An Intervention Study for Improving Pre-service Mathematics Teachers' Proof Schemes

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Abstract: This study aims to examine the effects of a course module aimed at improving pre-service mathematics teachers' (PMTs) proof schemes. This study used designed-based research. Participants of the course were 22 PMTs in a teacher preparation program. We obtained data from two surveys, one of which consists of proof questions. The other requires identifying the proof schemes of ten 9th grade students in a scenario including an excerpt of a hypothetical discussion with their teacher. We implemented both surveys before and after the intervention. In addition, semi-structured interviews were conducted with four PMTs. Findings indicated that the module had a significant and large effect on PMTs' proof schemes in favor of post-intervention. Most of the PMTs transformed their external and empirical proof schemes to the analytical ones. Findings also indicated that PMTs had difficulties identifying students' symbolic, example-based, transformational, and axiomatic proof schemes but overcame these difficulties after the intervention.

Keywords: Designed-based research, intervention study, proof schemes, proof teaching, teacher preparation

INTRODUCTION

Helping students understand proof and developing their proving techniques is a challenging field in mathematics education research (Marrades & Gutiérrez, 2000). Undergraduate courses often focus on how to write proofs rather than how to best use proofs in a high school classroom (Dickerson & Doerr, 2014). Likewise, PMTs reported little opportunity to deal with proof in their university experience and guidance to teach proof (Sears, Mueller-Hill, & Karadeniz, 2013). However, teachers' arguments should be strong, and at the same time, they should raise students with strong arguments. Therefore, mathematics teacher educators should look for different ideas to teach proof in teacher preparation courses (Stylianides & Stylianides, 2017). Lack of such courses and instructional materials in pre-service teacher education programs calls for well-designed intervention studies that foster PMTs' knowledge of proof. In response to this call, this study aims to report the findings of an intervention in the context of a course module that aims to develop PMTs' proof schemes.

Proof Schemes

Proof schemes are cognitive characteristics of the proving process (Harel & Sowder, 1998) and "consists of what constitutes ascertaining and persuading for that person (or community)" (Harel & Sowder, 2007, p. 809). The types of justification people use to prove a proposition determine their proof schemes (Harel & Sowder, 1998, 2007). These arguments may be weak exogenous justifications (Sowder & Harel, 1998), justifications based on experiences more advanced than exogenous arguments (Harel, 2007; Harel & Sowder, 1998). They may also be strong justifications such as generality, operational thought, and logical inferences (Harel, 2007; Sowder & Harel, 1998).

Harel and Sowder (1998) discovered undergraduate students' categories of proof schemes each of which "represents a cognitive stage and intellectual ability in students' mathematical development" (p. 244). Harel and Sowder (1998) offered three main categories of proof schemes and their sub-categories, each characterized by one's methods of justification. In the first category, the *external proof schemes*, an external source convinces the student. This source could be an authority (e.g., a teacher or a textbook). In this case, it is called the *authoritarian proof scheme*. The external source might also be the form or appearance of arguments, e.g., proofs in geometry must be in two columns (Harel & Sowder, 2007). In this case, it is called the *ritual proof scheme*. The last sub-category of an external proof scheme is the *symbolic proof scheme* which includes meaningless manipulations of symbols e.g. $\frac{x+y}{y+z} = \frac{x}{z}$ (Harel, 2007).

The second category is the *empirical proof schemes*. For this scheme, "conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences" (Harel & Sowder, 1998, p. 252). It has two sub-categories: (a) the *example-based proof scheme* that relies on evidence from one or more examples, and (b) the *perceptual proof scheme* that relies on intuition or perception to convince or to be convinced (Harel, 2007).

At the highest level of justification, the third category is the *analytical proof schemes* in which conjectures are validated using logical deductions (Sowder & Harel, 1998). It has two sub-categories: transformational and axiomatic. The *transformational proof scheme* relies on generality, operational thought, and logical inference (Harel, 2007; Sowder & Harel, 1998). Generality is concerned with justifying "for all" and operational thought occurs when a student "forms goals and subgoals and attempts to anticipate his/her outcomes during the proving process" (Harel, 2007, p. 67). Finally, mathematical justification should be based on the rules of logical inference (Harel, 2007; Harel & Sowder, 1998). In addition to these three characteristics, in the *axiomatic proof scheme*, proving processes are built upon an axiomatic system; therefore, they must start from accepted principles (Harel, 2007). Unlike the first two categories, analytical proof schemes require formal proofs and those who have this scheme use appropriate proof methods.

Both teachers' and students' proof schemes shape the teaching of proof in the classroom because proof schemes are characterized by one's methods of justification to convince himself or others about the truth or falsity of a proposition (Harel & Sowder, 1998; Sowder & Harel, 1998). Teachers might often use external and empirical arguments in their teaching. However, the use of this schemes causes difficulties in proof teaching. Because students have difficulties bridging the gap between informal and formal argumentation (Heinze & Reiss, 2003). For this reason, for teaching proof effectively, firstly, teachers should have analytical proof schemes when making proofs because they are at the top level of justification types and accept formal proof (Sowder & Harel, 1998). Nevertheless, research studies in the mathematics education literature indicated that pre-service and in-service mathematics teachers' proof schemes were not at the desired level (Manero & Arnal-Bailera, 2021; Sears, 2019). Therefore, this study aims to develop pre-service mathematics teachers' analytical proof schemes.

As future teachers, PMTs should determine the types of justifications used by their students and help them enhance their justification types to reach the analytical level. PMTs should identify students' valid and invalid arguments (Lesseig, Hine, & Boardman, 2018). However, research shows that they have difficulties in evaluating proofs (İmamoğlu & Yontar-Toğrol, 2015; Sears, 2019). Therefore, another aim of this study is to improve pre-service mathematics teachers' knowledge of identifying proof schemes used by students.

In line with the aims of the research, we designed a course module and teaching materials to address the deficiencies about proof schemes reported in the literature. The course module was integrated into a teacher preparation program. In doing that we aim to guide mathematics education researchers and teacher educators in designing similar modules on proof and proof schemes.

METHODOLOGY

This study is part of a wider study which used designed-based research (DBR). DBR "is an emerging paradigm for the study of learning in context through the systematic design and study of instructional strategies and tools" (Design-Based Research Collective, 2003, p.5). A proof course was designed and implemented in a teacher preparation program in a state university in Istanbul, Turkey. The current study focuses on a module on proof schemes. Within this module, DBR is used to develop and examine PMTs' proof schemes. The current study reports the first cycle of these processes and makes suggestions for a second cycle.

Participants and Context of the Study

Participants were 22 PMTs (fourteen females and eight males) who enrolled in an elective course called "Proof in Mathematics Teaching." Before this course, they were familiar with mathematical proofs. All participants signed a consent form that explains the aim of the study and ethical issues. We selected four PMTs using the theoretical sampling technique (Mason, 2002) based on their answers to the survey questions for a further qualitative investigation. PMT1 often used the analytical scheme, PMT2 the empirical scheme, PMT3 the external scheme, and PMT4 was the one who left the survey questions unanswered the most and could not complete the proofs (i.e., considered as not having a particular proof scheme).

Design of the Module

Therefore, we first specified the objectives of the module considering the related literature on proof schemes as reported above. We specified two learning objectives (Cihan, 2019; Cihan & Akkoç, 2019):

- PMTs will be able to use analytical proof schemes in their proofs.
- PMTs will be able to identify students' proof schemes.

In the wider study (Cihan, 2019), we designed a 15-week course. It had various modules (i.e., proof components, methods, identifying proof schemes, student difficulties with proving, reasons behind student difficulties, and teaching strategies to overcome student difficulties). Within the scope of this current study, we prepared an eight-week module to improve PMTs' proof schemes. The module's content consists of proof methods, proof classifications, and proof schemes. As a result of expert opinions, we decided to prepare worksheets and scenarios to improve proof schemes of PMTs.

Implementation of the Module

The first author was the tutor of the course module, while the second author had the role of a coordinator and a non-participant observer. The teaching methods included lecturing, questioning, discussion, problem-solving, case study, and scenario-based teaching. The course addressed various proof methods (proof by induction, direct proof, proof by cases, proof by contradiction, proof by contrapositive, proof by counterexample, proof by exhaustion) using hands-on activities as PMTs worked in groups of five or six. Later, the tutor explained Sowder and Harel's (1998) three main proof schemes and seven sub-proof schemes.

The tutor asked PMTs to prove theorems at high school and undergraduate levels using worksheets. We prepared worksheets following the axiomatic structure specific to each proof method. Each worksheet starts with a sentence stating the assumptions and ends with a closing sentence stating that the proof has been completed. The tutor asked PMTs to fill in the gaps between these sentences according to the proof methods. These gaps were specific to the proof methods. For example, for a theorem to be proved by the weak inductive proof method, worksheets had gaps for the basic step (or the initial step), the persuasion step, the inductive hypothesis (or the inductive assumption), and the inductive step (See Appendix 1). To exemplify the method of proof by cases, the worksheets had gaps to be filled by PMTs for proving all the cases. Worksheets for the proof by contrapositive had gaps to write the expressions $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$, and to prove $\neg q \Rightarrow \neg p$. Using these gaps in the worksheets, we aim to help them to decide about the steps in different proof methods. We devoted one week for each proof method. PMTs worked individually and as a group. Later, the tutor completed the proofs at the axiomatic level. The class discussed the gaps in the argument chain of different proofs used by PMTs.

We presented Sowder and Harel's (1998) classifications of proof schemes to the PMTs using two scenarios. PMTs worked first in groups and then individually to identify the students' proof schemes in these scenarios. Finally, the class discussed proof schemes of the students in the scenarios. PMTs took on the role of a teacher when determining students' schemes and were asked how they would respond to the students. After the intervention, we evaluated the learning objective related to proof schemes using a different scenario.

Evaluation of the Module

For this study, we evaluated the effect of the module on achieving the learning objectives. We considered pre-test, post-test, and interviews to evaluate them. As a result of these evaluations, which will be presented as the findings of this study, we revised the course module to be implemented in the next cycle to improve the intervention because DBR goes beyond designing and testing specific interventions and contributes to learning and teaching theories (Design-Based Research Collective, 2003). We will attend to these issues in the discussion section. Based on our experience in conducting the module and findings of our research, we have prepared a practical guide for practitioners who might want to use the module (See Appendix 2).

Data Collection Tools

We collected data using the proof survey, a scenario-based survey, and semi-structured interviews. We conducted the surveys and interviews twice, before and after the intervention (fifteen weeks later).

We designed a proof survey to explore the proof schemes used by PMTs (See Appendix 3). Proof survey consist of seven types of proving questions on mathematical topics such as Fibonacci sequence, matrices, inequalities, functions, divisibility, trigonometry, and numbers. Six of these questions require proofs of the theorems and the other a refutation of a false proposition. We took expert opinion about whether survey questions can be proved with seven different proof methods (proof by induction, direct proof, proof by cases, proof by contradiction, proof by contrapositive, proof by counterexample, proof by exhaustion), and whether the questions reveal PMTs' proof schemes. Experts gave their opinions during a workshop that introduced the purpose of the study, research questions, methodology, data collection tools, and data analysis techniques. We delivered expert opinion forms, and experts discussed their inputs. For the reliability of the survey, we conducted a pilot study with ten PMTs who were studying in the second year of a teacher preparation program in another university. After taking expert opinions and conducting the pilot study, we revised the survey questions so that PMTs prefer each of the proof methods above. Thus, the survey was revised to include seven questions for which seven proof methods are the most appropriate. We also wanted to observe whether PMTs could use each proof method using the analytical schemes.

The first author interviewed four PMTs individually using a semi-structured interview form. We asked PMTs follow-up questions during the interviews to obtain in-depth information for their answers to each question in the survey. Questions in the form explored whether PMTs had a proof scheme, and if so, which scheme they had. The interviewer asked PMTs whether the proof was sufficient to convince themselves or others, whether there were missing points in their proofs, whether a different proof was possible, and whether they had any difficulties. He also asked them to make sense of their steps in the proof and follow-up questions based on their proof schemes.

We also designed a scenario-based survey to explore how PMTs identified students' proof schemes (See Appendix 4). Scenarios include "short stories about hypothetical characters in specified circumstances" to whose situation the respondent is invited to respond (Finch, 1987, p. 105). In this study, the scenario includes an excerpt of a hypothetical discussion among a mathematics teacher and ten 9th-grade students (age of fifteen). The topic of the scenario is set theory, which is a typical topic in the 9th-grade curriculum in Turkey, the basis of many mathematical subjects (e.g., functions, derivatives) and has an essential place in the use of mathematical language. In addition, set theory is suitable for the use of different proof schemes (See Cadwallader-Olsker, 2011). The class discusses the truth of a proposition. PMTs are familiar with set theory from undergraduate courses. In the scenario, the teacher presents the following proposition and asks students whether this proposition is true or false, and to justify their answers:

"Let X, Y and Z be sets. If $X \subset Z$ and $Y \subset Z$ then $X \subset Z$".

Students' excerpts illustrate Sowder and Harel's (1998) proof schemes. The scenario in the survey was different from the one used in the module for validity concerns. The topic was also different. PMTs filled the scenario-based survey before and after the intervention (fifteen weeks later). They worked on handouts that included the scenario and the course objective. The survey included the question: "*Determine the arguments (justifications) used by the students in this scenario to prove the truth or falsity of the statement*."

We ensured the validity and reliability of the scenario-based survey in a similar way as for the proof survey. For validity concerns, researchers conducted a workshop with ten experts in mathematics and mathematics education. Considering the expert opinions, we specified the number of students as ten which is bigger than seven (the number of sub-proof schemes), to prevent PMTs from matching students' work to the proof schemes. Based on expert opinion, we prepared a coding key for proof schemes (See Appendix 5).

We used the same scenario and asked the following questions to the four PMTs during the semi-structured interviews: (a) identify students' argument (justification type), (b) whether students' answers convinced you or would convince others (c) how would you intervene with students' answers if you were the teacher.

Data Analysis

We used Sowder and Harel's (1998) categories for proof schemes to analyze data to explore the module's effectiveness. We used the Wilcoxon Signed Rank Test (Wilcoxon, 1945) to investigate whether the module significantly affected PMTs' proof schemes. Probability value $p < 0.05$ was decided to be enough to be a statistically significant difference with a confidence level of 95%. The effect (r) size was calculated using the formula $r = z/\sqrt{n}$ (Pallant, 2007). According to Cohen (1988), the effect size of $r = 0.1$ is considered small, $r = 0.3$ medium, and $r = 0.5$ big.

We analyzed the answers to the proof survey by descriptive analysis. One hundred fifty-four responses given by 22 PMTs to seven questions in the survey were coded as "external," "empirical," "analytical" and their sub-schemes. We coded no responses and incomplete proofs as "without a scheme." We considered answers that the prover is not convinced by himself also in this category because, for a person to have a proof scheme, he must first be convinced of the proof he made (Harel & Sowder, 1998; Sowder & Harel, 1998). We scored the analytical schemes as three points, empirical schemes as two points, external schemes as one point, and "without a scheme" answers as none. External scheme is ranked higher than "without a scheme" because one having an external scheme provides a justification for the truth or falsity of a proposition.

We analyzed the answers to the scenario-based survey also by descriptive analysis. Each participant identified 10 students' proof schemes. Therefore, there are a total of 220 answers (22 PMTs times 10 schemes). We coded quantitative data obtained from the survey as "correct" (proof scheme was identified correctly), "incorrect" (proof scheme was identified incorrectly), or "no response." We scored the correct answers as one point, others as none.

For the qualitative analysis, we used descriptive content analysis of interview data. We determined the thematic framework as analytical, empirical, external, and without a scheme to explore proof scheme using the interview data. If a PMT has doubts that the proof he made is not convincing for himself and others, or if he could not complete the proof, we coded it as "without a scheme." We considered PMTs who did not have any problem with persuasiveness, answers of those who resorted to authority for proof in the interviews, and who thought that this was valid for proof and gave answers in this direction to be having an "external (authoritarian) proof scheme." We coded the answers of the PMTs who used symbols without a meaning thinking that they were making valid proofs as "external (symbolic) proof scheme." The PMTs who tried to imitate the proofs they made in their past learning experiences focused only on the proof's form or appearance and not its content or logic. We coded them as an "external (ritual) proof scheme." We coded the answers of the PMTs who thought that one or more examples are sufficient for proof as an "empirical (example-based) proof scheme." We considered proving by drawing figures to indicate an "empirical (perceptual) proof scheme." We coded the answers reaching a generalization with logical inference rules and operational thinking and making sense of the steps taken as "analytical (transformational) proof scheme," the answers reaching generalization in an axiomatic structure and making sense of this as "analytical (axiomatic) proof scheme." We also used descriptive content analysis of the interview transcripts of PMTs' responses to the scenario-based survey to elaborate further on how they identified the students' arguments (justifications) in the scenario using Sowder and Harel's (1998) classification and the teacher's response guiding the student.

RESULTS

This section first presents the findings related to the effects of the intervention on PMTs' proof schemes. We will compare the proof schemes of 22 PMTs before and after the module based on the analysis of their responses to the proof survey (See Appendix 3). We will present the excerpts from the interviews to exemplify how PMTs' proof schemes evolved from the external and empirical schemes to the analytical schemes. The second section will present the findings related to the effect of the intervention on the way 22 PMTs determined the proof schemes of the students in a scenario.

The Effects of the Intervention on PMTs' Proof Schemes

We evaluated PMTs' written proofs and Table 1 presents the frequencies of the proof schemes used by 22 PMTs before and after the intervention.

		External			**Empirical**		**Analytical**			
Post-Pre	f and f_{sum}	Authoritarian	Symbolic	Ritual	Example-based	Perceptual	Transformational	Axiomatic	Without a scheme	Total
Pre	f	26 (30)*	35	5	27	5	25	4	27	154
	f_s	66			32		29		27	
Post	f	8	20	4	2	0	8	110	2	154
	f_s	32			2		118		2	

Table 1: Frequencies for PMTs' Proof Schemes (*Note.* *Before the intervention, more than one proof scheme was used in four questions. We considered those at the higher level as the scores.)

If PMTs had no response, could not complete their answers, or were not convinced, then we considered their answers as "without a scheme." The frequency of this category decreased from 27 to two throughout the intervention. PMTs used two different main proof schemes (external and empirical) in four of the responses given before the intervention. We coded these responses as the empirical scheme. After the intervention, the frequencies of the external and empirical proof schemes decreased while the frequency of the analytical schemes increased dramatically (from 29 to 118).

We explored the turn from external schemes and empirical schemes to analytical schemes in-depth during the interviews. For example, PMT3 mostly used external proof schemes before the intervention. In response to the question, "*Prove that* $x + \frac{1}{x} > 1$ *for* $\forall x \in R_{>0}$", he just manipulated the symbols. His approach was like problem-solving without using a proof method and displayed the characteristics of the external (symbolic) proof scheme. After the intervention, he chose the method of proof by cases and showed that the inequality was satisfied for the two cases ($0 < x < 1$ and $x \geq 1$) using the axiomatic structure. After the intervention, his proof scheme transformed into an analytical (axiomatic) scheme. Another question in the proof survey was "*Prove that* $\sqrt{5}$ *is not a rational number*". Before the intervention, he developed an argument based on a rule by stating that "*Root numbers are generally irrational numbers. Since* 5 *is not a perfect square, it is not rational*," and he was convinced about the correctness of the proposition in this way. In the interview, he displayed the characteristics of the external (authoritarian) scheme. After the intervention, he used the method of proof by contradiction; he assumed that $\sqrt{5}$ is rational, that is, $\sqrt{5} = \frac{a}{b}$, $a, b \in Z$ and $(a, b) = 1$. He then reached a contradiction that $(a, b) \neq 1$. During the interview, he made sense of the proof method and all the steps he took. With his answers, he showed all the characteristics of the axiomatic scheme.

PMT2's proof scheme was mostly empirical before the intervention. She did not use any specific proof method to prove that "$\forall m \in Z^{+}$, $m^3 - m$ *is divisible by* 6". She mentioned that $m^3 - m$ is a multiple of 6 for $m = 1, m = 2, m = 3$, and $m = 4$ until she convinced herself. After the intervention, she preferred the method of proof by exhaustion, considering that $m \in \{0,1,2,3,4,5\}$. In short, she moved from the example-based scheme to the axiomatic scheme. When responding to another question in the proof survey, she used the perceptual proof scheme (empirical) before the intervention, as shown in Figure 1.

Let $K, L,$ *and* M *be non-empty sets and* $g: K \to L$ *and* $h: L \to M$ *be two functions. Prove that if* hog *is a one-to-one function then* g *is also a one-to-one function.*

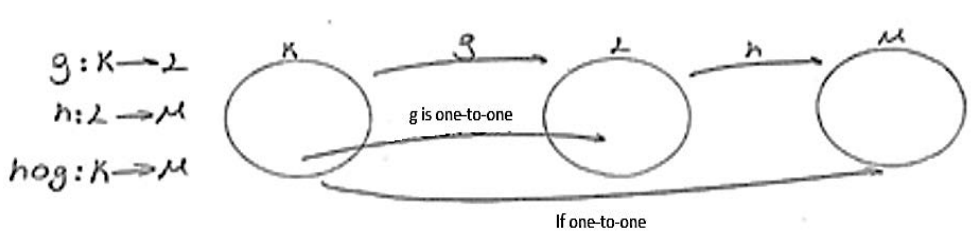

Figure 1: A drawing implying the perceptual proof scheme (PMT2, Pre-Intervention)

During the interview, PMT2 stated that the figure above was convincing for her and enough to convince others:

I: Which proof method did you use for this question?

PMT2: I drew a figure

I: How did you draw it?

PMT2: I checked the definitions of composite functions and one-to-one functions.

I: Any method?

PMT2: Actually, I did not use any method.

I: Is it possible to prove without a proof method?

PMT2: It was enough for me to do it, as in the diagram, intuitively.

I: Is this proof enough to convince you or others?

PMT2: Intuitively but if any other person did it then I would be convinced. I would also convince others.

The excerpt above is an indication of the perceptual scheme since PMT2 did not choose any method of proof and relied on a diagram. The following excerpt underlines her perceptual scheme further:

I: Are there any deficiencies in your proof?

PMT2: Even so, it is a convincing proof. Intuitively.

I: Is there a different proof other than yours?

PMT2: There are some others using mathematical expressions. But it explains this figure.

I: Is this enough to draw a figure in this question?

PMT2: Yes, if it is explanatory and meaningful.

After the intervention, she transformed into the analytical (axiomatic) scheme. She wrote the following proof:

Let g be a function that is not one-to-one. Let's prove that the function hog is not one-to-one.

If g is not one-to-one, then there exists $x, y \in K$ such that $g(x) = g(y)$. (1)

Then $(h \circ g)(x) = h\big(g(x)\big) = h(g(y)) = (h \circ g)(y)$ (2)

If $(h \circ g)(x) = (h \circ g)(y)$, *then* $h \circ g$ *is not one-to-one.* (3)

The proof ends here.

As her proof shows, accepting that the function g is not one-to-one, she showed that the composite function is not one-to-one in the axiomatic structure using the definition of one-to-one-ness. She applied the proof method correctly. The excerpt below indicates that she was aware of the proof method she chose:

I: Which proof method did you use for this question?

PMT2: Proof by contraction.

I: Why did you choose this method?

PMT2: Direct proof was also possible, but I think proof by contraction was the easiest for this question instead of looking for one-to-one-ness in composite functions.

I: Can you explain the method you chose?

PMT2: When the truth of the proposition $p \Rightarrow q$ *is asked, we prove that* $\neg q \Rightarrow \neg p$.

During the interview, she explained why her proof was convincing for her and others:

I: Is the proof method you choose suitable for this question?

PMT2: Yes.

I: Is this proof enough to convince you or others?

PMT2: It was very convincing for me.

I: Does it convince others?

PMT2: It does. If people know what a composite and one-to-one function is.

I: Can you make sense of each step in your proof?

PMT2: I can make sense. Suppose g *is not one-to-one. From here, using the definitions of composite and one-to-one functions, I proved that the composite function is not one-to-one.*

I: Did you have any difficulties when proving?

PMT2: No …at least I know the definitions and rules.

As the excerpt above shows, she could make sense of the steps she took in the proof. Based on the definitions, she reached a generalization using operational thought and logical inference which are indications of the axiomatic scheme.

PMTs who did not embrace any specific proof scheme tended to demonstrate the characteristics of the analytical proof schemes after the intervention. PMT4 left most of the questions in the proof survey unanswered before the intervention. In the interview, she reported that this situation was related to a lack of knowledge of proof methods and the content. For example, for the question, "*Let X be a matrix.* $\forall X$ *prove that* XX^T *is a symmetric matrix*", she stated that she could not prove it because she did not know the correctness of the proposition herself. Hence, she said that she did not have a convincing argument. PMT4 could not give any response which belonged to any scheme for this question. After the intervention, she chose the direct proof method and performed the following operation: $(XX^T)^T = (X^T)^T X^T = XX^T$, starting with the definition of a symmetric matrix. In other words, she used the axiomatic scheme.

PMTs failed to use the analytical scheme before the intervention. For example, when PMT1 was proving the statement "*Let* f_m *be an element of the Fibonacci sequence. If* $m \geq 2$ *then* $(f_m)^2 - f_{m+1} \cdot f_{m-1} = (-1)^{m+1}$" (Bloch, 2011) before and after the intervention, she proved the base case ($m = 2$) and assumed that the statement holds for $m = k$ for any k. However, when proving that it holds for $m = k + 1$, she did not use the axiomatic structure before the intervention. Instead of reaching the induction step from the induction hypothesis, she performed the necessary transformations of the induction step (logical inferences, procedural thinking, and generalizations) and completed the proof. After the intervention, she managed to do all these in an axiomatic structure by starting with the induction hypothesis, constructing a chain of arguments, and reaching the induction step. In other words, she moved from the transformational scheme to the axiomatic scheme throughout the intervention.

We can conclude from the findings presented in Table 1 and as emerged from the interviews that the module effectively reached the learning objective "PMTs will be able to use analytical proof schemes in their proofs. We used the Wilcoxon Signed-Rank Test to decide whether this effect was significant (See Table 2).

Post-intervention—Pre-intervention	***N***	**Mean Rank**	**Sum of Ranks**	***Z***	***p***	***r***
Negative Ranks	0^a	0.00	0.00	-4.113^*	<0.001	-0.88
Positive Ranks	22^b	11.50	253.00			
Ties	0^c					

Table 2: Wilcoxon Signed-Rank Test Outputs Regarding PMTs' Proof Schemes (*Note.* * Based on negative ranks, [a] Post-intervention < Pre-intervention, [b] Post-intervention > Pre-intervention, [c] Post-intervention = Pre-intervention.)

Table 2 shows a significant difference ($Z = -4.113$, $p < 0.001$) between scores obtained before and after the intervention. Since the absolute value of effect size (r) is 0.88, greater than 0.50, we can say that the module had a large effect size on the scores of proof schemes in favor of post-intervention.

The Effects of the Intervention on PMTs' Identification of Proof Schemes

In addition to PMTs' proof schemes, we also examined the effect of the intervention on the way PMTs determined the proof schemes of the students in the scenario and found that they better identified students' proof schemes after the module. Table 3 presents the frequencies of the PMTs' correct and incorrect answers for identifying the proof schemes of ten students in the scenario before and after the intervention. The scenario included 10 students and the table shows the answers from 22 PMTs for identifying each student's proof schemes.

Student number	Type of Proof Scheme	Correct (Pre-/Post-)	Incorrect or Empty (Pre-/Post-)
Student 1	Authoritarian	19/18	3/4
Student 2	Authoritarian	19/22	3/0
Student 3	Perceptual	17/22	5/0
Student 4	Ritual	18/22	4/0
Student 5	Symbolic	0/20	22/2
Student 6	Example-based (a single example of finite sets)	21/22	1/0
Student 7	Example-based (a single example of infinite sets)	13/21	9/1
Student 8	Example-based (multiple)	14/21	8/1
Student 9	Transformational	2/22	20/0
Student 10	Axiomatic	12/22	10/0
Total		135/212	85/8

Table 3: The frequencies of PMTs' correct and incorrect answers for identifying proof schemes

As can be seen in Table 3, findings indicate an improvement of the way PMTs identified students' proof schemes. The number of correct answers increased by 77. As Table 3 shows, PMTs had difficulties identifying symbolic, example-based, transformational, and axiomatic proof schemes before the intervention. They had overcome most of these difficulties after the intervention. Below, we illustrate these improvements with specific examples.

Student 5 in the scenario has the *symbolic proof scheme*. He justifies his answer considering the number of elements: "If $X \subset Y$ then $S(X) < S(Y)$ and if $Y \subset Z$ then $S(Y) < S(Z)$. Therefore $S(X) < S(Z)$ that is $X \subset Z$". Using this statement which is wrong, Student 5 uses shallow symbolic manipulation. None of the PMTs correctly identified that this student has a symbolic proof scheme because they also assumed it was a valid proof. PMTs thought that Student 5 in the scenario "*puts forth other cases based on known facts or data*" or "*justified it using logical inference*" and did not notice that Student 5 used symbolic manipulations. After the intervention, 20 out of 22 PMTs identified Student 5's proof scheme correctly. They used the terminology of the proof scheme framework. PMT1, before the intervention, mentioned that Student 5 completed the proof using the mathematical expression and did not notice the errors in the procedures, and she was convinced. After the intervention, she mentioned that Student 5 had the external (symbolic) proof scheme since the student manipulated symbols incorrectly, and she described the scheme's characteristics. Furthermore, she said that she was not convinced by Student 5's justifications and would tell the student his wrong symbolic manipulations.

In the scenario, we prepared three different cases of the *example-based proof scheme* using (a) a single example of finite sets (Student 6: Let $X \subset Y$ and $Y \subset Z$. Let $X = \{1,2\}$, $Y = \{1,2,3\}$ and $Z = \{1,2,3,4\}$, Since $\{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\}$ then $X \subset Z$), (b) a single example of infinite sets (Student 7: $N \subset Z$ and $Z \subset R$. Therefore $N \subset R$), and (c) multiple examples (Student 8: If each one of us in the class finds an example to show the truth of the proposition, then we can reach a generalization). Before the intervention, 21 out of 22 PMTs noticed that Student 6 relied on only one example. After the intervention, all PMTs identified the proof scheme of Student 6 correctly. For the cases of (b) and (c), frequencies of correct answers increased considerably after the intervention. For (b), after the teacher called out for a more general example, Student 7 justified his answer using the sets $N, Z,$ and R, which are infinite. We consider this as an "example-based proof scheme using a single example," as in the case of (a). However, before the intervention, nine PMTs could not identify the proof scheme in the case of (b) because they thought this was a generalization. Since they considered the student's proof scheme a generalization rather than the example-based scheme, we coded their responses as incorrect. However, after the intervention, they improved in identifying this scheme (21 out of 22 PMTs answered correctly). Student 8 suggested that if each student find one example, then there would be many examples to justify the truth of the proposition. 8 out of 22 PMTs could not identify Student 8's scheme as "*example-based*" before the intervention thinking that multiple examples were convincing for a generalization. After the intervention, 21 out of 22 PMTs correctly identified Student 8's justification as an "*empirical (example-based) proof scheme*." As an example, we present the responses from PMT3.

I: What is this student's argument (justification)? (Pre-intervention)

PMT3: If each student in the class gave an example, different examples would be sufficient for a generalization.

I: Did this student's justification convince you?

PMT3: So, what the student did, led to a proof. He convinced me.

Before the intervention, he thought that students' different examples were convincing to reach a generalization. He failed to refer to the properties of the example-based scheme prior to the intervention. However, he did not find them convincing and pointed out the need for a generalization after the intervention:

I: What is this student's argument (justification)? (Post-intervention)

PMT3: He gave a lot of examples.

I: Did this student's justification convince you?

PMT3: It has to be a generalization to persuade, but it is not. So he didn't.

I: What kind of approach would you take if you were the teacher in the scenario?

PMT3: I would say what he did was giving a lot of examples. I would say this is not a proof.

I: What is the main proof scheme this student has?

PMT3: Empirical proof scheme.

I: What is the sub-proof scheme that this student has?

PMT3: Example-based proof scheme.

I: What are the features of the scheme that this student has?

PMT3: Using examples.

As the excerpt above indicates, PMT3 emphasized the need for a generalization. Although he identified the example-based proof scheme of the student, his response to the student was instructive. He said he would say that giving a lot of examples is not a proof. He could not offer another approach to convince students about the limitations of the example-based proof scheme.

Student 9 has the *transformational proof scheme* since he reached a generalization through operational thought based on logical inferences:

Let $X \subset Y$ and $Y \subset Z$. (1)

Considering the rules we mentioned in our lessons, if $Y \subset Z$, then $Y \cup Z = Z$. (2)

$X \subset Y$ then $X \cup Z = Z$. (3)

If $X \cup Z = Z$, then $X \subset Z$. (4)

Before the intervention, only two PMTs could identify the proof scheme correctly because others did not refer to any components of this scheme (generalization, operational thought, or logical inference) in their explanations about Student 9's justification. After the intervention, all the PMTs identified the proof scheme correctly. For example, PMT2, who could not identify the student's proof scheme before the intervention, mentioned that Student 9's proof was valid and convinced him after the intervention. She added that "*Student 9 acted by the rules he knew. He did the things* *right and reached a generalization*". She also successfully identified that Student 9 had the analytical proof scheme as the main scheme and the transformational proof scheme as the sub-scheme. As a response to the question, "*If you were the teacher in the scenario, what would be your approach?*" she said that "*I would teach how to use the axiomatic proof scheme. They should start with the definition.*" Her response points out the difference between the transformational and axiomatic proof schemes.

Student 10 has the *axiomatic scheme* since he started the proof by using the definition of a subset and completed the proof:

Let $X \subset Y$ and $Y \subset Z$. (1)

In this case, from the definition of a subset, if $X \subset Y$ then for $\forall a \in X\ a \in Y$. (2)

If $Y \subset Z$ then for $\forall a \in X\ a \in Z$. (3)

Therefore, since for $\forall a \in X\ a \in Z$ then $X \subset Z$. (4)

It's proven.

Before the intervention, 12 out of 22 PMTs could identify this scheme. Others just mentioned that it was a mathematical proof. After the intervention, PMTs' knowledge of identifying this proof scheme has developed. For example, PMT3, before the intervention, could not identify Student 10's method of justification and just mentioned that the proof was similar to the ones they did in undergraduate mathematics courses. After the intervention, he identified the justification by saying, "*The student started with the definition and reached a generalization. He used the axiomatic structure which I would also encourage*". Furthermore, he considered the student's proof scheme as an analytical (axiomatic) scheme.

As presented in Table 3 and as emerged from the interviews, findings implied that the module effectively reached the learning objective "PMTs will be able to identify students' proof schemes". A Wilcoxon Signed Rank-Test showed that the module had a significant and large effect size ($Z = -4.144, p < 0.001, r = -0.88$) on the scores of ways PMTs identified students' proof schemes in favor of post-intervention.

DISCUSSION AND CONCLUSION

This study examined the effects of a course module to improve PMTs' proof schemes as they prove or refute propositions or identify students' proof schemes. The findings indicated that the intervention was effective in helping PMTs to use the analytical (especially axiomatic) proof schemes and overcoming their difficulties with identifying the symbolic, example-based, transformational, and axiomatic proof schemes. As the evaluation phase of our DBR, we will reflect on the developments of PMTs who participated in the intervention. In doing that, we aim to highlight the characteristics of the module (both related to its content and learning environment) that yielded the desired development of the learning objectives regarding proof schemes to be able to offer some guiding design principles for a course module on proof in the context of pre-service teacher education.

We consider the methodological explanations of proof components, proof methods, and all main proof schemes and their sub-schemes during the course module with examples as one of the reasons for the development. Likewise, according to Heinze and Reiss (2003), methodological knowledge is an important component of proof competence. In line with our research aim, the course module emphasized using the axiomatic scheme as one of the analytical proof schemes. As a result, PMTs converted from external and empirical proof schemes to the analytical ones. According to Weber (2010), improving understanding of proof depends on improving the methods of processing arguments. Therefore, another reason for the development may be that the module focused on proofs as a process rather than a product. According to Yoo (2008), a process-oriented approach should be preferred instead of a traditional product-oriented approach to proof teaching. During the module, we gave PMTs tasks to develop their ideas and focus on the critical ideas in proofs in an environment that ensured class interaction. For effective proof teaching, teachers should help students develop their ideas by focusing on the structure of proofs and key ideas in proofs (Heinze & Reiss, 2003; Raman, 2003). Following the process-oriented proof teaching, we followed a student-centered approach and provided feedback when necessary, during the course module.

Another characteristic of the course module regarding proof methods that yielded a development is how we used worksheets. It is necessary to know all the proof methods to develop the axiomatic scheme. Before the intervention, the PMTs did not know the proof methods, and they thought of proof more like problem-solving and did not need a method. Allocating a week to each proof method in the course and focusing many proof questions about these methods with worksheets can be considered as a reason for the development. In addition, each worksheet was devoted to a specific proof method algorithmically. Since mathematicians think that introductory and ending sentences should be included in proofs, that the main ideas should be formatted to emphasize their importance, and that unnecessary information should be removed so as not to distract or confuse the reader (Lai, Weber, & Mejía-Ramos, 2012), each worksheet had spaces for the beginning sentence containing the hypothesis, for the steps to be taken (specific to each proof method), and for the ending statement saying that the proof has finished. We suggest preparing worksheets specific to each method for an effective teaching of proof.

We also evaluated how PMTs' identified students' proof schemes in a scenario. Before the intervention, PMTs were more successful with identifying axiomatic proof schemes when compared to the transformational scheme, probably because they were more familiar with definitions and axioms. However, they could not identify students' shallow symbolic manipulation. Instead, they were convinced that the proof was valid just because it included symbols. The PMTs could not identify the symbolic proof scheme before the intervention because they did not notice the symbolic manipulations in the argument chain. In addition, the fact that some answers in the scenario start with the ones given in the hypothesis and ends with the judgment may have led PMTs to think that the proof was complete. However, the chain of arguments was incorrect. According to Weber (2010), undergraduate mathematics students who cannot recognize logical flaws consider invalid deductive arguments convincing. After the intervention, PMTs improved in noticing symbolic manipulation. They become more experienced with seeing the gaps in an argument chain which may have improved their identification of symbolic proof schemes.

Before the intervention, although the PMTs who participated in the course module identified an example-based proof scheme easily, they had difficulties identifying proof schemes of students who used infinite sets and multiple examples that might have aroused a sense of proof. After the intervention, PMTs improved in identifying the use of examples in a proof. The module, which included different cases of example-based proofs, helped them overcome their difficulties. This finding is in line with Weber's (2010) study in which undergraduate mathematics majors did not find the empirical arguments convincing.

The module also effectively overcame PMTs' difficulties in identifying transformational proof schemes by focusing on practices of generalizations using rules of logical inference and operational thought. Also, they became aware of the requirements of the axiomatic structure after the intervention. Using the analytical scheme, PMTs became aware of the limitations of the external and empirical schemes because they identified students' proof schemes.

The use of worksheets containing case studies and scenarios in which PMTs identify students' proof schemes might be an essential characteristic of a course module on proof schemes. After the intervention, PMTs developed both their proof schemes and the way they identify them in the scenario. This finding is parallel to Zazkis and Zazkis's (2016) study in which scenarios improved PMTs' images of proof and how they evaluated proofs.

With regard to the learning environment of the module, the course module included discussions of many examples of the three main and seven sub-proof schemes. Whole class discussions took place through the tasks given to PMTs. They defended their ideas in groups and individually. We consider these discussions an important feature of the module. Analysis of these discussions also indicated an improvement in PMTs' own arguments and the way they were convinced after they learnt about proof schemes. Their response to students' proofs also improved in guiding them towards analytical schemes.

Considering the potential of scenarios to improve proof schemes as implied by the findings of this study, we suggest that future studies could design scenarios focusing on proofs in different content areas. We also recommend using scenarios in transition courses in undergraduate mathematics programs as well as teacher preparation programs. However, one should consider potential limitations of assessing the knowledge of identifying proof schemes using scenarios that could not reflect the complexity of a classroom. Therefore, a possible second cycle of DBR could focus on teaching and learning situations of proof schemes in real classroom settings.

Acknowledgment

This study is part of a research project sponsored by Marmara University Scientific Research Projects Committee with project number EGT-C-DRP-120418-0202 and the first author's doctoral dissertation (Cihan, 2019) conducted under the supervision of the second author at the same university. This article is a full version of Cihan and Akkoç (2019) presented in the 11th Congress of European Research in Mathematics Education (CERME 11) which was held in Utrecht, the Netherlands.

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Appendix 1. Proof with weak induction method: A sample worksheet

<table>
<tr><td colspan="2">Theorem: Prove that $1+3+5+\cdots+(2n-1)=n^2$ for any positive integer n. Use proof with weak induction method.</td></tr>
<tr><td>Hypothesis:</td><td>Conclusion:</td></tr>
<tr><td colspan="2">Beginning sentence:</td></tr>
<tr><td colspan="2">Basic step:</td></tr>
<tr><td colspan="2">Persuasion step:</td></tr>
</table>

Inductive assumption:
Inductive step:
Closing sentence:

Appendix 2. Practitioners' guide to the module

Objectives	General objectives	Practitioners can apply this module as a stand-alone module or integrate it into different courses that can improve PMTs knowledge of proof or pedagogical content knowledge regarding proof.
	Specific objectives	• To improve the proof schemes that PMTs have. • To develop PMTs' knowledge of identifying students' proof schemes.
	Learning outcome	• PMTs will be able to use analytical proof schemes in their proofs. • PMTs will be able to identify students' proof schemes.
Content	Proof classifications	Include various proof classifications within the historical context.
	Proof schemes	Include main and sub-proof schemes: External proof schemes (authoritarian proof scheme, ritual proof scheme, symbolic proof scheme), empirical proof schemes (example-based proof scheme, perceptual proof scheme), analytical proof schemes (transformational proof scheme, axiomatic proof scheme).
	Proof methods	Include all proof methods (proof by induction, direct proof, proof by cases, proof by contradiction, proof by contrapositive, proof by counterexample, proof by exhaustion).
	Instructional materials	• Prepare presentations to give methodological information about proof classifications, proof schemes, and proof methods. • For the first learning outcome, prepare worksheets (See Appendix 1) specific to each proof method in the context of different topics in mathematics both at high school and undergraduate levels. • For the second learning outcome, prepare case studies and scenarios (See Appendix 4) specific to each proof scheme in the context of different topics in mathematics both at high school and undergraduate levels.
Learning-teaching processes	Teaching method	Use the teaching methods such as lecturing, questioning, discussion, problem-solving, case study, and scenario-based teaching where necessary.
	Roles	• The instructor is the person who imparts methodological knowledge and directs classroom practices. • During the tasks related to the first learning outcome, the PMTs are in the role of students. • During the tasks related to the second learning outcome, the PMTs are in the role of teachers.
	Learning-teaching environment and tasks	• First of all, make sure that the classroom where you will apply the module is designed appropriately for both individual and group work. In addition, create an online classroom environment where you will provide weekly tasks for the PMTs. • Introduce methodological information about proof classifications, proof schemes, and proof methods to the PMTs.

		• Use the teaching materials in a certain order (inductive proof methods before deductive proof methods, direct proof method before indirect proof methods). • Do the first proof for each proof method. • Have the next proofs done by group work first. Allow group discussions. When the discussions reach a certain maturity, move on to individual studies. • First of all, practice with the worksheets (See Appendix 1), which are the teaching materials prepared for the first learning outcome. In these studies, the PMTs are in the role of students and perform the proving tasks assigned to them. In all proof methods, when the PMTs have the analytical scheme, move on to the second outcome. • In case studies and scenarios (See Appendix 4), which are the teaching materials prepared for the second learning outcome, the PMTs are in the role of teachers and determine the proof schemes of the students in the scenarios given to them. They explain the characteristics of these schemes. They speculate on whether these schemes are convincing, whether they are proofs, and put themselves in the place of the teacher in the scenario and determine how they will guide that student and the class upon each student's answer. In all proof methods in the classroom environment, when the PMTs can identify the proof schemes of the students in the scenarios, end the module and proceed to the evaluation stage.
Assessment-evaluation	Pre-test	Before the module, apply the pre-tests to measure the PMT's prior knowledge for the two learning outcomes (See Appendix 3 and Appendix 4 as examples).
	Evaluations of the process	Observe the process during the implementation of the module and identify the factors that prevent PMTs from reaching the two learning outcomes, the difficulties experienced by them or unexpected situations. Accordingly, revise the design and other stages of the module.
	Post-test	After the implementation of the module, apply the post-tests to measure the level of PMT's achievement of the two learning outcomes (See Appendix 3 and Appendix 4 as examples). If the PMTs have achieved them, end the module. If not, proceed to the second cycle by revising the module.
	Interviews	Conduct interviews in order to evaluate the PMTs' pre- and post-knowledge for the two learning objectives in depth.

Appendix 3. The proof survey

1. Let f_m be an element of the Fibonacci sequence. If $m \geq 2$ then prove that $(f_m)^2 - f_{m+1} \cdot f_{m-1} = (-1)^{m+1}$
2. Let X be a matrix. $\forall X$ prove that XX^T is a symmetric matrix.
3. Prove that $x + \frac{1}{x} > 1$ for $\forall x \in R_{>0}$.
4. Let K, L and M be non-empty sets and $g: K \to L$ and $h: L \to M$ be two functions. Prove that if hog is a one-to-one function then g is also a one-to-one function.
5. Prove that $\forall m \in Z^+$, $m^3 - m$ is divisible by 6.
6. Prove that the proposition $\forall a \in R$, $1 + tan^2 a = sec^2 a$ is not true.
7. Prove that $\sqrt{5}$ is not a rational number.

Appendix 4. The scenario-based survey

Teacher: Is the proposition below true or false? If true, why? If false, why? Justify your answer. Proposition: "Let X, Y and Z be sets. If $X \subset Y$and $Y \subset Z$ then $X \subset Z$"

Student 1: I think it is false. We've seen a lot of rules about sets. But I don't remember this one.

Teacher: OK. We haven't seen it in a lesson. Couldn't it still be true?

Student 1: I've never heard of a rule like this. Therefore, I think it's false.

Student 2: Teacher, it is true. Because this theorem is in our maths textbook. So, it is definitely true.

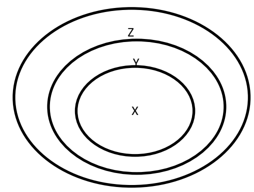
Teacher: Do you think this is enough for a justification? You didn't write or do anything about it.

Student 2: I think it's enough. Why do we need another kind of justification if it's in the textbook?

Teacher: It's very important for us to reason about the truth or falsity of a proposition.

Student 3: Teacher. May I draw a picture?

Teacher: Of course, you can.



Student 3: I think it's obvious from the picture.

Teacher: (heading towards the class) Is it enough for a proof? Just to draw a picture?

Student 4: Well, in fact. Every element in X is also an element of set Y. Every element in set Y is also an element of Z. I can express the truth of the theorem. But we should do something mathematical. But I can't do it. Theorems should be proven using mathematical statements. But it shouldn't be. Verbal expressions, as I use, convince me much more.

Teacher: How did you come to this conclusion that proofs consist of mathematical statements only?

Student 4: Because proofs I've seen so far are just like that.

Teacher: Is there anyone who could use mathematical statements?

Student 5: If $X \subset Y$ then $S(X) < S(Y)$ and if $Y \subset Z$ then $S(Y) < S(Z)$. Therefore $S(X) < S(Z)$ that is $X \subset Z$.

Teacher: If the number of elements of a set is smaller than the number of elements of another set, then does it mean that the first set is a subset of the second set?

Student 6: I think not. I think that there is an easier way.

Teacher: What is that?

Student 6: Let $X \subset Y$ and $Y \subset Z$. Let $X = \{1,2\}$, $Y = \{1,2,3\}$ and $Z = \{1,2,3,4\}$.
Since $\{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\}$ then $X \subset Z$.

Teacher: Well. Do you think that this example is sufficient?

Student 6: Now it is true. I think it is sufficient.

Teacher: (heading towards the class) Do you think that this is sufficient?

Student 7: Not that example. But it would be sufficient if we justify with a more general example.

Teacher: For example?

Student 7: $N \subset Z$ and $Z \subset R$. Therefore $N \subset R$.

Teacher: That is a more general example. But still, it is not sufficient for the generality issue of a proof.
Student 8: Teacher! If each one of us in the class finds an example to show the truth (of the proposition), then we can reach a generalization.
Teacher: When I talk about a generalization, it means it is true for all X, Y and Z. We can reach a generalisation through the rules of logical inference and operational thought. That is, using other rules we should reach a judgement from a hypothesis through operational thought. Is there anyone who could reach a generalization using what we've done in our previous lessons?
Student 9: Let $X \subset Y$ and $Y \subset Z$. Considering the rules we mentioned in our lessons, if $Y \subset Z$ then $Y \cup Z = Z$. $X \subset Y$ then $X \cup Z = Z$. If $X \cup Z = Z$ then $X \subset Z$.
Teacher: That is correct. However, it is better if we think of the modern components of proof. It is appropriate to start a proof with definitions and axioms. Is there anyone who could prove it using the definition of a subset?
Student 10: Let $X \subset Y$and $Y \subset Z$. In this case, from the definition of a subset, if $X \subset Y$ then for $\forall a \in X$ $a \in Y$. If $Y \subset Z$ then for $\forall a \in X$ $a \in Z$. Therefore, since for $\forall a \in X$ $a \in Z$ then $X \subset Z$. It's proven.
Teacher: I think it has been proven.

Appendix 5. Coding key for the scenario-based survey

Students	***Justification type***	**Main categories**	**Sub-categories**
Student 1	*He trusted what he wrote in that class*	External	Authoritarian
Student 2	*He trusted to the rule he saw in the book*	External	Authoritarian
Student 3	*She drew a simple shape*	Empirical	Perceptual
Student 4	*He focused on the form of the proof*	External	Ritual
Student 5	*He made meaningless symbol manipulation*	External	Symbolic
Student 6	*He verified with a single example of finite sets*	Empirical	Examples-based
Student 7	*He verified with a single example of infinite sets*	Empirical	Examples-based
Student 8	*He verified with multiple examples*	Empirical	Examples-based
Student 9	*He achieved generalization using logical inference and operational thought*	Analytical	Transformational
Student 10	*He completed proof in axiomatic structure, starting with the modern components of proof*	Analytical	Axiomatic