2 = 3n + 1$ and the appearance of the values a and b which are then constructed to obtain $GCD(a, b) \neq 1$ based on a and b both being divisible by 3. The material contradictions $a^2 = 3n + 1$ and $GCD(a, b) \neq 1$, respectively, are considered contradictions with $a^2 = 3m$ and $GCD(a, b) = 1$ indicating inconsistency with the supposition or what is to be constructed and manipulated (meta-theoretical hypothesis). The second logical foundation is characterized by the rift of the contradiction material resulting from the act of constructing and manipulating $a^2 = 3n + 1$ with the disproved material $a^2 = 3m$ (argumentation rift hypothesis), hence the effect of this action step rift on the third logical foundation, characterized by two contradictions that cannot infer the existence of a contradiction, because the contradiction material and reference both come from incompatible deductions (conflation hypothesis).

Failure in the Disproving Supposition

A disproving supposition or showing that supposition is wrong is not visible from the subject's construction results. The construction result in Figure 5 only shows a contradiction, even though it is false and there is no conclusion saying that the supposition is false.

Disproving a supposition is not the same as identifying a contradiction. Identifying contradictions is only in the domain of determining whether there is a contradiction occurring, i.e., between the construction result and the supposition. While refuting a supposition is stating that the supposition is false.

The statement "it contradicts the earlier assumption that a and b are relatively prime" based on $GCD(a, b) \neq 1$ contradicting $GCD(a, b) = 1$, cannot be said as disproving the supposition.



Translation
if $a = 3m + 1$, then … $a^2 = 3n + 1$, contradiction with the supposition
if $a = 3m + 2$, then … $a^2 = 3n + 1$ contradiction with the supposition
…
This is a contradiction the earlier assumption that a and b are relatively prime or $GCD(a, b) = 1$.

Figure 5. Subject FTR failed in disproving supposition

The non-occurrence of this disproof is then traced through the following interview activity.

Researcher	:	*Explain how your disproof the supposition with the construction $a^2 = 3n + 1$?*
Subject FTR	:	$a^2 = 3n + 1$ *contradicts* $a^2 = 3m$.
Researcher	:	*How does $GCD(a, b) \neq 1$ contradict $GCD(a, b) = 1$?*
Subject FTR	:	*a and b are both divisible by 3. This contradicts the earlier assumption, which says that a and b are relatively prime.*
Researcher	:	*Which earlier assumption are you referring to?*
Subject FTR	:	*The earlier assumption that says, "there exist is a rational number x that satisfies the equation $x^2 = 3$".*
Researcher	:	*Then what can be concluded if there is a contradiction with the earlier assumption?*
Subject FTR	:	*$GCD(a, b) \neq 1$ contradicts $GCD(a, b) = 1$. Please guide me, sir.*

The emotional response or view from the interview results above shows that the subject failed to conclude the construction results that contradict the supposition. The subject realized that he was unable to conclude the meaning of "$GCD(a, b) \neq 1$ contradicts $GCD(a, b) = 1$" which indicates that the proof construction process has been completed, so he asked for stimulation to activate ideas to get a valid proof construction (constructive/destructive hypothesis). Stimulation is important as a medium to conclude that the supposition is false (acceptability hypothesis). The failure to disprove the supposition is due to the subject's assumption of disproving the supposition is to conclude a contradiction, not to conclude that the supposition is wrong, or the original statement is true, so giving confidence through stimulation of important ideas to build/activate ideas is an action that needs to be considered (conviction hypothesis). Conviction hypothesis denial shows that the result of the construction contradicts the supposition so that the originally true negation hypothesis becomes false. Disproving a supposition can make the subject realize that they have succeeded in showing that the world they live in is false (false world hypothesis). Negating a supposition based on a construct that appears true but is false, shows that the premise or idea of the construct comes from an illogical idea (false premise hypothesis). Failure to refute a supposition can result from using the wrong refutation material, due to the wrong way of constructing the conflicting object (impossible object hypothesis).

The failure to disprove supposition has three logical bases spread over three foundational failure hypotheses. Firstly, the logical foundation of the failure to disprove supposition focuses on the conclusion that "there is no rational number x that satisfies the equation $x^2 = 3$". This conclusion can be formulated as the effect of having found a result that contradicts the supposition. This conclusion can also be a sign that the construction process has been completed or successful. However, the subject, in this case, failed to conclude that "there is no rational number x that satisfies the equation $x^2 = 3$" based on the construction results that are considered to have found a contradiction (meta-theoretical hypothesis). Then the second logical basis, which is characterized by the emergence of two refute formulations, namely: "$a^2 = 3n + 1$ contradicts $a^2 = 3m$" and $GCD(a, b) \neq 1$ contradicts $GCD(a, b) = 1$, both of which are derived from the logically unacceptable idea $x^2 = 3$ (argumentation rift hypothesis). The last logical foundation, seen from the two formulations of the negation "$a^2 = 3n + 1$ contradicts $a^2 = 3m$" and $GCD(a, b) \neq 1$ contradicts the supposition $GCD(a, b) = 1$, cannot be conflated to be used as material to conclude that "there is no rational number x satisfies the equation $x^2 = 3$" because the negation used to conclude the original statement is true is only one, and of course the negation is obtained from the construction and manipulation of the corresponding supposition (conflation hypothesis).

The four types of proof construction failures and the causes of their occurrence will then be depicted in the hypothetical framework of proof construction failures in proof by contradiction in Figure 6 below, to justify the source of the causes of each type of failure when FTR subjects engage in proof by contradiction.

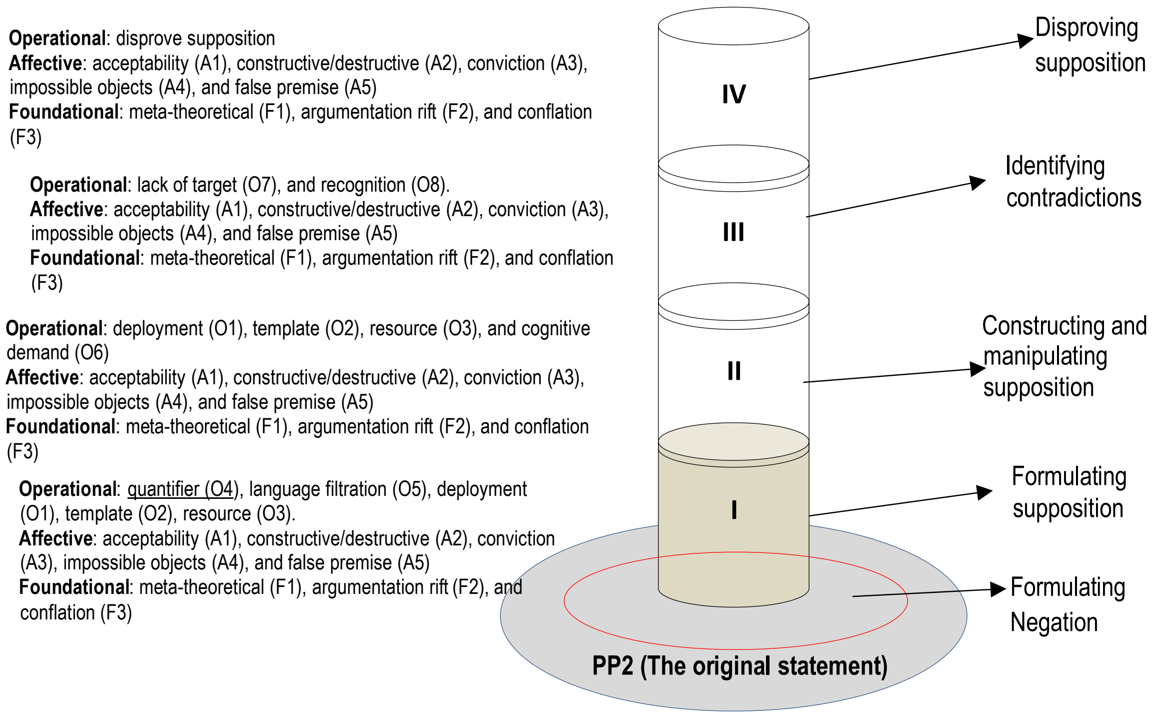


Figure 6. Hypothetical framework of proof construction failure and its causes by subject FTR

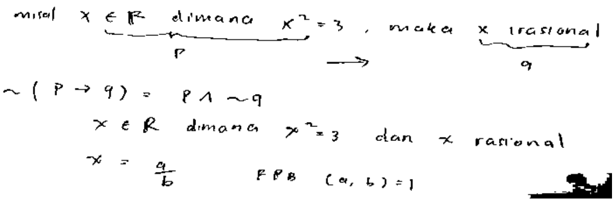
The framework for the failure of proof construction and its causes shows that the FTR subject is considered successful in formulating supposition (I), so the negation hypothesis (quantifier hypothesis) is considered successful, although some hypotheses such as the language filtration hypothesis are still inaccurate, but have enough substance to formulate supposition, so the FTR subject still cannot be said to have failed in formulating supposition. While the action step of disproving supposition (IV), the FTR subject is considered a failure because it does not formulate a disproof of the supposition, so the supposition is false or the original statement is true, and finally, the steps of constructing and manipulating supposition (II) and identifying contradictions (III) are considered a failure for several reasons as mentioned earlier.

Subject LSW

The results of the analysis of the subject's proof answer sheet on the action step of constructing proof in proof by contradiction, subject LSW stated: 1) failed to formulate the negation supposition of the original statement, 2) failed to construct and manipulate suppositions, 3) failed to identify contradictions, and 4) failed to disprove suppositions. The causes of failure as well as the logical basis in constructing this proof are then explained based on the following operational, affective, and foundational hypotheses.

Failure in Formulating Supposition

Failure to formulate the supposition, in this case, can be seen from the way (1) converting a single statement into a conditional/implication form, namely: $p: x \in \mathbb{R}$ where $x^2 = 3$ and $q: x$ irrational; (2) making the logical equivalence of the implication, namely $\sim(p \to q) \equiv p \wedge \sim q$, with $p: x \in \mathbb{R}$ where $x^2 = 3$ and $\sim q: x$ rational; and (3) writing the expression $x = \frac{a}{b}$, $GCD(a, b) = 1$ with incomplete conditions. All three construction activities failed in formulating supposition.



Translation
Let $x \in \mathbb{R}$ where $x^2 = 3$, then x is irrational
$\sim(p \to q) \equiv p \wedge \sim q$
$x \in \mathbb{R}$ where $x^2 = 3$ and x is rational

Figure 7. Subject LSW failed in formulating supposition

The construction results in Figure 7 show that: (1) the subject did not see PP2 as a single statement, but rather as a conditional statement, and the subject is suspected of not understanding the non-existential (universal) quantifier "not exist (all): ∀" contained in PP2 (quantifier hypothesis), did not understand the negation of the quantified statement (negation hypothesis), and thus could not formulate the negation of the sentence/quantified statement with proper mathematical language (language filtration hypothesis). The proof method to solve PP2's problem was only based on the question clues (deployment hypothesis). This is due to the lack of similar experiences related to the problem representation (template hypothesis), so understanding the problem representation along with the steps of proving proof of contradiction is the cause of failure in formulating supposition (resource hypothesis). The failure to formulate this hypothesis needs to be interviewed to explore the cause of the failure.

Researcher	:	Can you explain the true value of the statement "There is no rational number x that satisfies the equation $x^2 = 3$"?
Subject LSW	:	The statement is true because there is no rational number $\frac{a}{b}$ that can be expressed as $\left(\frac{a}{b}\right)^2 = 3$.
Researcher	:	What method of proof would you use to explain the truth of the statement?
Subject LSW	:	According to the question instructions, of course, I will prove by the method of proof by contradiction.
Researcher	:	How would you prove that this statement is true, using the steps of contradiction proof?
Subject LSW	:	First, determine the premises p and q of the statement: there is no rational number x that satisfies the equation $x^2 = 3$, with $p: x \in \mathbb{R}$ where $x^2 = 3$ and $q: x$ is

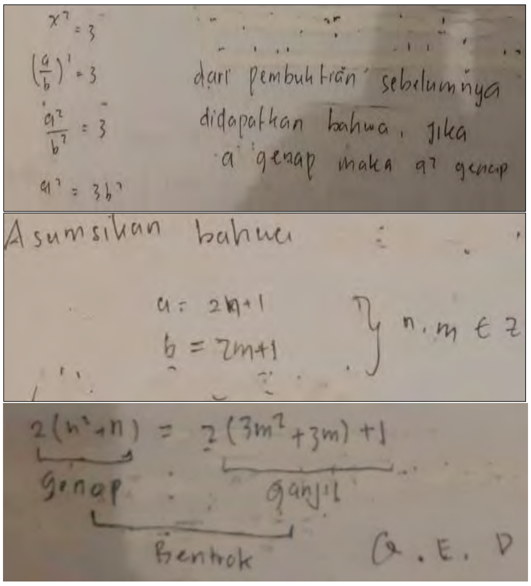
		irrational; then make a logical equivalence: $\sim(p \to q) \equiv p \wedge \sim q$; and prove to show the existence of a contradiction.
Researcher	:	How do you see statements as conditional statements?
Subject LSW	:	Initially, I saw the original statement as a causal statement, where the cause is $p: x \in \mathbb{R}$ where $x^2 = 3$ and since "there are no rational numbers" then the effect is $q: x$ is irrational.
Researcher	:	Okay, if you believe that the statement "there is no rational number x that satisfies the equation $x^2 = 3$" is true. Now try to formulate its negation!
Subject LSW	:	If p: There is no rational number x that satisfies the equation $x^2 = 3$; then $\sim p$: There is a rational number x that satisfies the equation $x^2 = 3$.
Researcher	:	Is $\sim p$ true, if not then explain why $\sim p$ is false.
Subject LSW	:	No, $\sim p$ is wrong. Because the values of x that satisfy the equation $x^2 = 3$ are $x = -\sqrt{3}$ and $x = \sqrt{3}$. While $x = -\sqrt{3}$ and $x = \sqrt{3}$ are both irrational numbers.
Researcher	:	Now suppose that $\sim p$ is true!
Subject LSW	:	How is it possible to suppose something that is considered false to be true? But I tried it already. Suppose $\sim p$ is true, then there is a rational number x that satisfies the equation $x^2 = 3$.
Researcher	:	What is the result of supposing $\sim p$ is true?
Subject LSW	:	Suppose there is a rational number x that satisfies the equation $x^2 = 3$, then there are $a, b \in \mathbb{Z}$ such that $x = \frac{a}{b}$.
Researcher	:	Since $b \in \mathbb{Z}$, what if $b = 0$? Can $x = \frac{a}{b}$ still, be said to be a rational number? and $GCD(a, b) = 1$?
Subject LSW	:	If $b = 0$, then $x = \frac{a}{b}$ cannot be said to be a rational number, but $GCD(a, b) = 1$. But if $b \neq 0$, then $x = \frac{a}{b}$ can be expressed as a rational number, and $GCD(a, b) = 1$.
Researcher	:	Thank you very much for your guidance, sir.

Being aware of the lack of conditions on the definition of a rational number shows that this interview can activate prior knowledge. Questions such as whether the original statement is true, then how to formulate the negation, and finally how to formulate the supposition of the negation of the original statement, led to the awareness that belief in the steps of proof by contradiction known beforehand was erroneous. The question arises: "How is it possible to suppose something that is considered false to be true", this gives the subject the awareness that he is inhabiting a false world (false world hypothesis). A false world that contains the supposition formulation of the negation of the original statement that is true (false premise hypothesis) and will require the subject to dare to work with the material idea as a representational effect of the supposition formulation (impossible object hypothesis). This demand can be a medium to then construct and manipulate the supposition to obtain results that can be identified as contrary to the supposition (acceptability hypothesis), as well as a constructive means of working with the supposition (constructive/destructive hypothesis). It thus requires a conviction to realize that working with supposition requires a strong mental effort so that the results obtained can be accepted as logically derived ideas (conviction hypothesis).

Failure in formulating suppositions has a logical basis spread over three foundational failure hypotheses. The first foundation, the failure to formulate the supposition is due to the subject's understanding of the form of the statement in PP2, thus leading the subject to make the logical equivalence of the conditional statement, and then formulate the negation of PP2 based on the logical equivalence (meta-theoretical hypothesis). The second foundation, the failure in formulating the supposition is caused by the subject's understanding of the problem at hand, namely the mistake of assuming that the focused object proved as a conditional statement is not a single statement (argumentation rift hypothesis). The third foundation, the failure in formulating supposition in PP2, provides an understanding of the types of statements. A single statement should not be viewed as a conditional statement, because it can affect the form of logical equivalence and logical structure of the mathematical sentence contained in PP2 (conflation hypothesis).

Failed in Constructing and Manipulating Supposition

The results of the construction and manipulation of supposition show that the subject made an error in stating that a^2 is an even number based on "if a is even, then a^2 is even", made the assumption that $a = 2n + 1$ and $b = 2m + 1$, where $n, m \in \mathbb{Z}$, then substituted them into the expression $a^2 = 3b^2$ and claimed to have obtained conflicting results.



Translation

Substitute $x = \frac{a}{b}$ into $x^2 = 3$, we get $\left(\frac{a}{b}\right)^2 = 3$

$$a^2 = 3b^2$$

From the previous proof, it is found that "if a is even then a^2 is even".

Assume $a = 2n + 1$ and $b = 2m + 1$, with $n, m \in \mathbb{Z}$

Substitute $a = 2n + 1$ and $b = 2m + 1$ into the equation $a^2 = 3b^2$, we get

$$\underbrace{2(n^2 + n)}_{even} = \underbrace{2(3m^2 + 3m) + 1}_{odd}$$

Q.E.D

Figure 8. Subject LSW failed in constructing and manipulating suppositions

The results of the construction and manipulation of supposition in Figure 8 show that connecting ideas, definitions, properties, and/or statements that have been proven before but not by the problem at hand, apparently gives cognitive load and makes cognitive strain in thinking about contradictory statements (cognitive demand hypothesis). Burden and cognitive tension are seen from the formulation of the statement "if a is even, then a^2 is even" based on $a^2 = 3b^2$. So, failure in the step of constructing and manipulating supposition needs to be traced with the following interview.

Researcher	:	In this step, what do you understand about even numbers and odd numbers?
Subject LSW	:	An even number is a number that is divisible by 2, while an odd number is a number that is not divisible by 2,
Researcher	:	What about the formal definition of even and odd numbers?
Subject LSW	:	An even number is a whole number that can be expressed in the form $n = 2k$, where k is an integer. While odd numbers are integers with the form $n = 2k + 1$.
Researcher	:	How to express a number that is divisible by 3?
Subject LSW	:	What I know about numbers that are divisible by 3, a number that is divisible by 3 is a number whose sum of digits is divisible by 3. But regarding how to express it, I still need to remember more. Please give me some guidance, sir.
Researcher	:	What made you substitute the value of $x = \frac{a}{b}$ into the equation $x^2 = 3$?
Subject LSW	:	To get the value of $a^2 = 3b^2$, sir.

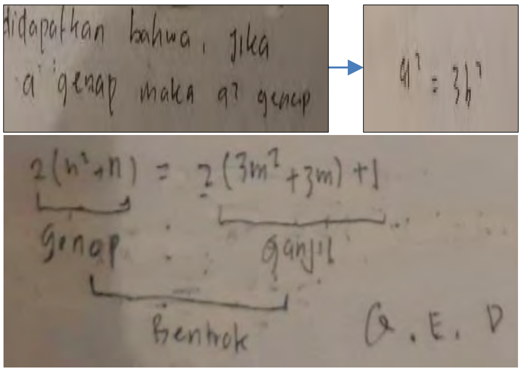
Researcher	:	What do you understand by $a^2 = 3b^2$?
Subject LSW	:	$a^2 = 3b^2$ is an even number or an odd number.
Researcher	:	What made you write "if a is even then a^2 is even" based on $a^2 = 3b^2$.
Subject LSW	:	It's like this sir, I want to justify the statement "if a is even then a^2 is even" by substituting $a = 2n+1$ and $b = 2m+1$ into the equation $a^2 = 3b^2$. The result will be whether a^2 is odd?
Researcher	:	Why do you assume $a = 2n+1$ and $b = 2m+1$ are odd?
Subject LSW	:	Because of the statement "if a is even then a^2 is even".
Researcher	:	I see. Now try to show the statement you mean.
Subject LSW	:	$a^2 = 3b^2$ $(2n+1)^2 = 3(2m+1)^2$ $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ *There is a conflict/contradiction sir.*
Researcher	:	*Do you know,* - *made the statement "if a is even then a^2 is even" based on $a^2 = 3b^2$, is it correct?* - *Assuming $a = 2n+1$ and $b = 2m+1$ based on the statement "if a is even then a^2 is even", then substituting into the equation $a^2 = 3b^2$, is appropriate?*
Subject LSW	:	*I don't think so, sir. It's just that I originally thought that the proof by contraposition could help in proving the proof by contradiction.*

The results of the interview above show that the failure in formulating a supposition affects the way of constructing and manipulating a supposition. The subject failed in constructing and manipulating the supposition because he assumed that $a^2 = 3b^2$ was the definition of an even number, not the definition of a number divisible by 3. The supposition of a^2 as an even number then led the subject to assume $a = 2n+1$ and $b = 2m+1$ and substitute them into the equation $a^2 = 3b^2$ to obtain the contradictory result (constructive/destructive hypothesis). In this case, knowledge of the representation of numbers divisible by 3 greatly influenced the construction result (acceptability hypothesis) and could not give confidence that a contradiction had been found (conviction hypothesis). The subject's awareness of the problems that will be faced in the false world (if the negation is true), has not been able to stimulate the subject to show that it is true that they are living in a false world, by showing a contradiction (false world hypothesis). Failure to construct $a^2 = 3b^2$ to obtain a contradictory result is a representation of a false premise (false premise hypothesis), because it comes from the idea of incompatible knowledge (impossible object hypothesis).

The logical underpinnings of this constructing and manipulating step are also spread across three foundational failure hypotheses. The first logical foundation, characterized by a focus on the supposition "there is a rational number x that satisfies the equation $x^2 = 3$" that will be constructed and manipulated to obtain a result that does not match the claim of the supposition (meta-theoretical hypothesis). The second logical foundation, characterized by the rift of the initial construction step (substitution of $x = \frac{a}{b}$ into the equation $a^2 = 3b^2$) with the unformulated supposition and the rift of manipulation of the initial construction result with the wrong premise of "if a is even then a^2 is even" (argumentation rift hypothesis). The last (third) logical foundation, is characterized by the conclusion that a is an even number, because a^2 is even based on the equation $a^2 = 3b^2$ which is understood as the definition of an even number, not a number divisible by 3, and the equation $a^2 = 3b^2$ is used to obtain a result that is considered conflicting (conflation hypothesis).

Failure in Identifying Contradictions

The results of the contradiction identification, in this case, show that the subject failed to identify the contradiction from the argument he wrote because the construction results that show incompatible results (left-right segment) are considered a form of contradiction and do not refer to the supposition (see Figure 9).



Translation

" if a is even then a^2 is even". Hence, $a^2 = 3b^2$ is even.

From the substitution of assumptions a and b, we get

$$\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$$

Clashes. Q.E.D

Figure 9. Subject LSW failed in identifying contradiction

Failure to identify contradictions as a basis for disproving supposition is called failing according to contradiction hypotheses. The construction result $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ is considered a contradiction, because the left and right segments of the construction result are different (lack of target hypothesis). By not referring to the supposition (because no supposition was formulated), the result of this construction shows that the subject struggled enough to identify that the contradiction was achieved, even though the result failed (recognition hypothesis). The three types of hypotheses were identified as contradictions through the following interview.

Researcher	:	*Based on the given problem (proof problem 2), which of your constructions is said to be a supposition?*
Subject LSW	:	*Since the proof problem is "there is no rational number x satisfies the equation $x^2 = 3$", it is known that $p: x \in \mathbb{Z}$ where $x^2 = 3$ and $q: x$ is irrational. Then the supposition of the negation of this statement is $\sim(p \rightarrow q) \equiv p \wedge \sim q$.*
Researcher	:	*What is the meaning of $\sim(p \rightarrow q) \equiv p \wedge \sim q$ that you consider being this supposition?*
Subject LSW	:	*Suppose $x \in \mathbb{R}$, where $x^2 = 3$ and x is rational.*
Researcher	:	*What is your basis for saying $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ as a conflicting result?*
Subject LSW	:	*The two segments of $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ are different.*
Researcher	:	*What is your reason for substituting $a = 2n+1$ and $b = 2m+1$ into the equation $a^2 = 3b^2$ to conclude that there is a clash?*
Subject LSW	:	*Because $a^2 = 3b^2$ is an even number, and $a = 2n+1$ and $b = 2m+1$ are both odd numbers.*
Researcher	:	*Isn't the supposition you have formulated "suppose $x \in \mathbb{R}$, where $x^2 = 3$ and x is rational". Why doesn't your result refer to that supposition?*
Subject LSW	:	*Because the construction result shows a clash, so there is no need to refer to the supposition. Please give me some guidance. Sir.*

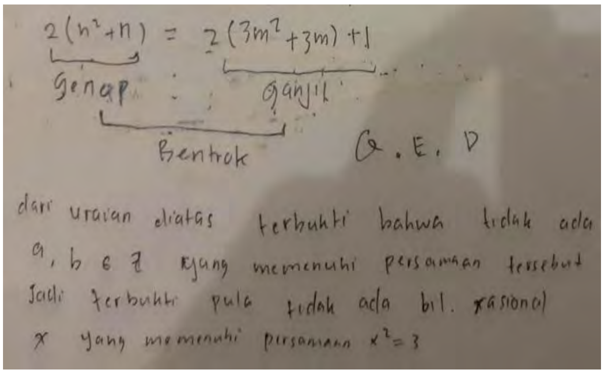
Materializing $a^2 = 3b^2$ as a representation of even numbers based on the understanding of the proof of contraposition resulted in the subject failing to construct the proof by contradiction. Interestingly at this stage, the subject was aware of having formulated a supposition, namely "if $x \in \mathbb{R}$, where $x^2 = 3$ and x is rational". However, the supposition was not used as a reference for contradiction. The appearance of the supposition "if $x \in \mathbb{R}$, where $x^2 = 3$ and x is rational", although wrong, makes it interesting because it appears at the stage of identifying contradictions, not at the stage of formulating suppositions. The emergence of this new supposition was possible due to the interview question at this stage and the subject's awareness of the supposition he believed in. Furthermore, the assumption of having successfully found a contradiction through the construction result of $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ as a form of construction of the assumptions $a = 2n+1$ and $b = 2m+1$ by substituting into the equation $a^2 = 3b^2$ cannot be justified (constructive/destructive hypothesis). The result of this construction cannot be used as a medium to find contradictions (acceptability hypothesis). Thus, it requires activating ideas to give confidence in constructing and manipulating suppositions with logical deduction rules (conviction hypothesis). The identification of construction results that cannot be said to be contradictions has awakened the subject to the territory it inhabits (false world hypothesis), because the construction results cannot be used as material to identify contradictions (false premise hypothesis), and these construction result materials are not built based on appropriate logical ideas as the representational effect of the supposition formulation (impossible object hypothesis).

The logical foundations that led to the failure to identify the contradiction were also scattered in the three foundational failure hypotheses. The first logical foundation, though, fails. The failure is because the construction $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ cannot be used to show the existence of a contradiction (meta-theoretical hypothesis). The second logical foundation is that the construction does not refer to the supposition "if $x \in \mathbb{R}$, where $x^2 = 3$ and x is rational" because the construction already shows a clash (different right-left segments) (argumentation rift hypothesis). The last (third) logical foundation, the conflation/contradiction in the construction $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ does not require a reference to the supposition "if $x \in \mathbb{R}$, where $x^2 = 3$ and x is rational" (conflation hypothesis).

Failure in the Disproving Supposition

The results of the analysis of the subject's construction failure in the step of disproving supposition were declared a failure. Failure to disprove a supposition is caused by failure to produce a contradiction. The construction result $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ as in Figure 10 cannot be used to show that the supposition is false.

The assumptions $a = 2n+1$ and $b = 2m+1$ with $n, m \in \mathbb{Z}$ which are substituted into the expression $a^2 = 3b^2$ and obtain $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$, are then used to show that "there is no rational number $a, b \in \mathbb{Z}$ satisfies the equation (unknown)" and was judged unacceptable or considered a failure, because the end of the proof by contradiction was trying to disprove the supposition of the negation of PP2, not based on the definition of integers (disproving supposition).


$\underbrace{2(n^2+n)}_{\text{genap}} = \underbrace{2(3m^2+3m)+1}_{\text{ganjil}}$
bentrok Q.E.D
dari uraian diatas terbukti bahwa tidak ada $a, b \in \mathbb{Z}$ yang memenuhi persamaan tersebut
Jadi terbukti pula tidak ada bil. rasional x yang memenuhi persamaan $x^2=3$

Translation
Because the result of substituting assumptions a and b into the equation $a^2 = 3b^2$ are

$$\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$$

Shows that both segments are different (clash) Q.E.D
It is proved that "there is no $a, b \in \mathbb{Z}$ that satisfies the equations".
So it is proved that "there is no rational number x that satisfies the equation $x^2 = 3$".

Figure 10. Subject LSW failed in the disproving supposition

The results of this construction were confirmed through the following interview to justify the source of the failure in disproving the supposition.

Researcher	:	*Based on the construction you worked out. How do you say that the result:* $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ *can be inferred that* "there no is $a, b \in \mathbb{Z}$ *satisfies the equation"? Which equation are you referring to?*
Subject LSW	:	*The equation* $a^2 = 3b^2$ *and the result* $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ *which indicates a clash.*
Researcher	:	*How can the statement "there is no* $a, b \in \mathbb{Z}$ *that satisfies the equation* $a^2 = 3b^2$" *inferred/disprove "there is no rational number* x *that satisfies the equation* $x^2 = 3$"?
Subject LSW	:	*Because the substitution of* $a = 2n+1$ *and* $b = 2m+1$ *into the equation* $a^2 = 3b^2$ *does not produce different results (left and right segments are different).*
Researcher	:	*So, if* a, b *is even then the result of substitution in the equation* $a^2 = 3b^2$ *is even?; and if a,b is odd then the result of substitution into the equation* $a^2 = 3b^2$ *is odd?*
Subject LSW	:	*Yes sir, because* $a^2 = 3b^2$ *is the definition of an even number.*
Researcher	:	*Going back to the statement "there is no* $a, b \in \mathbb{Z}$ *that satisfies the equation* $a^2 = 3b^2$" *can disprove "there is no rational number* x *that satisfies the equation* $x^2 = 3$". *Is there a difference between the equations* $a^2 = 3b^2$ *and* $x^2 = 3$*, or otherwise?*
Subject LSW	:	$a^2 = 3b^2$ *and* $x^2 = 3$*, different variables, same degree (power of 2) but different numbers.*

The interview results showed that the failure to disprove the supposition was due to the assumption that the left-right segments of the construction $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ were different (clashing) so that the conclusion "there is no $a, b \in \mathbb{Z}$ that satisfies the equation $a^2 = 3b^2$", and then back to conclude the supposition is false (constructive/destructive hypothesis). There is no reference to the supposition of contradiction in the construction result $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$, because the construction result already shows the existence of a different (conflicting) result (acceptability hypothesis). The assumption that the construction result $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ is a contradiction so that "there is no $a, b \in \mathbb{Z}$ that satisfies the equation $a^2 = 3b^2$", assumes that the inhabited world is a false world (false world hypotheses), because the construction result obtained shows a contradiction (false premise hypothesis), but this result cannot give confidence that the original statement is true (conviction hypothesis).

The failure in disproving supposition has three logical bases spread over three foundational failure hypotheses. The first foundation hypothesis, the construction $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ which shows a clash, comes from the erroneous idea that $a^2 = 3b^2$ is the definition of an even number based on "if a^2 is even then a is even", and does not refer to the meta-theoretical hypothesis, and since disproving supposition is concluding the hypothesis is false, this construction cannot refer to the hypothesis because the construction already shows a contradiction. This condition can lead to argumentation rift because it is based on incompatible ideas and construction steps (argumentation rift hypothesis). The two statements "there is no $a, b \in \mathbb{Z}$ that satisfies the equation $a^2 = 3b^2$" and "There is no rational number x that satisfies the equation $x^2 = 3$" cannot be conflation as a successful disproving supposition, because the former statement does not come from the corresponding logical idea (conflation hypothesis).

The four types of proof construction failures and their causes are then depicted/formulated in the hypothetical framework of failures proof construction with proof by contradiction in Figure 11 below, to justify the source of causes of each type of failure when LSW subjects engage with the contradiction method.

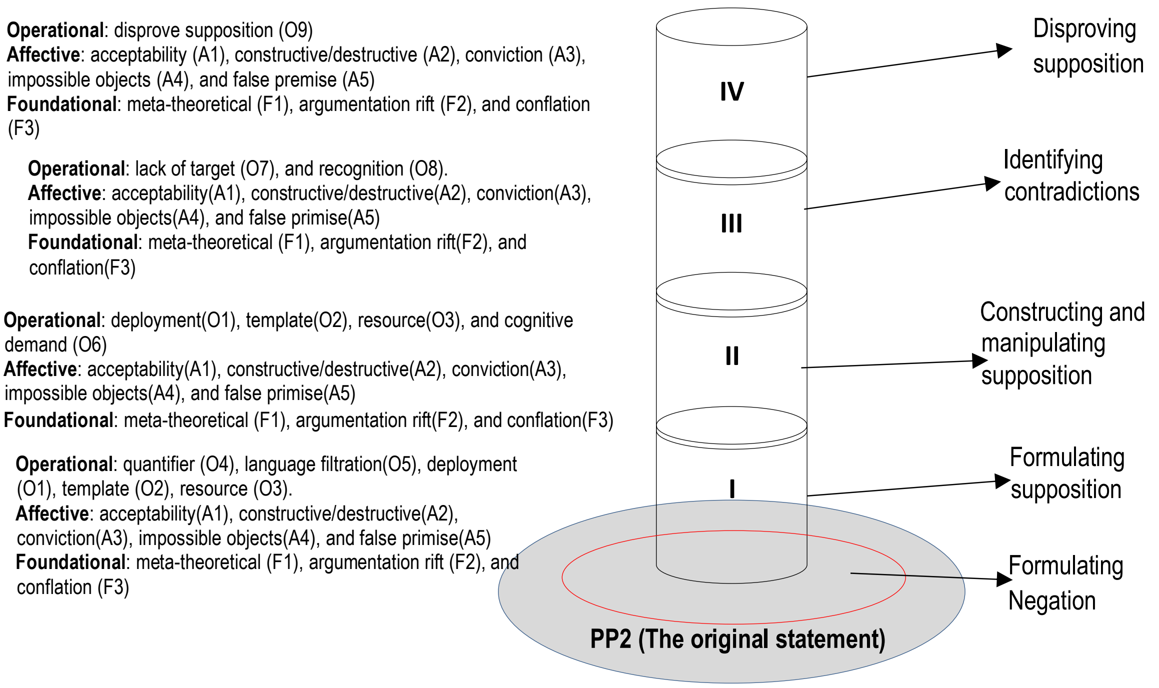


Figure 11. Hypothetical framework of proof construction failure and its causes by subject LSW

In contrast to the failure hypothesis framework belonging to subject FTR. The proof construction failure hypothesis framework in Figure 11 shows that LSW subject failed in all steps of proof construction actions in proof by contradiction, so all the nodes in the operational, affective, and foundational hypotheses are scattered in each type of failure. Subject LSW's failure hypothesis framework adequately describes all the hypotheses contained in the failure types and is the same as the hypotheses contained in the proof construction failure hypothesis framework in Figure 1.

The discussion of the results of the data analysis of this study begins with the types of proof construction failures with contradictions. Previously, the types of failures were classified based on construction failures in the steps of formulating supposition (F1), constructing and manipulating supposition (F2), identifying contradictions (F3), and disproving supposition (F4). The classification of these four types of failure is based on: (1) difficulty in negating intuitive claims (Dubinsky et al., 1988; Reid & Dobbin, 1998; Sellers, 2018), (2) constructing and manipulating contradictory mathematical objects (Leron, 1985; Antonini & Mariotti, 2008), (3) identifying contradictions (Barnard & Tall, 1997; Antonini & Mariotti, 2009; Brown, 2018; Antonini, 2019; Chamberlain & Vidakovic, 2016), and (4) results showing that disproving suppositions do not appear to be a focus in developing understanding in constructing proof by contradiction (Chamberlain & Vidakovic, 2016; 2021), and disproving suppositions does not appear to be the main basis for developing hypothesis frameworks of difficulties with proof by contradiction (Quarfoot & Rabin, 2021; Rabin & Quarfoot, 2021).

The development of the proof of construction failure hypothesis framework with the building blocks of four failure types has resulted in 17 failure hypothesis nodes, whereas previously the nodes of the difficulty hypothesis framework were 16 (Quarfoot & Rabin, 2021). These 17 nodes are spread across the four construction failure types. The distribution of failure hypothesis nodes in affective hypotheses and foundational hypotheses has not changed from before they were developed. Hypothesis nodes only changed in the operational hypothesis, where previously there were 8 nodes and after being developed there were 9 nodes. The development of this hypothesis framework aims to justify the source of the cause of failure at each step of the action of constructing proof by contradiction.

Justification of the cause of failure based on the action of constructing proof by contradiction (operational hypothesis), starting from the action step of formulating supposition is influenced by students' prior knowledge of the form of the statement formulated in PP-2. Subject FTR succeeded in formulating supposition, although the mathematical sentence structure still needed to be refined, especially in the hypotheses of language filtration, deployment, templates, and resources. Meanwhile, subject LSW failed to formulate a supposition, because he failed in the quantifier hypothesis, negation hypothesis, language filtration hypothesis; deployment hypothesis, template hypothesis; and resource hypothesis. Furthermore, the justification of failure based on the action steps of constructing and manipulating hypotheses shows that both subjects failed. Subject FTR made a mistake in formulating a^2 divisible by 3 based on $x^2 = 3$ and b^2 divisible by 3 based on $a^2 = b^2$, and then chose $a = 3m + 1$ or $a = 3m + 2$ to get $a^2 = 3n + 1$ (cognitive demand hypothesis). In contrast to FTR, subject LSW made a mistake in stating a^2 as an even number and assumed that $a = 2n + 1$ and $b = 2m + 1$ which were then substituted into $a^2 = 3b^2$ (cognitive demand hypothesis). This result shows the lack of experience in constructing and manipulating related ideas (template hypothesis) and is the source of the subject's failure to construct and manipulate supposition (resource hypothesis).

Justification of failure in the next step of identifying contradictions showed that both subjects failed. This is because each subject both had the results of identifying contradictions that came from the wrong idea. Subject FTR identified two contradictions, namely: 1) the construction result $a^2 = 3n + 1$ contradicts the supposition $a^2 = 3m$, and 2) the construction result $GCD(a, b) \neq 1$ contradicts $GCD(a, b) = 1$ (contradiction hypotheses), because $a^2 = 3n + 1$ comes from the construction $x^2 = 3$, and $a = 3m$ cannot be used to get $b^2 = 3m^2$ (lack of target hypothesis). Meanwhile, subject LSW considered that the construction result $\underbrace{2(n^2 + n)}_{even} = \underbrace{2(3m^2 + 3m) + 1}_{odd}$ which showed a clash as a contradiction (contradiction hypotheses). The contradiction construction results do not require a supposition reference, so it is not in accordance to identify contradictions (lack of target hypothesis), and a series of constructions shows that the subject has struggled in identifying contradictions has been achieved, although in reality it has not been achieved or failed (recognition hypothesis). Then the justification of failure according to the action step of disproving supposition shows that both subjects failed. It's just that at this stage the FTR subject did not disprove the supposition. Subject FTR did not make a refutation because he considered that finding a contradiction was enough to show that the construction of contradiction proof was successful, while the construction of proof by contradiction required a refutation of the supposition, namely concluding that the supposition was wrong (disproving supposition hypothesis). While the LSW subject did disprove the supposition, it is just that the subject is considered to have failed in disproving the supposition. This is because the construction $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ which shows the difference in values (left-right segment) as a contradiction without referring to the supposition, and then saying that "there is no $a, b \in \mathbb{Z}$ that satisfy the equation $a^2 = 3b^2$" and conclude that "there is no rational number x that satisfies the equation $x^2 = 3$" (disproving supposition hypothesis), let alone showing it to be true.

To obtain the subject's mental emotional response/view in constructing the proof in proof by contradiction, it is necessary to justify the source of the cause of failure in each step of constructing the proof in proof by contradiction. The first justification starts with the subject's emotional response to the failure to formulate a supposition. The response of subject FTR who succeeded in formulating a supposition (false world hypothesis), which contains the formulation of a true supposition (false premise hypothesis) requires the subject to dare to work with opposite objects, such as rational numbers (impossible object hypothesis), and becomes a medium for constructing and manipulating suppositions (acceptability hypothesis), as well as a constructive means of activating knowledge (constructive/destructive hypothesis), to give confidence to the deductive logic present in the construction of proof by contradiction (conviction hypothesis). In contrast to the response of the LSW subject who failed to formulate a hypothesis, refusing to inhabit a false world (hypothesis), due to a lack of confidence in something that is considered false to be true (false premise hypothesis), and will certainly require working with impossible objects (rational numbers) as a representational effect of the hypothesis formulation (impossible object hypothesis). This uncertainty then cannot be a medium to continue working to show that the original statement is true (acceptability hypothesis), as seen from the attempt to formulate the supposition in the form of logical equivalence because it assumes that PP2 is a conditional statement, not a singular one (constructive/destructive hypothesis), so that a different way that is believed can be used to justify the contradiction step (conviction hypothesis).

The next emotional response is the subject's emotionality when constructing and manipulating suppositions. Constructing and manipulating supposition is working with supposition material to obtain results that can be identified as contradictions. The emotional response at this stage shows that the emotions of the FTR subject who succeeded in formulating the supposition are less convincing because at this stage the subject failed to work with the supposition to get a result that can be identified as a contradiction. The subject's decision to work not using supposition, however, appears correct, but not logical (constructive/destructive hypothesis), produces results that are identified as a medium rather than a way to find contradictions (acceptability hypothesis), and does not give confidence in the construction results to identify contradictions (conviction hypothesis). The construction results obtained by not using supposition forced cognition to think about how to get contradictory results (false world hypothesis), such as starting to work with the equation $x^2 = 3$ to produce two contradictory results (false premise hypothesis), which might be a contradictory or incompatible result (impossible object hypothesis). In contrast to the response of subject LSW, although it failed to formulate the supposition. The subject's initial response was correct, but the final failed due to the wrong definition (constructive/destructive hypothesis), the lack of understanding of the definition of a number affects the construction result (acceptability hypothesis), because the wrong representation certainly cannot provide confidence in finding the result identified as a contradiction (conviction hypothesis). The subject's awareness of the problem to be faced in the false world is a problem that demands contradictory results (false world hypothesis) and requires the subject to produce construction results that do not need to refer to supposition or statements that can cancel the results (false premise hypothesis) because it will only give cognitive load to work again with contradictory objects and choose different ideas and can produce contradictions (impossible object hypothesis).

Continuing the exploration of the subject's emotional response when constructing and manipulating the supposition, namely identifying contradictions. Identifying contradictions is the work of finding contradictions from the activities of constructing and manipulating supposition results, which are then used to disprove the supposition. The emotional response at this stage shows that the FTR subject's emotion when finding two contradictions from the results of the construction and manipulation of illogical ideas is doubt (constructive/destructive hypothesis). The two contradictions then cannot be a medium to conclude that they have found a contradiction (acceptability hypothesis), and therefore cannot provide confidence in finding a contradiction (conviction hypothesis). Therefore, contradiction (refutation) cannot be used to disprove a supposition (false world hypothesis), because the construction material of contradiction comes from a false idea (false premise hypothesis), so it requires an effort to reconstruct a supposition (impossible object hypothesis). In contrast to subject LSW, subject LSW's emotional response to the contradictory construction result $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ (constructive/destructive hypothesis), provides a way to obtain the contradiction without referring to the supposition, but does not include the construction medium to identify the contradiction (acceptability hypothesis). The result of a contradictory construction without reference to a supposition cannot give confidence in having found a contradiction (conviction hypothesis). The identification of contradictions by tracing the process of supposition construction makes the subject aware of being in the territory of the false world (false world hypothesis) because the identification of contradictions provides space to think back about the constructed premise material (false premise hypothesis), and logical premise material is not built on a flawed idea (impossible object hypothesis).

The last emotional response justification is the subject's emotionality when disproving supposition. Disproving a supposition is to conclude that the supposition is false, based on the existence of construction results that contradict the supposition. The emotional response at this stage shows that the emotional response of FTR subjects who failed to disprove the supposition was that they realized they did not give/conclude that the negation supposition was wrong (constructive/destructive hypothesis) because the lack of prior knowledge could not be a medium to disprove the supposition (acceptability hypothesis). The assumption that disproving a supposition is inferring a contradiction, not inferring a false supposition, shows the importance of activating ideas, especially the steps of constructing a proof with a contradiction (conviction hypothesis). Disproving supposition is a medium to show that the inhabited world is false (false world hypothesis). The false world hypothesis is based on a construction or premise that contradicts the true premise (false premise hypothesis). The use of disprove material that is wrong/not based on logical ideas causes invalid construction results and cannot be used as disprove material (impossible object hypothesis). In contrast to subject LSW, the interview results showed that the failure to disprove a supposition was because the subject considered that $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ as a contradictory construction result, resulting in "there is no $a, b \in \mathbb{Z}$ satisfies the equation $a^2 = 3b^2$ and then concluding "there is no rational number x satisfies the equation $x^2 = 3$" (acceptability hypothesis). The failure of construction $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ by substituting $a = 2n+1$ and $b = 2m+1$ into the equation $a^2 = 3b^2$ (constructive/ destructive hypothesis) gives confidence that there is a mistake in defining $a^2 = 3b^2$ as an even number to get a contradiction (conviction hypothesis). The statement there is no $a, b \in \mathbb{Z}$ that satisfies the equation $a^2 = 3b^2$ and then concluding "There is no rational number x that satisfies the equation $x^2 = 3$" cannot be used to disprove the false world hypotheses, because both are impossible objects analogous to disprove the hypothesis and the statement "there is no $a, b \in \mathbb{Z}$ that satisfies the equation $a^2 = 3b^2$ is a premise that has no logical basis (false premise hypothesis).

Not only an exploration of the operational and emotional (affective) actions when dealing with contradictions, the logical (foundational) basis dealing with why failure occurs, what causes failure to occur, and what impact it has on other indirect arguments is an important part of this discussion. Starting with the logical foundation of the causes of failure to formulate suppositions. Subject FTR succeeded in formulating the supposition while subject LSW failed. The cause of subject FTR's success in formulating the supposition is the understanding of the problem representation of PP2 (meta-theoretical hypothesis), such as the representation of the problem of irrationality 3 contained in PP2. However, the success in understanding the problem representation of PP2, in this case, did not support success at other stages, thus causing a rift in what was understood (argumentation rift hypothesis), and a review of the success of formulating suppositions is needed to see what ideas have not been fully associated or conflated (conflation hypothesis). In contrast to subject FTR, subject LSW at the stage of formulating supposition was considered a failure, due to the subject's understanding of the form of statement PP2 (meta-theoretical hypothesis). The assumption that PP2 is a conditional statement rather than a single statement is one of the main causes of failure to formulate supposition (argumentation rift hypothesis), so the form of the statement contained in a proof problem can affect the form of logical equivalence and logical structure of the mathematical sentence contained in PP2 (conflation hypothesis).

The next foundation is the logical basis for the cause of failure in constructing and manipulating the supposition. FTR and LSW subjects both failed and used different ways of construction and manipulation which certainly affected the logical basis that caused them to experience failure in constructing and manipulating supposition. Subject FTR failed to construct and manipulate the supposition because the subject chose a seemingly correct but incorrect step to produce a valid construction result (meta-theoretical hypothesis), such as assuming the idea $x^2 = a^2$, thus obtaining a^2 divisible by 3 and a is also divisible by 3, and because a is divisible by 3 then $a = 3m$, consequently $b^2 = 3m^2$ (argumentation rift hypothesis). This then resulted in a and b both being divisible by 3, which appeared to be correct but was incorrect (conflation hypothesis). Subject LSW, on the other hand, chose a different way and managed to find the value of $a^2 = 3b^2$, but eventually failed to get the value of a due to a lack of understanding of the definition of a number divisible by 3 (meta-theoretical hypothesis). The success in obtaining $a^2 = 3b^2$ but failure according to the result $\underbrace{2(n^2+n)}_{even} =$

$\underbrace{2(3m^2+3m)+1}_{odd}$, indicates an argumentation rift at the step of construction and manipulation of supposition (argumentation rift hypothesis). Between the results $a^2 = 3b^2$ and $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ cannot be conflated to show that the argumentation step or the construction and manipulation of supposition is correct (conflation hypothesis).

The next logical foundation is the logical foundation for the cause of failure in identifying contradictions. This foundation shows how the two subjects found a contradiction from the results of the construction and manipulation of supposition. Starting with subject FTR who was finally considered to have failed to identify a contradiction, because $a^2 = 3n + 1$ cannot be said to be a contradiction with $a^2 = 3m$, as well as $GCD(a, b) \neq 1$ cannot be said to be a contradiction with $GCD(a, b) = 1$, because both construction results come from an incompatible or illogical idea (meta-theoretical hypothesis). These two contradictory results cause a rift so that it cannot be said to have found a contradiction (argumentation rift hypothesis). The material and the basis of contradiction derived from illogical deduction caused the two contradictory results could not be conflated (conflation hypothesis). In contrast to subject FTR, subject LSW used one contradictory result to show the existence of a contradiction, namely the construction result $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ (meta-theoretical hypothesis). This result suffers from an argumentation rift hypothesis, so the construction result that shows a clash/contradiction but does not refer to the formulated supposition cannot be said to successfully identify the contradiction (conflation hypothesis).

Finally, the logical basis for the failure to disprove the supposition of the two subjects. The logical basis at this stage is slightly different because subject FTR did not formulate a disproof that showed the supposition was wrong, while subject LSW formulated a disproof but was considered to have failed to disprove the supposition. Subject FTR did not make a disproof of the supposition, because he considered that finding a contradiction was the same as disproving the supposition (meta-theoretical hypothesis). Identifying a contradiction by not concluding the supposition is false (argumentation rift hypothesis) cannot be conflated to be used as material to conclude that the original statement (PP2) is true (conflation hypothesis). In contrast to subject FTR, subject LSW had formulated a false supposition but used an erroneous construction result to signify the existence of a contradiction, namely $\underbrace{2(n^2+n)}_{even} = \underbrace{2(3m^2+3m)+1}_{odd}$ (meta-theoretical hypothesis). The result of this construction then stimulates the subject to state "There is no $a, b \in \mathbb{Z}$ that satisfies the equation $a^2 = 3m$" so that "there is no rational number x that satisfies the equation $x^2 = 3$" (argumentation rift hypothesis), and the two arguments cannot be said to affect each other, because the first statement comes from a fractured argumentation (conflation hypothesis).

Failure to construct proofs, especially contradiction proofs, is not an easy failure to avoid because the act of supposing a negation of a false statement, then working with that supposition to get a contradictory result, to conclude that the supposition is false, mostly leads to construction failure. These construction results are not without reason, with some researchers suggesting that understanding the step of working with proofs of contradiction is partly due to proof learning resources that do not provide enough access to practice reasoning through formal logic (Brown, 2016), representation of proof-related problems with contradictions (Lin et al., 1998; Barnard & Tall, 1997), the experience of solving proof problems with proof by contradiction steps (Antonini & Mariotti, 2008; Brown, 2018; Hanna & de Villiers, 2021; Tall, 1980; Thompson, 1966), the difficulty of formulating the negation of a quaternary statement with appropriate mathematical sentences (Inglis & Simpson, 2008; Sellers, 2018; Sellers et al., 2021; Alacaci & Pasztor, 2005; Dawkins, 2017; Lin et al., 2003; Shipman, 2016; Epp, 2003), cognitive demands to prove suppositions false (Antonini & Mariotti, 2008; Leron, 1985), difficulty identifying contradictions (Antonini & Mariotti, 2009; Barnard & Tall, 1997; Antonini, 2019; Brown, 2018; Chamberlain & Vidakovic, 2016, 2021), lack of understanding the purpose of finding contradictions (Chamberlain & Vidakovic, 2016, 2021), and not understanding the function of disproving suppositions (Balacheff, 1991; Reid et al., 2008; Lew & Zazkis, 2019).

Both subjects' emotional responses to the action resulted in constructions that were judged to have failed, not without obvious reasons either. Some research results have also supported the failure of proof construction by contradiction, such as research results showing that: the understanding of the coherence of proof steps by contradiction is not yet accepted as an appropriate way to solve proof problems, and an important part of mathematical proof (Antonini, 2019), the lack of understanding that countering ideas/refuting suppositions is the main purpose of proof by contradiction (Leron, 1985; Antonini & Mariotti, 2008; Brown, 2013; Brown, 2018; Harel & Sowder, 1998), proof by contradiction is still difficult to give strong confidence in the construction result (Brown, 2018; Tall, 1980), unable to use appropriate definitions, theorems, and axioms in constructing and manipulating supposition (Antonini & Mariotti, 2006; Koichu, 2012; Dumas & McCarthy, 2015), and construction and manipulation of true negation supposition causes cognitive tension, because the subject does not know whether the construction result is true or not? (Hine, 2019).

The failure of PP2's proof construction in this study is also not without a logical basis that supports it, starting from formulating the supposition until it is finally judged to fail because it cannot conclude that the supposition is false, or the original statement is true. Some research results that become the logical-theoretical basis supporting the failure of the results of proof construction with contradiction proof state that the proof problem constructed with the contradiction proof step, starting from formulating the supposition to finally disproving the supposition, has become the theoretical basis of the proof (Brown, 2018; Antonini & Mariotti, 2008; Epp, 2003, 2020), the difference in the underlying or composed logic in proofs by contradiction (such as PP2) causes knowledge/understanding of previous problems (either direct proofs, proofs by contraposition or proofs of other problems) to be unable to be activated when constructing proofs by contradiction, especially the lack of symbolic elements in the problem (Lin et al., 2003; Reid & Dobbin, 1998; Otani, 2019), so that different proof logics cannot always be combined to solve proof problems (e.g. PP2), especially if each proof logic is learnt in a short time (Thompson, 1966; Stylianides et al., 2004; Bleiler et al., 2014; Jourdan & Yevdokimov, 2016; Doruk & Kaplan, 2018; Lee, 2019; Doruk, 2019).

Observing the initial problems encountered in constructing proofs by contradiction, such as understanding the problem representation, has required cognition to recognize signs that proof by contradiction can be used as a means of solving a problem. The results of this study showed that FTR subject and LSW subject failed to understand the problem representation of PP2. Both subjects did not know that PP2 was a representation of the proof problem of irrationality 3, and only knew proof by contradiction as a method of solving proof problems based on the problem instructions. Failure to understand the representation of proof problems by contradiction is also found in several studies such as research conducted by Lin and Barnard (Lin et al., 1998; Barnard & Tall, 1997), and by Quarfoot and Rabin (Quarfoot & Rabin, 2021), so understanding the representation of proof problems related to proof of contradiction requires a struggle to recognize several criteria before being formally proven. In addition, failure to construct proofs is caused by understanding the steps of proof by contradiction. The results of the exploration of the subject's emotional response and logical foundation showed that subjects FTR and LSW did not seem to understand the four steps of constructing proofs as proposed in this study. Other possible causes as indicated by the results of research that say common ways of reasoning in proof by contradiction are less numerous, less understandable by students, or less trained by teachers (Antonini & Mariotti, 2008; Brown, 2018; Hanna & de Villiers, 2021; Tall, 1980; Thompson, 1966).

Furthermore, observations on the step of disproving supposition and hypothesis lack of understanding the purpose of finding contradiction (lack of target), in this study cannot be said to be the same two hypotheses, but they are different. Disproving supposition works based on the discovery of contradiction (contradiction as target), and lack of understanding of the purpose of finding contradiction (lack of target) results in not being able to refute the disproving supposition, so the supposition is false, or the original statement is true. This distinction is not without reason, from the results of this study it was found that FTR subjects who were judged to fail to identify contradictions, did not refute the supposition, because they only knew that the purpose of proof by contradiction was to find contradictory results which indicated that the proof was successful. While the LSW subject, although judged to have failed to identify the contradiction, then disproved the supposition so that the original statement was true.

Although the failure hypothesis framework illustrated in Figure 1, Figure 6, and Figure 11 can help justify the types of failures along with the sources of causes, and the logical reasons that led to the failure of proof construction in this study. It is possible that the framework can still be questioned because when the proof problem is done by participants from different population groups, it is possible that the hypothesis node can be developed again or otherwise. This argument is supported by research results that show that proof is conceptually different from what has been shown previously and is intuitively considered different if it combines different ideas or tactics (Rav, 1999; Dawson, 2006; Tall & Mejia-Ramos, 2010). This research shows that the failure of proof construction in proof by contradiction is caused by the action step of producing proof by contradiction, and the difference in the logic underlying proof by contradiction with other proofs causes cracks in the argumentation so that it cannot always be conflated (foundational hypothesis). The lack of understanding of the action steps and the logic underlying proof by contradiction may be due to resources (textbooks, videos, lessons, etc.) that do not adequately train students' understanding of the contradictory problems encountered (training hypothesis). Although the subject believed that the proof problem, he constructed was considered correct, it turned out that after being validated based on the ideas he constructed and manipulated, it was finally concluded that the construction results failed. This means that the logical reason why the construction result is considered to fail, what causes it to fail, and what is the result of the failure (foundational hypothesis), cannot be a measure to instill confidence that the resulting construction result (disproving supposition) is correct or otherwise (conviction hypothesis).

Thus, the development of this proof construction failure hypothesis framework is possible based on different populations, especially if the participants' mathematical language content is very concerned with quantifiers, logical equivalences, or rules of inference of a mathematical sentence (language filtration hypothesis) and have studied the formal logic of proof of contradiction for longer (conflation hypothesis), conflation without argumentation rift may occur. The different populations are also closely related to how the proofs in a proof by contradiction are presented, in what semester, in what course, and for what purpose.

CONCLUSION

There are four types of failures, namely failures formulating supposition, constructing, and manipulating supposition, identifying contradiction, and disproving supposition. These four types of failures were then used to develop a hypothesis framework for proof construction failure in proof by contradiction, which consists of 17 hypothesis nodes divided into three main hypotheses, namely: operational (action) hypothesis, affective (emotional) hypothesis, and foundational (logical reasoning) hypothesis.

The framework of the failure hypothesis shows that the sources of the failure of proof construction in proof by contradiction are understanding of the act of producing a proof by contradiction, emotions towards the coherence of the construction steps; disproving suppositions; beliefs; use of appropriate definitions, theorems and axioms; and cognitive tension in proof by contradiction; and logical reasons for the act of producing a proof by contradiction; as well as differences in the underlying logic causing cracks in the argumentation so that it cannot always be conflated.

Furthermore, there are six open problems for further studies, such as the steps leading to the act of disproving supposition in different population groups, how resources (prior knowledge, intuition, experience, books, etc.) influence the way of reasoning when engaging with problems that demand proof by contradiction, how to measure the cognitive demands of inferring supposition, how to measure the cognitive demands of inferring supposition, how to measure the cognitive demands of inferring false supposition, how proof by contradiction can be accepted as a method of solving proof problems, and the development of hypothesis nodes for each type of failure.

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