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Research Article

Characterizing mathematical discourse according to teacher and student interactions: The core of mathematical discourse

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The discussion on the development of mathematical discourse plays a key role in the determination of the in-classroom interactions in mathematics learning and instruction. The present study aims to present a theoretical framework for the nature of mathematical discourse that addresses the teacher and student interaction in the classroom. Previous studies attempted to discuss the theoretical structure in mathematical discourse with the embedded theory approach. The findings revealed the core of mathematical discourse that reflected the structure of mathematical discourse based on open, axial and selective codes determined based on the embedded theory approach. The external structure of this core reflects the types of in-classroom interaction, while the internal structure reflects the development of the mathematical discourse. The external structure included four types of interaction: teacher, teacher-class, teacher-student, and student-student. The internal structure includes mathematical discourse movements associated with three stages: motivation, explanation of mathematical ideas, and achievement of mathematical ideas. The external structure of mathematical discourse core revealed the general state of in-classroom interaction core, and the internal structure revealed the specific mathematical discourse based on the mathematical content. It could be suggested that the discourse movements in the mathematical discourse core determined in the present study would provide guidelines for mathematical communications. The study also includes recommendations for future studies on the employment of this general and specific theoretical mathematical discourse framework.

Keywords: Mathematical discourse; Classroom interaction; Embedded theory; Video analysis

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1. Introduction

Mathematical communication is an important instrument in teacher-student interaction to express, reveal and develop thoughts and ideas, explain misconceptions and demonstrate mathematical connections (Ballard, 2017). As stipulated in international standards, students' skills to express their ideas with mathematical communication in mathematics classes are attributed importance (National Council of Teachers of Mathematics [NCTM], 2000). Mathematical communication allows students to understand and analyze mathematical ideas, and leads to the development of

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mathematical thought (Vui, 2007; Yang et al., 2016). Thus, mathematical communication is necessary for student's comprehension of classroom conversation (Viseu & Oliveira, 2012). In this context, mathematical discourse serves as a bridge in mathematical communication (Baki & Celik, 2019). Because through mathematical discourse, students can speak, think and discuss about mathematics (Nathan & Knuth, 2003). Mathematical discourse is a key component of quality classroom experiences, as it involves explanation and discussion and the debate on mathematical ideas (Walshaw & Anthony, 2008). Authors who study mathematical discourse should emphasize the association between discourse and communicative action rules, not only the proposition and rule content of discourse (Sfard, 2000). Therefore, mathematical discourses should be considered within the context of in-classroom interactions (discussion, question-answer, etc.). Ryve (2011), in a review of the studies on mathematical discourse, stated that future researchers on mathematical discourse should focus on the definition of the concept of discourse. Thus, the conceptual structure of mathematical discourse and its nature should be emphasized. In this study, the structure of mathematical discourses according to teacher and students conversations was examined both according to mathematical content and according to discourse types.

2. Conceptualizing the Mathematical Discourse

Mathematical conversation is also classified based on the fact that the discursive framework is a mathematical conversation instrument in the classroom (Richards, 1991; Chapin, O'Connor, & Anderson, 2003; Hufferd-Ackles et al, 2004; Kazemi & Hintz, 2014). Thus, when students talk about mathematics, they use one or more of these types of conversation to express mathematical or other ideas (Matson, 2010). Mathematical discourse that allows in-classroom mathematical conversations is produced in different forms and types (Knuth & Peressini, 2001). In math classrooms, the discursive approach is the general classification; however, other classifications such as mathematical discourse and metacognitive discourse are also available (Mercer, 1995; Brendefur & Frykholm, 2000; Shilo & Kramarski, 2019). On the other hand Drageset (2015) divided the mathematical discourses between students into two basic categories. The first of these categories is the discourses between the students' mathematical explanations and the teacher's focusing actions, and the second is the discourses between the teacher's progressive actions and the teacher-led students' responses. However, in this study, in which the structure of mathematical discourses is examined, the building blocks that make up the mathematical discourse are determined by looking at the analysis of mathematical discourse from a wider perspective. Adler and Ronda (2015) defined an analytical framework for mathematical discourse based on mathematics learning and instruction. In this framework, mathematical discourse was characterized by three interactive components: exemplification, explanatory speech, and student participation based on the learning objectives in the mathematics course. However, the indicators that characterize mathematical discourse movements were not included. On the other hand, Kalathil (2004) investigated the nature of mathematical discourse, and defined a novel framework that reflected the classroom process in which discourse structures are developed. The framework utilized the mathematical discourses between the teacher and the individual student, the teacher and all students, and the students and other students. However, the development of mathematical discourse between the teacher and a student, or between the teacher and all students, was not compared. Thus, the specific development of these discourses was not investigated; only a theoretical framework for the general structure of mathematical discourse was determined. Sfard (2012) reported that mathematical discourse had four attributes (Special terms, visual mathematical functions, routines, approved narratives) when she characterized the specific structure of mathematical discourse. With the help of these elements, the mathematical content was determined. Due to the different grade levels and different mathematical classrooms being observed, there was a need to determine the mathematical content. Thus, using Sfard's (2012) theory, a part of the mathematical discourse structure has been determined in this study. Because, mathematical discourse includes more than linguistic structures and depends on students' skills in explaining their ideas to others (Shortino-Buck, 2017). Mathematical language does not include only mathematical symbols and terms but rather is a comprehensive approach that includes mathematical discourse that focuses on face-to-face communications in the classroom (Morgan et al., 2014). Because in-depth comprehension of the terms used to convey mathematical concepts by the students is important (Kim & Lim, 2017).

The mathematical dialogue between students or between the teacher and the students depends on the questions asked in the classroom and the responses. Rich and meaningful mathematical discourse is defined as an interactive and sustained discourse between teachers and students (Piccolo et al., 2008). Lemke (1990), who studied in-classroom dialogues, reported that the type of dialogue described as the tripartite pattern was the most employed type of dialogue by the teachers in the class. This dialogue entails a teacher's question, student's answer, and teacher's evaluation. Furthermore, according to Mercer's (1995) discourse analysis framework, in-classroom discourse can be analyzed in three groups: controversial, cumulative and exploratory discourse. While individual ideas without a consensus prevail in controversial discourse, cumulative discourse is made positive, and cumulative discourse leads to common knowledge. Cumulative discourse is characterized by repetitions, affirmations, and scrutiny. In exploratory discourse, on the other hand, statements, suggestions and different hypotheses are presented to reach a consensus. Questioning of the ideas, reasoning and justifications are clearer when compared to the other two types of discourse. Furthermore, the determination of teacher and student roles based on the in-classroom discourse was analyzed with a different approach to classroom interaction models. One of the interaction classroom models is the classroom interaction model in which various structures were introduced by Mortimer and Scott (2003). Since the data of the research and the components in this model are compatible, this model was preferred in evaluating the mathematical discourse in terms of interaction. In addition, teacher and student conversations are divided into four components, it is thought to be more inclusive than other discourse analysis models. Thus, using Mortimer and Scott's theory (2003), a part of the structure of mathematical discourse was also determined in terms of teacher and student interaction in this study. Previous studies investigated the development of dialogue in the classroom or the unique structure of mathematical discourse (Sfard, 2000; 2012); however, no theoretical framework that characterized both the structure of mathematical discourse and its development in the classroom was previously investigated. Furthermore, it was suggested to develop professional learning-instruction environments, not only based on lexical and syntactic language, but also mathematical discourse in future studies on mathematics education (Erath et al., 2018). Thus, the present study aimed to discuss the structure and development of mathematical discourse. The development of mathematical discourse was based on the classification of mathematical discourse. It is known that the classification of discourse is important to determine the correlation between mathematical comprehension and mathematical discourse (Pirie & Schwarzenberger, 1988). Thus, it would be possible to determine the development of the mathematical discourse between the teacher and the students in a mathematics course. Furthermore, mathematical discourse could be shaped by mathematical content.

The above-mentioned facts emphasize the need to identify mathematical discourse indicators specific to mathematical content and discourse types. It could be suggested that the indicators that determine the types of discourse would give an idea about the interpretation and development of similar discourses in any classroom setting. Thus, the present study could shed light on the development of discourse types in other courses, not only in the mathematics course. It could be suggested that the identification of specific indicators that explained the discourse types based on the mathematical content would contribute to the development and explanation of the development of interactions in mathematics classes. This study is important since it would give mathematics teachers an idea about the types of discourse that could develop based on the mathematical content. Thus, the present study would contribute to the literature by providing guidelines on the employment of different types of discourse by teachers in the planning and instruction processes. In the literature, the number of studies that theoretically explained the natural structure of mathematical discourse based on the teacher and student discourses in the classroom is quite limited.

The analysis of the mathematical discourse based on teacher discourse data would also provide a framework for future researchers for data analysis (Hale et al., 2018). It could be suggested that the development of a broader framework based on the analysis of the mathematical discourses of both the teachers and students would provide diverse means of analysis for future researchers. This study seeks to answer the following research questions:

RQ 1) How could the structure of mathematical discourse in in-classroom interactions be described based on mathematical terminology?

RQ 2) How could the structure of mathematical discourse in in-classroom interactions be described based on visual mediators?

RQ 3) How could the structure of mathematical discourse in in-classroom interactions be described based on question/problem-solving?

3. Methodology

3.1. Research Design

The structure of the discourse in mathematics classrooms was determined with the grounded theory based on the in-classroom interaction. The grounded theory is different from other research approaches since the data collection and analysis continues throughout the research (Charmaz, 2006; Christensen et al., 2011). Because in the grounded theory approach, micro-level events are seen as the basis of an explanation at a more macro level (Neuman, 2013). In this study, the theoretical structure, the theory that existed in the structure of mathematical discourse, and the mathematical discourse core that reflected this theory emerged only after the observation-based micro cases conducted in six classes.

3.2. Participants

Natural observations were conducted made in middle school mathematics classes to determine the structure of mathematical discourse. Thus, the study participants included middle school mathematics teachers and students. The study was conducted with six mathematics teachers and students attending the classes of these teachers in three middle schools. There are approximately 30 students in a class.

3.3. Data Collection Method and Instrument

In grounded theory research, a strong theory could be achieved with rich data (Charmaz, 2006). Thus, reproducible and generalizable data were collected in the present study to determine the theoretical mathematical discourse framework. More than one natural observation was conducted and recorded in the mathematics class instructed by the same teacher. About 12 (two lessons of teacher) mathematics classes were observed per week and recorded during the academic year. However, certain classes were not recorded due to exams or social activities. A total of 140 course hours were observed and recorded in the classes of six mathematics teachers. Each parent signed a "Parental Permission Form" before the classes were recorded, and all observations were conducted based on ethical principles. The mathematical discourses in the classes were recorded with a video camera during the learning-instruction process to obtain all mathematical terms, symbols and other visual expressions, as well as the verbal expressions employed in the classroom (Tanısļı, 2016). Furthermore, field notes were taken to support the observation data in each observed class. What was done in the learning-teaching process for the observed classes was written in the field notes according to the teacher's organization in the classroom. Learning-instruction activities conducted in the classrooms were noted based on the organization in the classroom. For example, the employment of the smartboard by the teacher and instruction of the topic based on the mathematics textbook were considered as different class organizations and were comprehensively noted based on the order and time of the events. Furthermore, while the textbooks problems were solved, everything was noted, including the number of the pages where these problems were included in the textbook.

After the study data were collected with video records, a video analysis form was developed to assist the analysis. The video analysis form included two sections: "classroom organizations, classroom conversations." The recorded teacher and student discourses were transcribed in the conversations section. The conversations about mathematical content were considered mathematical discourse. A separate Excel file was created for each participating teacher in the study. Then, the mathematical discourse observed during the instruction by each teacher were written on a separate Excel sheet based on the date and mathematics learning area. Thus, the video analysis was more systematic.

3.4. Data Analysis

After the data collection, all mathematical discourses observed between the teacher and the students were transcribed. Then, the data for all teachers and students were analyzed with the MAXQDA Analytics Pro 2018 software. Mathematical discourses were grouped based on classroom organization and named dialogue. These dialogues were enumerated based on the time of the event. According to the data obtained from the research, approximately 2500 dialogs were determined. The findings were presented with dialogue numbers and mathematical discourses within dialogue numbers. For example, the code "1.15" represented the line 15 in dialogue 1.

In the separate analysis of each discourse group data, it was observed that the development of mathematical discourses was different based on the mathematical content and in-classroom interaction. Thus, it was determined that there was an grounded theory that reflected the development of mathematical discourses in the clustered dataset. Then, the implementation of the grounded theory steps revealed the core of mathematical discourse. The open coding, axial coding and selective coding data analysis based on the grounded theory approach is presented below.

3.4.1. Open coding

In embedded theory, the researcher attempts to reach theoretical generalizations through the explanation, interpretation and making sense of the observed data (Neuman, 2013). Thus, in the present study, where the development of mathematical discourse was observed and analyzed, initially, the dialogues were analyzed based on the discourse type and mathematical content and then interpreted. The interaction approach model developed by Mortimer and Scott (2003) was employed in the general analysis of the development of mathematical discourse. Because the overall review of the data set revealed that the participation of the teacher and the students in communication varied based on in-classroom interaction. This interaction model was considered more inclusive when compared to the other models or theoretical frameworks in the analysis of in-classroom discourses (Lemke, 1990; Cazden, 2001). Furthermore, the components that Sfard (2012) used to characterize mathematical discourse based on mathematical language were employed. Because it was considered that there was a correlation between classroom mathematical discourse and mathematical language, and the theoretical framework specific to the mathematical discourse was utilized. Thus, the initial coding phase in data analysis was based on the literature review. The literature review is necessary in the initial data coding stage to facilitate the confirmation of the findings in the later stages of grounded theory research (Saban & Ersoy, 2016). The theoretical framework employed in the open coding process to determine the discourse types and the mathematical content are presented in Table 1.

The data analysis was based on the literature presented in Table 1. Because the focus was on the stakeholders during the development and the method and topic of the mathematical discourse. Thus, the types of discourse were determined based on the stakeholders and the method, and the mathematical content was defined based on the topics of the discourse. In other words, the mathematical content could be considered as mathematical content. As seen in Table 1, the

Table 1
The present study elements that corresponded to the components reported in the literature

Mathematical Discourse Components (Sfard, 2012)	*Mathematical background (Mathematical Content)*
Terminology	The mathematical discourses that include mathematical expressions such as mathematical terms, symbols, signs, concepts, and equations.
Visual mediator	It consists of mathematical discourses on the tables, graphics, mathematical modeling, and shapes.
Routines Validated narratives	Problem solving. As Sfard (2012) stated, discourses on mathematical terminology and visual mediators become routine and include validated narratives. Discourses on problem-solving also include discourses on mathematical terminology and visual mediators.
Communication Model (Mortimer & Scott, 2003)	Discourse Types
Noninteractive / Authoritarian	Teacher: The dominance of teacher's discourse
Noninteractive / Dialogical	Teacher-Class: No interaction between the students and the teacher, but concurrent participation of more than one student in the discourse
Interactive / Authoritarian	Teacher-Student: The development of discourse between the teacher and a student under teacher's control. In a dialogue, more than one student could interact with the teacher, but the students do not participate in the mathematical discourse among themselves; it is a conversation between the teacher and a student.
Interactive / Dialogical	Student-Student: A multi-dimensional interaction in the classroom and the participation of various students in the discourse.

components had certain aspects that did not perfectly match the elements in the mathematical content and discourse types. For example, the repetitive employment of mathematical terminology and visual tools in routines and validated narratives as determined by Sfard (2012) was observed in mathematical problem solving. Thus, routines and validated narratives were combined in the problem-solving category in the present study.

3.4.2. Axial coding

It was observed that the discourse type and the mathematical content determined based on the literature were similar to the outer framework of the development of mathematical discourse. This outer framework was more clear in the axial coding phase, the second phase of grounded theory analysis. The mathematical discourses for the determination of the outer frame in the axial coding process are presented in Table 2.

Tablo 2
An axial coding example

In-classroom observation example	Axial Coding
Teacher: Yes, children, let's complete what is written in your notebook, so that I do not write the same expressions again. Pay attention. In the first expression, it is asked 7 plus 3 times 4. What should be the first operation? Addition? Subtraction? Multiplication? Division? What do you think? **Sevgi:** Subtraction. **Teacher:** I now do the first, Sevgi, when we say 7 plus 3, which operation to be done? **Sevgi:** Addition. **Teacher:** If in this verbal expression, the first directive is to add 3 to 7, in other words, addition, how can we prioritize addition? To put the operation within brackets. Because, when do we add? In fact, addition and subtraction are done the last. But this expression tells us to add 3 to 7 and then multiply the product with 4. OK, if I write the same operation as follows *(Teacher writes 7+3x4)*. **Teacher:** Why? *(Teacher waits a while for an answer)* Why? *(Murat takes the floor)*. **Murat:** Because the first operation is multiplication, we have to put brackets here. *(Around 3 and 4)* Then we add this to the product, but the result is not the same. **Teacher:** What is the result of this operation? Yours? **Murat:** 4 times 3 is 12 , 12 plus 7 is 19.	**Teacher-Student Discourse Type**
Teacher: Let us check my operation. 7 plus 3. **Class:** 10 **Teacher:** I multiplied it by 4 **Class :** 40 **Teacher:** The results are different; thus, brackets are important in an operation. When there is none, which has the priority? Multiplication or division. But this operation is different. The first directive is addition, and should be between what? Brackets. **Class:** Brackets.	**Teacher-Class Discourse Type**

As seen in Table 2, axial coding revealed a structure that surrounded the core of mathematical discourse. In other words, the external structure of mathematical discourse included mathematical discourse types and mathematical content. Then, selective coding was conducted to investigate the internal structure.

3.4.3. Selective coding

Selective coding was employed to investigate the development of mathematical discourse within itself. The development of mathematical discourse within itself included the motivation, explanation of mathematical ideas and achievement of mathematical ideas stages. Motivational discourses were considered as mathematical discourses that initiate the discourse types. The discourses to explain mathematical ideas included the discussed term, visual mediator, and problem-solving. Finally, discourses aimed to achieve mathematical ideas were determined as the mathematical discourses aimed at conclusions on mathematical content and finalization of the discourse types. Although these three associated stages were the building blocks of mathematical discourse, certain scattered building blocks were also identified. Higher level building blocks were determined by grouping the lower building blocks that reflect the correlations between mathematical discourse types. For example, the discourses such as the comparison of two visual mediators and determination of adequate drawing rules were combined as the mathematical discourses for the evaluation of drawing rules. Additionally, in the Teacher-Student discourse type, there are many sub-codes about giving feedback to the wrong. Sub-codes about giving feedback to the wrong; giving feedback on the order of operation, asking the reason for the mistake, asking the other way to the solution, making a conceptual-logical mistake, recognizing the mistake, etc. such as sub-codes. Similarly, other sub-codes under the horizontal codes were eliminated, and it was aimed to make coding more simple. Thus, it was investigated whether the scattered building blocks that determined the discourse type were repeated to reveal the theory that would guide the mathematics learning-instruction process.

3.4.4. Validity and reliability

To determine the reliability of the study, two randomly selected classes instructed by each teacher were coded (total twelve lessons). The course hours where the teachers instructed different mathematics topics were selected. Lessons where each learning area was instructed were selected to determine the general structure of the mathematical discourses. Because it was considered that mathematical discourses that were specific to a certain topic would not reflect the general theoretical framework. Coding lasted for about one year and conducted by the and two specialists. The selected classes were on different learning areas and instructed in different semesters (fall or spring semester). Meetings were organized about every two weeks between the authors and specialists to determine intercoder agreement. The class videos were watched by the authors and specialists to decide on the structure of the dialogues. The study data were coded both internally and externally. In external coding, discourse types and mathematical content (terminology, visual mediator, and problem- solving) that reflected the parties of mathematical discourse in the dialogues were coded. In order to calculate the reliability of the research, the percentage of agreement for the first six courses determined was calculated as 0.81 (Miles & Huberman, 1994). Since the researchers made backward and forward-looking arrangements regarding the codes based on the decisions taken at the previous meeting, a consensus was sought. However, in this meeting, it was decided to agree on the codes and to rearrange the codes by re-discussing on the type of discourse and mathematical contents that are still not compatible. Afterwards, it was decided to code the remaining 6 courses. The reliability coefficient between the coders for the remaining six courses was calculated as 0.84 (Miles & Huberman, 1994). In the internal coding process, which was conducted later, the focus was on the development of mathematical discourse. The consistency in motivation that initiates mathematical discourse and the discourse indicators in the expression of mathematical ideas and the development of these ideas was investigated. The percentage of agreement among the coders in the coding for all lessons was calculated as 0.87 (Miles & Huberman, 1994). After the reliability analysis of the data, it was determined that certain discourse indicators should be merged, separated or renamed. For example, before the reliability analysis, several sub-codes (feedback on the operation order, asking the reason for the mistake, asking for another solution, conceptual-logical mistakes, recognition of the mistake, etc.) were proposed by the authors in teacher-student discourse type and feedback to errors during the acquisition of mathematical ideas. After the reliability analysis, these sub-codes were eliminated and only the subcode of feedback to errors during the development of mathematical ideas was included.

4. Findings

The characteristics of mathematical discourse was employed to determine the structure of the discourses in mathematics classes. Thus, it was observed that the discourse types that developed based on the in-classroom interaction constituted the external structure of the mathematical discourse. The internal structure of mathematical discourses generally includes three layers: motivation, discussion of mathematical thoughts and acquisition of mathematical ideas. Actually, these three layers can be thought of as the beginning, middle and end of the dialogue. These layers, namely the internal structure of mathematical discourse, include a set of discourse indicators that characterize the development of mathematical discourse. Each internal layer of mathematical discourse varies based on the mathematical content and the type of discourse. Thus, the presence of intersecting discourse indicators demonstrated that the transition between these discourse indicators that allow mathematical communication was flexible. Therefore, the transitions in the external structure of mathematical discourse and the internal layers resemble a core. The study findings on the development of mathematical discourse core (MDC) that reflects the structure of mathematical discourse are presented based on the mathematical content in the following paragraphs.

4.1. MDC on Mathematical Terminology

In the terminological teacher discourse type, it was observed that there were either total lack of teacher motivation or the discourse only included the instruction of mathematical discourse on terminology before the development of mathematical discourse. For example, teacher could inform that she or he will instruct on the terminology, and let the students to turn on the page on the textbook or the smartboard and allowed the students to be ready for the instruction of terminology. Thus, after the students were ready to listen about the terminology, the teachers instructed the terminology themselves by discussing the symbols, the function or the properties of the term, rules, and definitions. Mathematical expressions that reflect this case are presented below.

> 144.1. T5: What could be my research problem? When I left home, I wondered about the most loved frit in the class 5K. Based on this research problem, I wrote a few examples and applied them in the class. What do we call the information I collected? We call it data (Wrote DATA on the board with capital letters). What do we call the total information I acquired?
> 144.2. S1: Data.
> 144.3. T5: Data. The group I collected that data by asking the research questions is called the sample (Wrote sample on the board with capital letters). Who are included in the sample? The 5K class.

As seen in line 1 of dialogue 144, there were no motivational discourse that allowed the students to participate in the mathematical discourse. Thus, students were not active in the stage of the explanation of mathematical ideas. It was observed that the teacher T5 answered the questions and stated the mathematical definitions herself/himself. In the teacher discourse type, the process was the same in the stage of the development of mathematical ideas, the final stage in the terminological discourse. In this stage, repetitive notices were observed on terminology such as a summary the terminology, re-emphasis of the employment of the symbols/signs, reminders and further study topics on terminology. Thus, it could be suggested that mathematical discourse on terminological notices were prevalent in the development of the last layer in the internal structure of the core of mathematical discourse. For example, phrases such as "... let us be careful, the most common mistake students make is ..., do not make this mistake..." etc. were common.

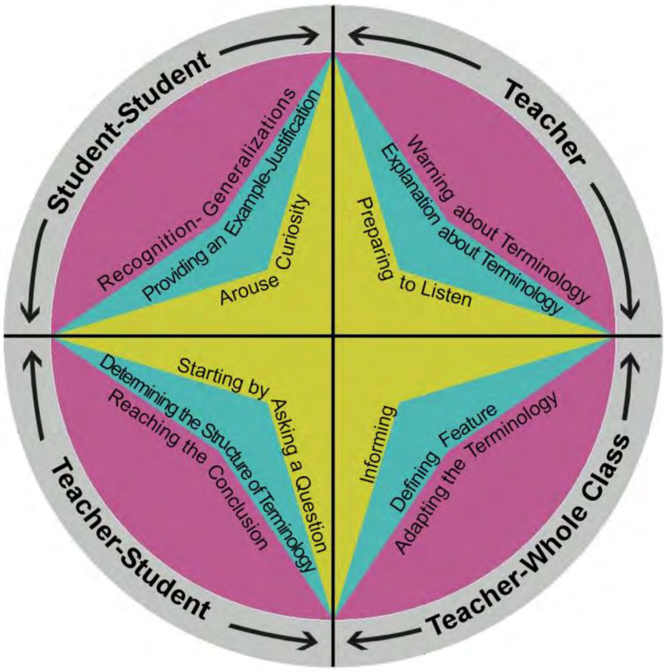
The analysis of the mathematical discourses on terminology in the teacher-class discourse type revealed that the teacher initiates the mathematical discourse by informing the students that the students will participate in the mathematical discourse. It was observed that question strategies were employed effectively in the discourse that aimed to explain mathematical ideas, which is the next stage where students participate in the mathematical discourse. For example, it was observed that the properties of a term were determined by asking a simple terminological question, or a confirmative question. In the stage of the development of the mathematical ideas, the last layer of the mathematical discourse core, it was observed that the properties of a term were transferred to another term and the mathematical discourse was terminated.

In the teacher-student discourse type on terminology, it was determined that either the teacher or the students initiated the mathematical discourse. Thus, it could be suggested that the teacher and the student played a role in the development of the first internal layer of the mathematical discourse core in this type of discourse. The mathematical discourse on terminology was initiated by the teacher or the students by asking a question about the incomprehended terminology. For example, the development of discourses such as "Teacher, I did not understand this (initiated by the student)" or "Ayşe, you tell him (initiated by the teacher)..." supported this case. Then, it was determined that discourses such as giving/not giving examples for terminology, explanation of symbol/sign use and determination of the formulas available in the structure of the term developed between the teacher and students. The review of these mathematical discourses demonstrated that the structure of the term was determined in the stage of the explaination of mathematical ideas, the second layer of the mathematical discourse. After the explanation discourses between the teacher and a student, a mathematical discourse on the results associated with the terminology was developed. It was determined that these discourses included the requestioning the terminology, and reaching the preassumed outcome. For example; "Teacher, if all the diagonals in the figure are inside the figure, it is convex, if they are not..." In other words, students reach mathematical ideas with discourses such as "Thus,...", "If..., then..."

The initiation of the development of mathematical discourse on terminology varied significantly in the student-student discourse type. In this type, the teacher aroused student curiosity about terminology (mathematical discourses such as stating that it is an fun-abstract topic) and allowed the students to develop a mathematical discourse among themselves. In this type of discourse, students initiated the discourse by asking hypothetical questions on terminology or their item of curiosity in terminology. For example, in the class of the teacher T6, a student asked the following: "Teacher, can we exchange a variable and a coefficient?" After the question, it was observed that a discourse on the concept of variable developed among the students. Sometimes, it was observed that the teacher initiated the participation of the students in mathematical discourse by asking questions. For example, in the discourse closter 765, the teacher T3 asked the following: "Same numbers, same operations, multiplication and subtraction, there is only one difference between the two. One has only parentheses. Do you think the results will be the same?" The review of the other motivational mathematical discourses revealed that the next mathematical discourse, where the teacher pre- explained or pre-defined a mathematical, developed among the students. It was determined that the students gave daily-life examples in the mathematical discourse, described the term with examples from the classroom, or discovered the attributes of the term. Furthermore, it was a significant finding that students based their discourses on justifications and causality when discussion mathematical ideas. After the expression of mathematical ideas, it was determined that disorganized ideas became clear, the students reached a conclusion about the term, and they rationalized the terminology. Thus, the development of the last internal layer of MDC was effective since students developed a mathematical discourse among themselves to reach definitions and generalizations based on consensus.

Thus, it was determined that the internal and external layers of MDC were affected by the other. First, the internal layers of the core of the mathematical discourse determined the external structure. MDC that reflects the development of mathematical discourse on terminology is presented in Figure 1.

Figure 1
The core of mathematical discourse on terminology



Note: The yellow in the internal layers reflects motivational mathematical expressions, green reflects the discourse that aimed to explain mathematical ideas, pink reflects the discourses that aimed to develop mathematical ideas. This was similar to other cores of mathematical discourse.

4.2. MDC on Visual Mediators

In the teacher discourse type on visual mediators, it was observed that the mathematical discourse started when the teacher drew attention to the visual mediator. The teacher drew the attention of the students to the visual mediator by directing them to turn to the related page in the textbook or directing them to write the title on their notebooks. For example, the teacher introduced the visual mediator at the stage of the explanation of mathematical ideas, explained the visual mediator, or discussed the drawing rules for the visual mediator herself or himself. In the final stage of the development of mathematical ideas, the discourse is dominated by the teacher. It was determined that the teacher warned the students about the drawing rules such as reemphasizing the drawing rule for the visual mediator, or emphasizing the equal spacing between the visuals. For example, the teacher T1 stated the following in dialogue 1512: "...Now there is a KLM angle here. Look inside the KLM angle, how many II are there? 1, 2, 3, 4, 5. Let us say 4 by 5 square and draw it like that (The teacher drew the shape on the board). I can even make it larger, five by six. Two II, starting from here (continued drawing). Let us take a look at what we should pay attention to when drawing congruent angles on a squared paper. The important thing is to count the squares correctly. Now look at the side, I will draw another one right away. Yes, now look at what it says in the textbook. It tells us draw an angle equal to the KLM angle. Now, let me draw KLM in blue first..." In this context, it was concluded that the teacher's mathematical discourse on warning the students about the drawing rules were specific rules for the implementation of the visual mediator, as well as general warnings such as equal spacing/equal distance. Thus, it was observed that the discourse that aimed to warn the students about the drawing rules for the visual mediator was effective in the development of the last internal layer of the mathematical discourse core.

In the teacher-class discourse type on visual mediators, the teacher stated that they will develop or interpret the visual mediator with the students, announcing that the students in the classroom will participate in the mathematical discourse. Thus, the development of the second and third internal layers of MDC occurs when the students participate in mathematical discourse. For example, in the stage of the explanation of the mathematical ideas, the second layer of the core, it was observed that the students in the classroom simultaneously participated in the mathematical discourse, determined the graphic structure and read and placed the tables/graphics together. Finally, in the stage of the development of the mathematical idea, it was observed that the flexibility or the required conditions in the drawing were determined by the comparison of visual mediators.

In the teacher-student discourse type on visual mediators, it was determined that the teacher or students play a role in the initiation of the mathematical discourse, similar to other mathematical content. As the teacher gave the student the right to speak, also motivated the student to draw the visual mediator. On the other hand, it was observed that students initiated a mathematical discourse on visual mediator by asking about the issues they did not understand about the visual mediator. Thus, it was observed that the discourse that included teacher's or the students' questions about the visual mediator were effective in the development of the first internal layer of the core of mathematical discourse. In the second layer where the mathematical ideas were explained, the characteristics of the visual mediator and the attributes that are desired or not desired in the visual mediator were discussed between the teacher and a student. In this stage, it was observed that the mathematical discourse between the teacher and a student included explanations about certain conditions of the visual mediator. Finally, discourses such as the comparison of two visual mediators, emphasizing the function of the visual moderator, and determination of adequate drawing rules were observed in the discourses that aimed to develop mathematical ideas. Thus, it could be suggested that the mathematical discourses between the teacher and the student on the assessment of drawing rules were effective in the development of the last internal layer of the mathematical discourse core.

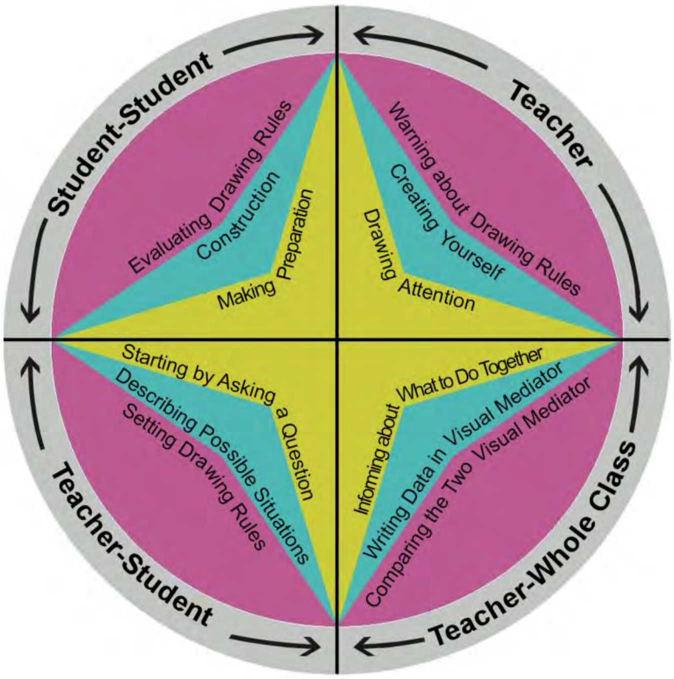
It was observed that the teacher motivated the students to participate in the mathematical discourse on the visual mediator before the student-student type mathematical discourses on visual mediators. It was observed that the teacher provided preliminary explanations and instructions on the visual mediator to improve student participation in the mathematical discourse. Thus, students were mentally prepared for participation in the mathematical discourse on visual mediators by asking intriguing questions or providing explanations-instructions. Furthermore, students were encouraged to participate in the mathematical discourse by asking hypothetical questions that could arouse curiosity. Thus, it could be suggested that mathematical discourse on preparation for visual moderators were effective in the development of the first internal layer of the core of mathematical discourse. It was determined that the second internal layer of the mathematical discourse core included mathematical discourse that aimed to determine the structure of the visual mediator and construct the visual moderator via discourses such as questioning the development of the visual moderator, associating the moderator with other background or daily life. Rejective or supportive discourses were observed among the students about the construction of the visual mediator. Finally, it was determined that the discourses that aimed to develop a mathematical idea included the student decisions, conclusions, implementation and interpretation of the visual mediator. Thus, it was observed that the mathematical discourse on the evaluation of drawing rules were effective in the development of the last internal layer of the mathematical discourse core. A mathematical discourse example between the teachers and students that support this finding is presented below.

1589.10 T1: Vertical, OK, in the axis that includes vertical or horizontal numerical II data, how should the numbers succeed) (students raised hands)
1589.11 S14: Consecutively.
1589.12 T1: Absolutely.
1589.13 S9: Why?
1589.14 S18: As such.
1589.15 S14: The rule

1589.15 S21: The number will be more confusing.
1589.17 T1: The rule. What would happen if do not obey the rule?
1589.18 S28: (before the teacher finished the sentence) Will be confusing.
1589.19 S15: When comparing the TV series, III, Ekin gave an example about the series (talking about the examples about line graphs). Now, do the ratings have to be consecutive?

As predicted from the final lines of dialogue 1589, the students evaluated graph drawing rules. As seen in line 19, students questioned how to implement the drawing rules in different examples. It was observed that the discourse was effective in the development of the student-student discourse type on visual mediators. In other words, it was observed that the types of discourse was different when compared to the instruction of the drawing rules by the teacher, since the students discovered these rules in this case. The mathematical discourse core that reflects the development of the mathematical discourse on the visual mediators is presented in Figure 2.

Figure 2
The core of mathematical discourse on visual mediator background



4.3. MDC on Problem-Solving

It was determined that in the teacher discourse type, the teacher did not motivate the students to participate in mathematical discourse on problem-solving. Similar to mathematical discourse on terminology, the teacher started solving the problem by indicating the problem in the textbook or smartboard or by mentioning related examples. It was determined that as the teacher started to solve the problem, the discourse included the repetition of the solution. Thus, it was determined that the discourse on reinstruction of the solution was effective in the development of the first internal layer of the mathematical discourse core. The discourse that started with reinstruction, continued with the mathematical discourse on the solution such as demonstrating different solutions, explaining the rule employed in the solution in the stage of the explanation of mathematical ideas. Thus, it could be suggested that the mathematical discourse on the explanation of the solution method was effective in the development of the second internal layer of the mathematical discourse core. Similarly, teacher discourses were prominent in the stage of the development of mathematical ideas. After the solution was discussed, the teacher conducted repetitive discussions about the solution such as summary and causality. Furthermore, mathematical discourses where the teacher decided what was necessary for the solution and the comprehension levels of the students were identified. Thus, a mathematical discourse that included continuous warnings and recommendations about the inaccurate and wrong student solutions was observed. Thus, it could be suggested that the discourse on warnings about the problem solution were effective in the development of the last internal layer of the core of mathematical discourse.

In the teacher-class discourse type on problem solving, the teacher motivated the students in the classroom to participate in the mathematical discourse. The teacher allowed student participation in the mathematical discourse by providing clues about the rule, or directly instructing the problem. The analysis of the mathematical discourse on all discourse indicators revealed that the teacher informed the students, either implicitly or explicitly, that they could participate in the mathematical discourse. In the next stage where mathematical ideas were explained, the teacher oriented the students to participate in the mathematical discourse. It was determined that the orientation was associated with the effective use of questioning strategies by the teacher. Various questioning strategies such as asking leading questions, asking affirmative questions and asking simple questions were used more frequently. The dialogue that included the participation of the teacher and the students in the dialogue 1571 is presented below.

1571.3. T2: I will repeat the question. When a circle is placed in a square to touch the sides, forcing the limits, the diameter of the circle is equal to which element of the squate? Tell me the answer.
1571.4. Class: One side.
1571.5. T2: One side, you are wonderful. Then, the diameter is equal to what?
1571.6. Class: 8.
1571.7. T2: You found that, too. Super. Is the question about the shaded area? What did we do in shaded area problems? We subtracted the two areas. We subtract the smaller area from the larger one. What is the outer area?
1571.8. 1571.9 . shape? 1571.10. 1571.11. the circle? 1571.12 1571.13 include? 1571.14.
Class: 64.
T2: It should be 64 based on the problem. Wonderful. Minus. What is the inner
Class: Circle.
T2: Circle. Now I will calculate the area of the circle. What is the area formula for
Class: pi times r squared.
T2: (Teacher wrote the formula on the board) Wonderful. What does the formula
Class: The radius.
1571.15. 1571.16. 1571.17. subtraction? 1571.18.
T2: Radius, the lower case r.
Mehmet: Then it is 4.
T2: Yes, I have to select 4, not 8. 3 times 16 is 48. What is the result of the
Class: 16.

Thus, it was observed that the teacher's questioning strategies led to the participation of most of the students in in mathematical discourse. However, it could not be argued that students simultaneously participated in the mathematical discourse that aimed to develop the mathematical idea. However, the teacher often asked if the students understood, and they replied positively or negatively. In this stage, the teacher provided advice about the solution by emphasizing what needs to be known by heart, listed what needs to be done to solve the problems, and stated what do to learn better.

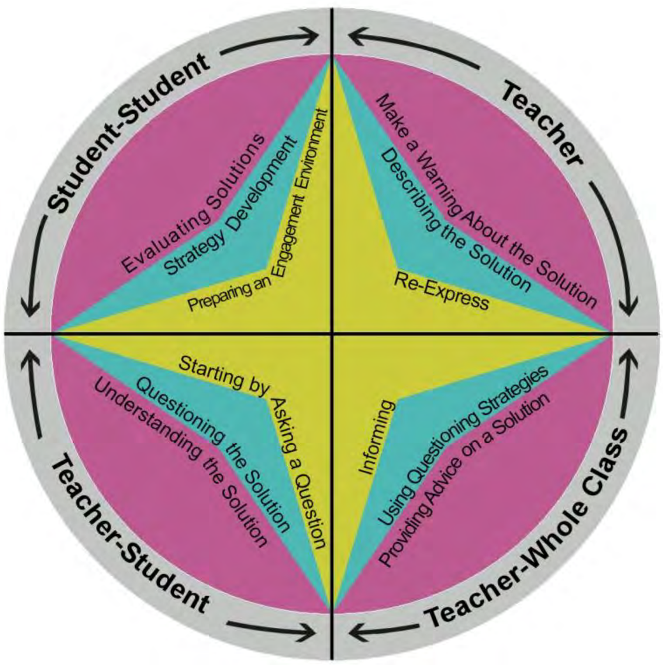
It was determined that the teacher-student discourse type on problem solving started with student or teacher discourse, similar to other mathematical contents. The discourse indicators such as confusing- remembering the rule, reflecting flexibility in the solution were observed in the mathematical discourse between the teacher and the students in the stage of the explanation of mathematical ideas. Based on the indicators, it was concluded that the problem solution method was questioned by the students. After the mathematical discourse developed between the teacher and a student, the dialogue was terminated with mathematical discourses for the comprehension of the solution such as providing feedback for right-wrong answers and recognition of the mistakes. The mathematical discourse between the teacher and the student that reflect this case is presented below.

1077.15.Student 4: Will we learn the subtraction operation as we do addition?
1077.16.T2: Yes. Wen ever do subtraction. We always add.
1077.17.Student 5: We turne even subtraction into addition.
1077.18.T2: Yes.
1077.19.Student 4: Is not there a plus sign at the bottom? Why did we multiply?
1077.20.T2: It told us to go back. It told us to stop, the road has ended, go back one step. What is going back? It was minus, wasn't it honey?
1077.21.Student 4: Can we use this transformation with all operations??
1077.22.T2: Yes. You can in every minus. Yes, not in plus but minus.

Thus, in the stage of the development of mathematical ideas, students questioned the solution. On the other hand, in the student-student discourse type, different students participated in the mathematical discourse by determining the problem solving steps with other students. In this stage, it was observed that the students also asked about the issues they were curious about the problem. This improves student motivation by discussing about the solution of the problem. Furthermore, the teacher encouraged the students by promoting curiosity during problem-solving. The teacher informed the students that problem-solving would entail a game/activity to raise their curiosity. Students, on the other hand, initiate the mathematical discourse by asking about the issues they did not understand. Furthermore, students intervened in incorrect solutions of the students, and asked about the issues they were curious about in the solution of the problem. Then, in the stage of the explanation of mathematical ideas, it was determined that the students came up with a discourse on the development of a solution strategy, determined the solution step by step, questioned the solution by reasoning, and discussed about these solutions. In other phrases about making the solution step-by-step, "first.....I will find it; It has been observed that the phrase such as "I will do it later" is frequently used.

It was concluded that the students produced various solutions through mathematical discourse on problem-solving. It was determined that the stage of the development of mathematical ideas included indicators such as finding unique results with the comparison or association of two results through mathematical discourse, re-questioning the solution, and finding unique results. Thus, it could be suggested that the mathematical discourse that entailed the analysis of possible solutions was effective in the development of the final internal layer of the MDC. The development of mathematical discourse on problem-solving is presented in Figure 3.

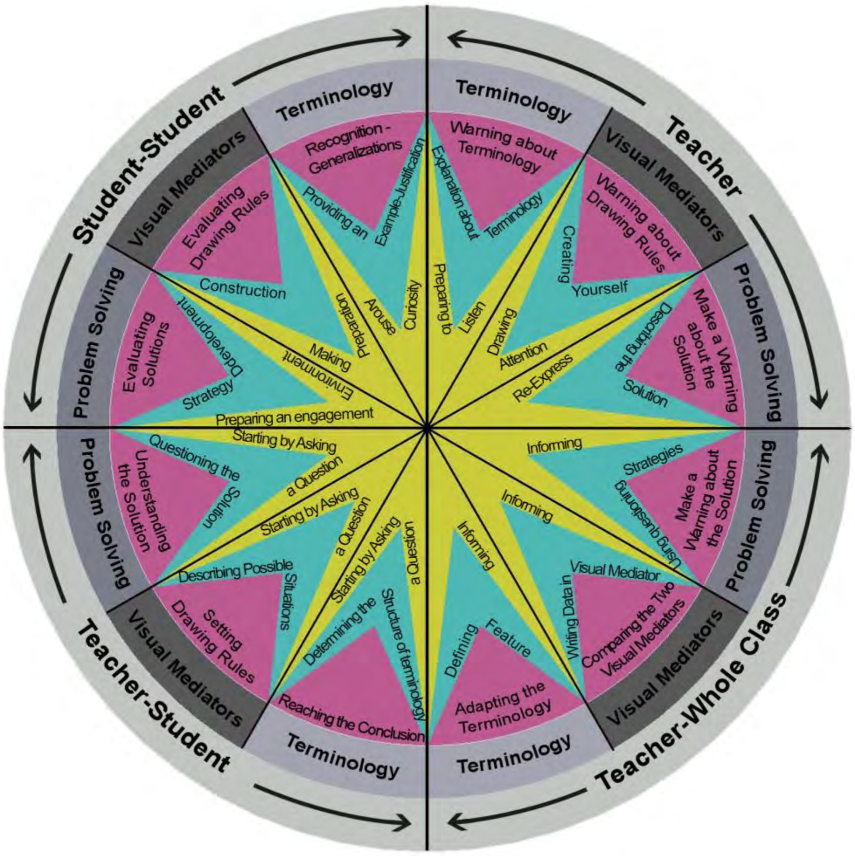
Figure 3
The core of mathematical discourse on problem-solving background



5. Discussion

In the present study, the theoretical framework that reflects the development of mathematical discourse was determined through the observation of mathematical learning and instruction processes in natural environment. Initially, small cores of mathematical discourse on mathematical terminology, visual mediators and problem-solving were identified and discussed in the findings section. Mathematical discourse cores included discourse indicators that reflect the development of mathematical dialogues. These indicators include moving discourses that lead to the development of mathematical discourse. Then, the main mathematical discourse core was obtained with the small mathematical discourse cores developed based on mathematical content. The main MDC is significant since it is a general theoretical structure that reflects the development of mathematical discourse in the classroom based on the mathematical content and the discourse types. In other words, the development of mathematical discourse that include in-classroom dialogues was holistically characterized by a main MDC. The main MDC is presented in Figure 4.

Figure 4
The general structure of the mathematical discourse core (The main mathematical discourse core)



The main MDC demonstrates the development of mathematical discourse based on the mathematical content. It could be observed that the internal structure of mathematical discourse varied based on the external structure of mathematical discourse. It was observed that the development of the mathematical core deepened from the teacher type mathematical discourse, which is an external element of the mathematical discourse core, to the student-student discourse type. Furthermore, there was a correlation between the internal and external structures of mathematical discourse. In the study, the development of mathematical discourse was discussed in detail by determining both the internal and external structures of mathematical discourse. In the literature, several studies investigated the development of mathematical discourse based on various perspectives. Hufferd-Ackles et al. (2004) discussed the development of mathematical discourse based on four components: asking questions, explanation of mathematical ideas, the source of mathematical ideas, and learning responsibility. Similarly, Adler and Ronda (2015) determined a theoretical framework that explained the development of mathematical discourse based on video recorded observations. They described the theoretical framework with three interactive components: sampling, explanatory dialogue, and student participation, based on the objective of learning mathematics. Thus, the theoretical framework that explained the development of mathematical discourse reflects an overview of the mathematics learning-instruction process. Similarly, Legesse et al. (2020) discussed the mathematics learning- instruction process by describing the stages of mathematical discourse-based instruction as planning the instruction, designing tasks, independent study, small group discussions, large group discussions, and post-class evaluation. However, the mathematical discourse core that emerged in the present study addressed the development of mathematical discourses both externally and internally. The external structure of the mathematical discourse core included the general structure of in-classroom dialogues, and the internal structure included the development of mathematical discourse based on the mathematical content. Thus, MDC explained the development of the mathematical discourse during learning and instruction of mathematics more clearly.

In the present study on the nature of mathematical discourse, it was observed that student participation in mathematical discourse determined the discourse types. It was identified that student participation in mathematical discourse was associated with the mathematical discourse that motivated the students. In the study, where the observations were conducted with video recordings of the problem-solving processes, the different participation levels of the students in problem-solving was explained by their motivation levels (Abele, 1998). In another study, where mathematical discourse that lead to interaction between the teacher and the students were observed, a theoretical framework was developed about the motivation of the students based on the interaction between the teacher and the students. Based on this theoretical framework, it was concluded that the organization of the class by the teacher was effective and ensured continuity (Durksen et al., 2017). Thus, it could be suggested that motivating classroom organizations were effective on student participation in mathematical discourse. On the other hand, it was observed that conflicting ideas increased student participation in mathematical discourse. Larsson (2015) reported that incorrect or incomplete solutions were effective in the initiation of class discussions in mathematics course based on the observations and the interviews conducted with the teachers. Furthermore, Blanke (2009) argued that student mistakes deepened mathematical discourse. Thus, contrasting examples and questions should be provided for the students to allow them to experience mathematical imbalances in mathematics instruction. Based on the approach by Piaget, it was observed that the student-student discourse type developed when students experienced imbalances in learning in the mathematics course.

In addition to studies on only the external student-student discourse type in the mathematical discourse core (Johnson, 1981), it was observed that certain components matched the student-student discourse type in other studies (Mercer, 1995; Muto-Humprey, 2010; Mortimer & Scott, 2003). The review of these discourses and the student-student type mathematical discourses observed in the present study revealed that certain discourses were open to discussion among students. Meaningful mathematical discourse is open to debate or interactive and includes an inclusive strategy that allows the development of mathematical concepts (Bennett, 2014). In the present study, it was observed that the student-student discourse type allowed the students to discuss mathematical ideas among themselves. In a study conducted by Ticar et al., (2020) on the argumentative discourse-centered classroom model in mathematics course, it was claimed that the model improved student comprehension and confidence in the mathematics course. They also suggested that the model could be employed not only in mathematics but also in other courses associated with mathematics. The discourse developed in other courses could be analyzed based on the types of discourse in the external mathematical discourse core. On the other hand, it was observed that mathematical discourse was also analyzed based on general discourse frameworks that were not specific to mathematics (Richards, 1991; Mercer, 1995; Brendefur & Frykholm, 2000;

Knuth & Peressini, 2001; Sabbagh, 2014). However, it could be suggested that MDC that emerged in the present would guide both the analyses specific to the mathematical content and in-classroom interaction.

Thus, MDC revealed the structure of mathematical discourse. It could be suggested that especially the discourse moves employed by the mathematics teachers determined the teacher-student dialogue. In other words, it was observed that the mathematical discourse of the teachers was more effective in the development of the extremal discourse types in the mathematical discourse core. Similarly, Demirbağ (2017) emphasized that different types of discourse contributed to the development of arguments by pre-service teachers and the role of the teacher was significant in every stage between the discussion of ideas and the conclusion. Thus, it could be suggested that the teacher's mathematical discourse determine the participation of the students in the class to express their views. Ceron (2019), in a study where mathematical dialogue movements that facilitate mathematical discourse were investigated, reported that teacher intervention was necessary for the students to produce meaningful mathematical discourse. It was observed in the study that the performance of students who recognized the discourse movements when conducting mathematical tasks improved. DuCloux (2020), on the other hand, investigated the mathematical discourse employed by two middle school mathematics teachers to encourage the students in online learning, and reported that the components of mathematical discourse based on questions and answers as follows: analysis, verification, guşdance, expansion and termination. In the present study, where MDC and the structure of mathematical discourse were determinded, it was observed that discourse types developed based on the demand for explanatory answers for the questions. In a study on the types of questions employed in the instruction of middle school mathematics course, Bozkurt et al. (2017) determined that the employment of questions with long answers and require deep comprehension and described as justification, inference, and criticism/interpretation was limited. Thus, questions that students could respond by justification should be asked. It was determined in the present study the that the justified mathematical discourse of the students were effective in the development of the second layer in the internal structure (explanation of mathematical ideas) of the mathematical discourse core. The analysis of MDC revealed that this was even more prominent in mathematical discourses on terminology and problem- solving.

6. Conclusion

The development of discourses in different internal layers of the mathematical discourse core was among the significant findings of the present study. For example, in the teacher discourse type on terminology, the teacher provided daily-life examples in the first stage of the horizontal dimension (motivation), while students provided daily-life examples in the second stage of the horizontal dimension in the student-student discourse type (explanation of mathematical ideas). Similarly, in the second stage of the horizontal dimension in the teacher discourse type, the teacher defined the mathematical terms, while in the third stage of the horizontal dimension (development of mathematical ideas) in the student-student discourse type, the students provided the definitions. Thus, the development of horizontal dimensions in the internal structure of mathematical discourse determined the development of vertical dimensions in the external structure. In other words, the development of the internal structure of mathematical discourse played a key role in the development of the external structure.

It was determined that students would not actively participate in the discourse in all teacher type motivational discourses. Similarly, in other discourse types, it was determined whether the students in the first layer would participate in mathematical discourse. Thus, it was observed that motivational discourse in the first layer of MDC affected the development of mathematical discourses in other layers. Another significant finding about the core of great mathematical discourse was the fact that terminological definitions were observed in the explanation of mathematcial ideas stage in the teacher discourse type, while these were observed in the development of mathematical ideas stage in the student-student discourse type. Furthermore, it was determined that questioning strategies were different in the vertical dimensions of mathematical discourse. It was determined that questions such as why and how were more effective in the development of student-student discourse type, a dialogic discourse, when compared to the development of mathematical discourse in the teacher-student discourse type. Because these questions cause the students to reflect on the answers and improve the interaction between the students. Thus, why-questions could be employed to motivate the students to think. It was determined that the employment of various questioning strategies such as asking orientative, confirmative and simple questions were effective in the development of teacher-class discourse type. Thus, various questioning strategies were employed in the development of teacher-class type mathematical discourse, which were the nature of group discussions.

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