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What enabled the production of mathematical knowledge in complex analysis?

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ABSTRACT

With the objective of identifying intrinsic forms of mathematical production in complex analysis (CA), this study presents an analysis of the mathematical activity of five original works that contributed to the development of Cauchy's integral theorem. The analysis of the mathematical activity was carried out through the identification of the types of expressions used and the way they were used by the historical subjects when communicating their results, to subsequently identify transversal elements of knowledge production. The analysis was refined by the notion of confrontation, which depicts the development of mathematical knowledge through the idea of building knowledge against previous knowledge. As a result of the study we established epistemological hypothesis, which are conceived as conjectures that reveal ways in which mathematical knowledge was generated in CA.

Keywords: mathematics education, epistemological analysis, complex analysis, qualitative documentary research, epistemological hypothesis, confrontation

INTRODUCTION

According to Remmert (1991), in the 19th century Augustin-Louis Cauchy, Bernhard Riemann, and Karl Weierstrass developed the modern theory of complex functions (i.e., complex analysis [CA]) based on the concept of holomorphic function by addressing it via three different approaches. Remmert (1991) states that Cauchy's "function theory is based on his famous integral theorem and the residue concept" (p. 4), which enable Cauchy to attend to holomorphic functions in terms of integral representations. Departing from this perspective, Riemann's (1851) geometric point of view allowed him to work with holomorphic functions via mappings between domains in the complex plane. Finally, Remmert (1991) reports that Weierstrass theory of complex functions is the theory of holomorphic functions developed locally into convergent power series; a theory that relies solely on the algebraic manipulations of such power series.

Even though Remmert (1991) points out that the theory of complex function was permeated by geometric and algebraic approximations, according to Garcia and Ross (2017), the current teaching of CA is characterized by a debate in terms of the level of detail necessary to prove the results framed in this branch of mathematics. On one hand, the authors claim that the technical details that support the proofs must meet the standards that make mathematics concise and complete. On the other hand, they emphasize that attending to this type of details hinders the applicability of results in other branches of scientific knowledge.

Studies in Mathematics Education have been carried out to address this type of problems. For example, some research has incorporated digital technologies to attend to different mathematical objects inscribed in the theory of complex functions (D'Azevedo & Dos Santos, 2021; Dittman et al., 2016; Ponce, 2019), while some other studies investigate how undergraduate mathematics students and professional mathematicians work geometrically with them (Hanke, 2020; Oehrtman et al., 2019; Soto & Oehrtman, 2022; Soto-Johnson & Hancock, 2019; Soto-Johnson et al., 2016).

Distancing themselves from this type of research, where mathematical objects of CA are endowed with meaning in the contemporary scenario, there are studies in the discipline that question them in their context of origin (Cantoral et al., 1987). Studies of this nature have made it possible to identify hegemonic arguments of the mathematical school discourse that restrict the way in which students confront new mathematical ideas from CA (Cantoral & Farfán, 2004), as well as forms of validation that do not adhere to standard proofs in mathematical language, such as the use of calculators in engineering communities (Martínez, 2017).

Most of the research in Mathematics Education that has problematized knowledge in CA in its context of origin has revolved around the origins of this field of mathematics; its further development it's address in this paper with the aim of unveiling forms of knowledge production that may have been overlooked, if not forgotten, in the contemporary mathematical school discourse.

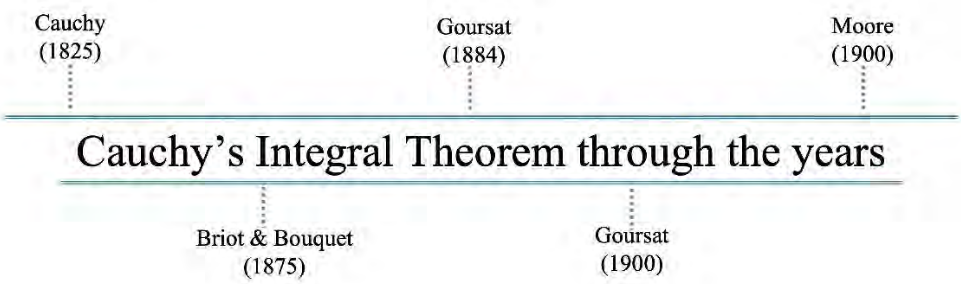

Figure 1. Development of Cauchy's integral theorem (Source: Authors' own illustration)

In this sense, the question guiding the study is how knowledge was produced in CA. To provide an answer, a result known as Cauchy's integral theorem (CIT) was problematized by means of analyzing original mathematical works that enabled its development. As stated by Nahin (1998), the theorem establishes that "the contour integral around *any* non-self-intersecting closed path C, of a function that is analytic everywhere inside and on C, is always zero" (p. 197). According to Gray (2000), this result is of vital importance to the theory of complex functions because it allows to distinguish CA from real analysis (RA) in two dimensions.

In Piña-Aguirre and Farfán (2022), different forms of mathematical production regarding CIT are conjectured through a qualitative content analysis of Cauchy's (1814) *Mémoire sur les intégrales définies*. According to the authors, in this memoire, the production of knowledge refers to the generalization of results of RA through the introduction of complex quantities conceived as the simultaneous handling between two real quantities through the symbolic-formal notation $z = x + iy$, in which the validation of the results derived from the extension process is based on a work by analogy in the field of real numbers.

Several studies (Bak & Popvassilev, 2017; Bottazzini & Gray, 2013; Neuenschwander, 1981) attribute the development of the theorem to different historical subjects in different time periods (**Figure 1**). To grasp a full picture of intrinsic forms of mathematical production in CA, and to further develop the results in Piña-Aguirre and Farfán (2022), the following original works were analyzed.

Cauchy's (1825) *Mémoire sur les intégrales définies, prises entre des limites imaginaires*; a section of the book by Briot and Bouquet (1875) titled *Théorie des fonctions elliptiques*; Goursat's (1884, 1900) works *Démonstration du théorème de Cauchy* and *Sur la définition générale des fonctions analytiques, d'après Cauchy*, and Moore's (1900) article *A simple proof of the fundamental Cauchy-Goursat theorem*.

Guided by Bråting and Pejlare (2015), we analyzed the original works bearing in mind the context of the authors' respective time periods when addressing the theorem. The analysis of the intellectual productions was further supported by the notion of confrontation (Farfán, 2012), which allows to conceive the development of mathematical knowledge by relying on and going against previous knowledge. As a result of the analysis, we propose a conjecture, which enables a description of how mathematical knowledge was produced when the authors addressed CIT; ergo, this conjecture depicts a way in which mathematical knowledge was produced in CA. In this investigation said conjecture is known as an epistemological hypothesis.

It is important to emphasize that the main objective of this article is to communicate how we have conjectured an epistemological hypothesis based on the analysis of original works. This hypothesis is conceived as a theoretical result that will be empirically tested and contrasted with the contemporary mathematical school discourse that permeates the teaching and learning of CA as our research develops.

THEORETICAL BACKGROUND: A WAY OF CONCEIVING THE PRODUCTION OF MATHEMATICAL KNOWLEDGE

In this research the study of the intellectual productions of historical subjects is conceived in what Bråting and Pejlare (2015) describe as the history approach to the interpretation of original works. The authors adopt the idea from Grattan-Guinness (2004), who categorizes the interpretation of modern mathematical notions from the history approach and the heritage approach. For Bråting and Pejlare (2015), a heritage interpretation is characterized by inserting a given modern mathematical notion–a definition, a theorem, a concept, among others–in the study of its own historical development, where the aim is usually to elucidate a path of development that culminates in the mathematical notion under study. On the other hand, a history interpretation is characterized by analyzing the development of a given mathematical notion based on the context of its time, without being influenced by its modern context.

Accepting that the production of mathematical knowledge depends on the context in which it was produced allows us to recognize as valid the processes that the historical subjects used for the communication of their results. Derived from this, the analysis of the original works is limited to an interpretation of the mathematical activity in terms of the types of expressions (iconic, symbolic, narrative, tabular, etc.) used by the historical subjects, detached of external judgments from the contemporary mathematical apparatus. Unveiling the types of expressions used by the authors and identifying the objectives of these expressions in the totality of their original works allow us to recognize how the authors did mathematics.

In addition to the recognition of the context in the production of mathematical knowledge, the proposed analysis is complemented through the notion of confrontation as a way of conceptualizing the development of mathematical knowledge. This notion is based on the idea of *building by relying on … and going against the same thing on which it was built* (Farfán, 2012). According to Farfán (2012), this idea has its origins in Bachelard (1934), who in his seminal work proposes that epistemological obstacles depict a way in which scientific knowledge is developed, recognizing that such development does not occur continuously and without errors; instead, in the production of knowledge some previous knowledge acts simultaneously as support and as an obstacle.

In this order of ideas, Brousseau (2006) presents a different meaning of epistemological obstacles framed in the field of Mathematics Education, which explains that the mistakes made by students are not the result of ignorance–lack of knowledge–but the effect of knowledge that up to that moment is successful, and that eventually prevents the acquisition of new knowledge. It should be noted that an important feature of an epistemological obstacle, in Brousseau’s (2006) sense, is that it presents itself both in history and in the mind of the current individual who wants to learn.

A distinctive feature of our analysis is that the notion of confrontation corresponds solely to the mathematical knowledge that in history hindered the development of the mathematical apparatus surrounding CIT, therefore we detach ourselves from the notion of epistemological obstacle in Brousseau’s (2006) broad sense and adhere to Bachelard’s (1934) ideas, to the extent that the notion of confrontation refers to the production of mathematical knowledge *based* on a certain knowledge that eventually *prevented* the production of new knowledge. In this sense, the notion of confrontation refers to the idea that in order to develop mathematical knowledge in CA, historical subjects had to *confront* their dominant ways of doing mathematics.

In conclusion, the epistemological study carried out does not seek a historical reconstruction of the events surrounding CIT; instead, it proposes a reconstruction that allows to conjecture how knowledge in CA was constructed by acknowledging the processes of knowledge production of the historical subjects in their time periods. Incorporating the notion of confrontation to the theoretical and methodological framework describes a way in which we conceived mathematical knowledge was developed. These ways of envisaging knowledge and its development is what enabled the structuring of an epistemological hypothesis.

METHODS: HOW TO CONJECTURE EPISTEMOLOGICAL HYPOTHESIS

The configuration of the epistemological hypothesis of this study is based on the analysis of the intellectual productions, mentioned in the introduction of this paper, by means of a *vertical analysis* and a *horizontal analysis* of them.

The vertical analysis of the original works is characterized by establishing a *dialogue* with each mathematical work by answering two questions. An answer to the question *what type of expressions (iconic, symbolic, narrative, tabular, etc.) allowed the authors to communicate the mathematical notions (definitions, theorems, concepts, mathematical objects) that structure the theorem?* Attends to value the ways in which the historical subjects did mathematics. Since each type of expression carries an intentionality of communicating the mathematical work of the authors, an answer to the question *what are these expressions used by the authors conveying?* is provided.

It is important to emphasize that the answers to the second question of the vertical analysis depend on how the expressions used by the authors are interpreted; therefore, the answers provided in this research are permeated by our mathematical knowledge. Our research aims to communicate our understanding of what the historical subjects were doing without judging it as erroneous, lacking, brilliant, cunning, or with any other adjective that positions the mathematical activity of the authors as inferior or superior to contemporary mathematics, meaning that in this research we do not compare the way in which mathematics was done with the way in which mathematics is currently done.

Carrying out a dialogue of this nature allowed the identification of two conceptualizations[1] of complex quantities (complex numbers and the complex variable) that permeated the mathematical production of the historical subjects. On one hand, the *symbolic-formal* conceptualization of complex quantities, i.e., the recognition of complex quantities through expressions of the form $x + iy$, made it possible to extend and give other meanings to results established in the field of real numbers. On the other hand, the *graphical-descriptive* conceptualization of complex quantities, understanding as such the recognition of a geometric setting to address complex quantities via points in the plane, allowed the production of mathematical knowledge concerning CIT.

Following the vertical analysis, the horizontal analysis has a twofold objective. To present some reasons that led the authors to address the theorem in their intellectual productions, the first objective was to provide an answer to the question *What was the authors’ purpose to write and publish their original mathematical works?* Secondly, because the development of the theorem depends on two conceptualizations of complex quantities, we recapitulate relevant elements of each conceptualization.

Interested in the transition between both conceptualizations, where the graphical-descriptive conceptualization was used for the first time by Briot and Bouquet half a century after Cauchy worked from the symbolic-formal conceptualization, questions arise as to how and why the change occurred. Because this research relies on the premise that in the production of mathematical knowledge the historical subjects faced confrontations, it is necessary to describe how to look for them in the analysis of their original works, and for this Cid (2015) was taken as a reference. The author, based on Gascón (1993), conceives that in the historical development of the mathematical apparatus the notion of epistemological obstacle can be inquired in the origins of a bifurcation, understanding as such a modification in the mathematical activity predominantly used until some specific moment in history.

[1] The word conceptualization is being used to highlight the plurality of points of view that made possible the production of mathematical knowledge regarding the theorem.

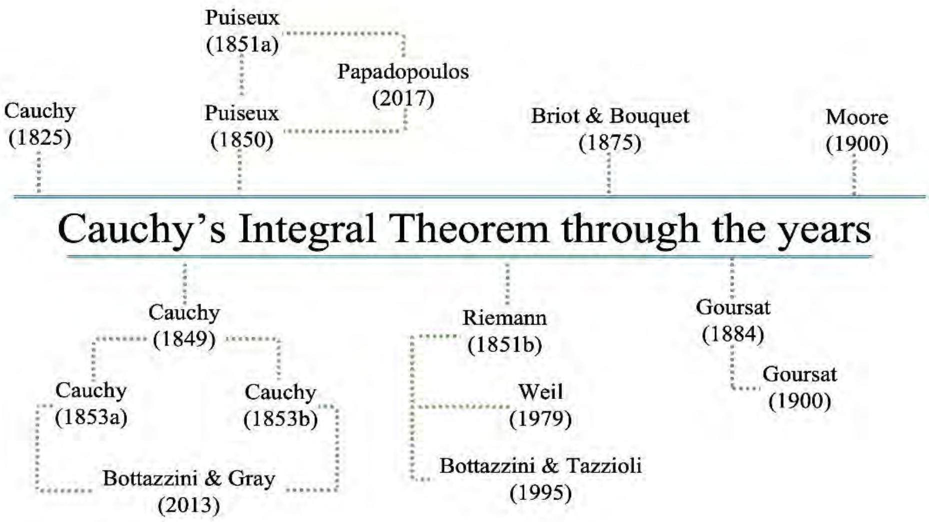

Figure 2. Primary and secondary sources that support the epistemological hypothesis (Source: Authors' own illustration)

From her work it is possible to find these moments of rupture–in the nature of the mathematical activity– in terms of the change of techniques being used by the historical subjects, as well as in the field of problems that developed once the obstacle was faced.

The above allows us to propose the search for confrontations through the following two aspects.

1. By identifying the *changes in the techniques* used by the historical subjects through the sudden introduction of different forms of mathematical activity that allow them to explain their mathematical work. For such task it is important to be aware of the ways in which the authors communicated their results, with the aim of showing if there is a change in them. In this order of ideas, Cid (2015) remarks that it is important to pay attention to their omissions and hesitations, which can be expressed by the excess of explanations, since these new ways of mathematical work still lack the status of being recognized by their communities; hence, the historical subjects sought to explain them in some way in the mathematical work of their time.
2. By revealing the *fields of problems that emerged* when the new forms of mathematical activity were in the process of acquiring an accepted status in their communities. These new field of problems are characterized by the use of the new forms of mathematical activity that were hindered by the predominant view that prevented their acceptance; in principle, they are fields that had not been previously addressed.

The primary and secondary sources that contribute to evidence a confrontation in the development of mathematical knowledge regarding CIT are presented in **Figure 2**, an extension of **Figure 1**, which showcases the totality of works that contributed to the configuration of the epistemological hypothesis.

RESULTS: THE ANALYSIS OF THE ORIGINAL WORKS IMBUED WITH THE NOTION OF CONFRONTATION

Vertical Analysis of the Original Works

To communicate our findings, under the headings *On [author's name] mathematical activity (year of publication of the original work)*, a summary of the mathematical production of each original work is presented in our own mathematical terms, which are as close as possible to the terminology used by the historical subjects. We recommend consulting the original intellectual productions for a detailed description of the authors' mathematical activity.

Under the headings *The how and what for of [author's name] mathematical activity (year(s) of publication of the work(s))*, the types of expressions used by the authors are presented to answer the inquiry 'what type of expressions allowed the authors to communicate the mathematical notions that structure the theorem?' Under the same headings their respective objectives–highlighted in *italics*–are configured as possible answers to the question 'what are these expressions used by the authors conveying?'

We start the communication of our results with the conclusions derived from the analysis of the first three sections of Cauchy's (1825) memoire in which CIT is addressed.

On Cauchy's mathematical activity (1825)

One of Cauchy's (1825) main objective is to propose ways to compute integrals comprised between complex numbers. In his memoire, Cauchy conceptualizes complex quantities as the simultaneous handling of two real quantities by means of symbolic-formal expressions of the form $x + yi$. By relying on this conceptualization, Cauchy extends his definition of integral comprised between real numbers–presented in his 1823 *Calcul infinitésimal*–outside the field of validity where it was configured.

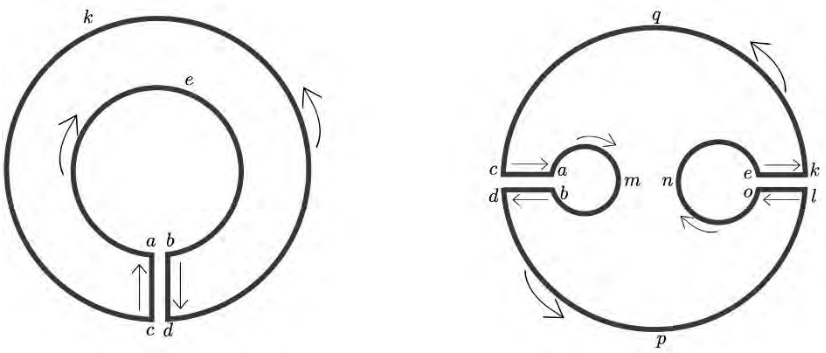

Figure 3. Transformations of complex contours (Source: Authors' own illustration)

By establishing a path of integration $\gamma(t) = \varphi(t) + \sqrt{-1}\chi(t)$, Cauchy shows that the integral comprised between complex numbers can be computed through an integral comprised between real numbers by means of the following equality.

$$\int_{x_0+y_0\sqrt{-1}}^{X_0+Y_0\sqrt{-1}} f(z)\,dz = \int_{t_0}^{T} [\varphi'(t) + \sqrt{-1}\chi'(t)] f[\varphi(t) + \sqrt{-1}\chi(t)]dt$$

After assuming that the value of the previous integral is $A + iB$ and by using a Taylor series, Cauchy proves that this value is independent of the curve (restricted to rectangular domains of the form $x_o \leq x \leq X; y_o \leq y \leq Y$) joining the initial and final points of integration $x_0 + y_0 i$ and $X + Yi$. Proving that the integral comprised between complex numbers is path independent helps to establish CIT, since the integral around closed curves can be conceptualized as the sum of integrals connecting the same initial and final points of two paths of integration traversed in opposite directions.

The how and what for of Cauchy's mathematical activity (1825)

Cauchy's (1825) mathematical activity consists of justifying his statements by means of the simultaneous use of symbolic and narrative expressions. Symbolic expressions are understood as equations and functional relations, while narrative expressions refer to the description of the author's symbolic procedure insofar as the narrative expressions describe the passage from one equation to the other.

The simultaneous use of symbolic and narrative expressions allows the author to *generalize* results from RA through the introduction of complex quantities, seen as the *simultaneous manipulation* of two real quantities. The generalization of the integral caught between two real quantities seeks to *fix an integration path* that connects the initial and final points of the complex integral. The substitution of this path, in the variable of integration of the complex function, *reduces the complex integral to an integral of a complex function comprised between real numbers*, i.e., the meaning that Cauchy (1825) attributes to a complex integral is defined from what he already knows from the real case. The justification that allows to prove the uniqueness of the value of the integral relies on the fact that *a small variation in the path of integration does not alter the result of the integral*.

It should be noted that Cauchy (1825) does not relate his symbolic mathematical activity to a graphical setting. CIT is addressed through a Taylor series expansion, which allows to prove that the change in the value of the integral after introducing a small variation in the path of integration is zero.

On Briot and Bouquet's mathematical activity (1875)

50 years after Cauchy's (1825) publication, in their book titled *Théorie des fonctions elliptiques* (Briot & Bouquet, 1875), Briot and Bouquet recapitulate the progress that had been made in the theory of complex functions based on Cauchy's ideas.

In the chapter *Propriétés fondamentales des intégrales définies*, the authors prove CIT by conceptualizing complex quantities as points in the plane. Prior to the establishment of the theorem, they begin by describing ways in which complex contours can be transformed into simple contours, exemplifying the transformations through narrative descriptions and with the display of figures (**Figure 3**).

To prove the theorem, the integral of a holomorphic function f is considered around a simple contour C. Contour C is scaled by a factor α, caught between zero and one, to describe the behavior of two integrals of f over two concentric simple contours. Comparing the values of both integrals shows that the integral of f over any simple contour A, where the function is holomorphic, depends only on the scaling factor α. The functional dependence of the integral value of the function f over the curve A, and the value α, is denoted by ψ(α).

Briot and Bouquet (1875) proceed to show that $\psi'(\alpha) = 0$, which implies that ψ(α) is constant. This entails that if the scaling factor α is reduced to a point, the value of the integral is zero; and since ψ is constant for every value of α, then the value of the integral of f is zero despite the integration curve, thus, proving CIT.

The how and what for of Briot and Bouquet's mathematical activity (1875)

Briot and Bouquet (1875) use narrative, symbolic and iconic expressions to communicate their results. Symbolic expressions are understood as functional relations and equations in which narrative expressions describe the passage from one equation to another, and in turn–when accompanied by figures–serve as a description of what the figure describes. The introduction of figures

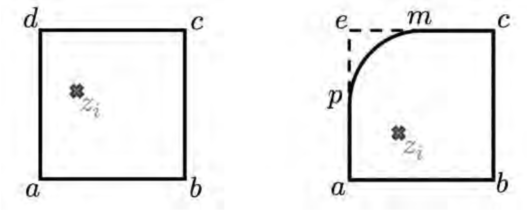

Figure 4. Types of curves C_i used by Goursat (Source: Authors' own illustration)

through narrative expressions, or figures per se, acquires the status of iconic expressions because rather than representing what is stipulated in the symbolic setting, they are conceived as a particular way of working with complex quantities. Conceptualizing complex quantities by means of points also enables the *manipulation of curves in the plane*, thus, permitting an *alternative description* of what is elicited in the symbolic setting.

On Goursat's mathematical activity (1884, 1900)

Goursat's (1884) work proves CIT by means of an argument that requires subdividing region R bounded by the curve C of integration. Subdividing the R region yields the computation of the integral f over C by integrating f over curves C_i (**Figure 4**) that make up the entire curve C. Goursat (1884) shows that the modulus of the integrals over the curves C_i can always be made as small as desired; and hence, the modulus of the integral of f over the curve C can be made as small as one wishes. That is to say:

$$\forall \varepsilon > 0, \left| \int_C f \, dz \right| < \varepsilon$$

Thus, obtaining the result concerning the theorem.

Due to a critique by Pringsheim (Gray, 2000) on the argument of the subdivision used by Goursat (1884) to prove the theorem, Goursat's (1900) work revolves around proving a lemma that allows the justification of the subdivision used in his publication of 1884. Namely, in his article from 1900 CIT is not proved per se, but the proof of the lemma helps towards its establishment. In short, the lemma states that the region bounded by the curve C can always be squared in such a way that the length of the sides of the squares can be made smaller than an arbitrary positive prefixed quantity. To prove it, Goursat (1900) proceeds by *reductio ad absurdum*, which enables the construction of nested regions through an infinite iterative process.

Goursat's (1900) work is inscribed in Cauchy's mathematical framework concerning CA insofar as the author concludes that "it follows that from CAUCHY's point of view it is sufficient, to construct the theory of analytic functions, to assume the continuity of f(z) and the existence of its derivative"[2] (Goursat, 1900, p. 16).

The how and what for of Goursat's mathematical activity (1884, 1900)

In the style of Briot and Bouquet (1875), Goursat's (1884, 1900) arguments are supported by means of symbolic, narrative, and iconic expressions. The narrative expressions make possible the *subdivision of the region delimited by the integration curve C*, as well as the *construction of an infinite succession of regions* in which each new region is contained in the previous one. Since the narrative expressions describe the *construction of curves and points in the plane*, these are related to the use of iconic expressions because they allow a conceptualization of complex quantities that distances itself as a means of representation of what is stipulated in the symbolic setting. In this sense, the construction of curves in the plane *orients the symbolic work* insofar as the figures are the basis for posing upper bounds for the modules of the integrals under study.

On Moore's mathematical activity (1900)

Unlike Goursat's (1884) work, where the author proves CIT via a direct process, Moore's (1900) proof is through an indirect process. Moore begins by assuming that

$$\left| \int_C f \, dz \right| = \eta_0 > 0$$

then, the author considers a square $\mathrm{S_0}$ that encompasses the integration curve C, and through an infinite iterative process subdivides region $\mathrm{R_0}$ bounded by C. This infinite iterative process leads to the construction of nested squared regions $\mathrm{S_m}$, which contain sections of the curve C (denoted by $\mathrm{C_m}$) and its interior. The author proofs that there is a point that lies in each of the regions constructed by the infinite iterative process, which allows him to bound the value of the modulus of the integrals over the curves $\mathrm{C_m}$, and this leads him to bound with an arbitrary and positive quantity ε the value of η_0, thus, obtaining a contradiction of the initial assumption.

The how and what for of Moore's mathematical activity (1900)

Like in the works of the other authors who consider complex quantities as points on the plane, Moore's (1900) work is mediated by narrative, symbolic and iconic expressions. The narrative expressions allow the description of an *infinite iterative process* in which a succession of nested squares are constructed. This succession of squares divides the integration curve C and region R,

[2] In the original work it reads as follows: On voit par là qu'en se plaçant au point de vue de CAUCHY il suffit, pour édifier la théorie des fonctions analytiques, de supposer la continuité de $f(z)$ et l'existence de la dérivée.

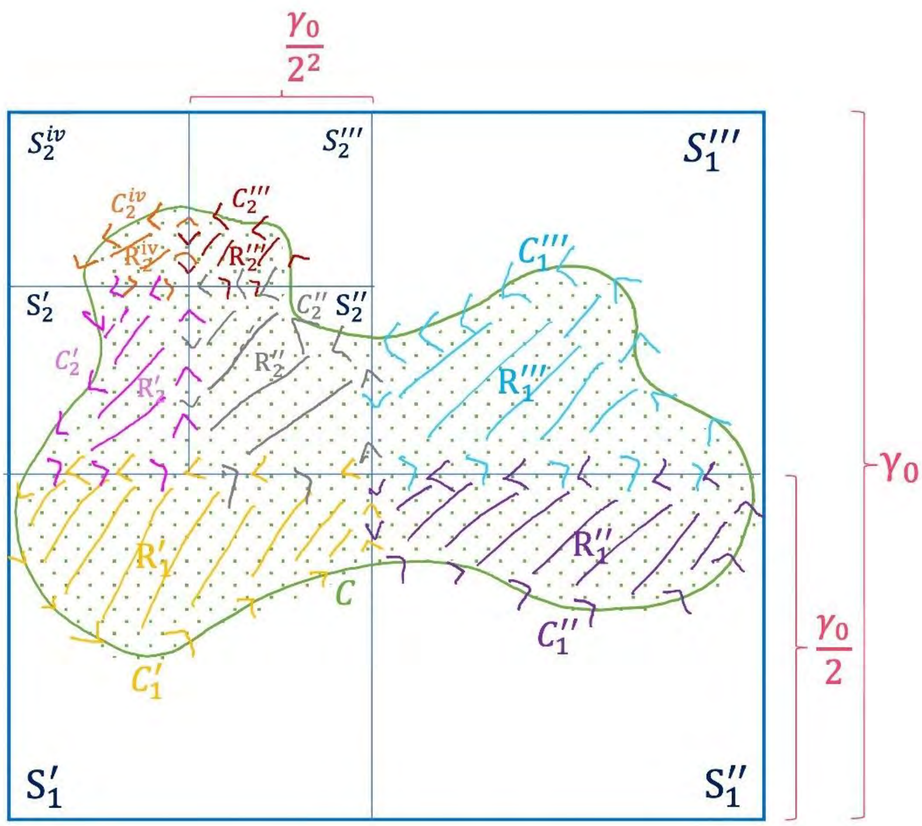

Figure 5. Subdivision of the curve C and its interior R via an infinite iterative process (Source: Authors' own illustration)

which it delimits. We claim that picturing such regions and integration curves is interpreted as the use of iconic expressions since these figures are the ones that *guide and structure* the work with symbolic expressions, such as equations and functional relations.

Figure 5 depicts our interpretation of Moore's (1900) narrative expressions that refer to the subdivision of curve C and its interior R through an infinite iterative process.

Horizontal Analysis of the Original Works

In the Methods section of this paper it is claimed that the development of the theorem depends on two conceptualizations of complex quantities; this result is sustained by identifying how the authors use narrative expressions to communicate their results. As shown in the sections titled *The how and what for of [author's name] mathematical activity (year(s) of publication of the work(s))*, the use of symbolic expressions and iconic expressions depend on how the authors use narrative expressions. For instance, in Cauchy's (1825) memoire the narrative expressions come into play to describe how to transit between equations that are obtained by the extension of results in RA. Whereas in the original works from Briot and Bouquet (1875), Goursat (1884), Moore (1900), aside from the fact that the narrative expressions describe the passage from one equation to the other, they communicate how the complex variable z refers to points and curves in the plane.

The reasons why the authors attended to the theorem are presented in the sections titled *On [author's name] mathematical activity (year(s) of publication of the original work(s)).* We claim that all the authors were looking for internal consistency of the mathematical apparatus, understanding as such that what enabled the production of mathematical knowledge in each original work was the development of the mathematical apparatus without the need to attend to other epistemological domains beyond mathematics. This search for internal consistency is the proffered answer to the question *what was the authors' purpose to write and publish their original mathematical works?* A detail description of how the authors attended to the internal consistency of the mathematical apparatus of their time periods is presented below.

In Cauchy's (1825) memoire, the symbolic-formal conceptualization of complex quantities made it possible to provide internal consistency to the mathematical apparatus by generalizing results from RA in which the validation of the results obtained from the generalization process is based on a work by analogy in the field of real numbers. Thus, Cauchy's (1825) method of extension and justification does not depend on other epistemological domains outside of mathematics since it is intrinsic to the mathematical apparatus of his time. These results are consistent with those reported by Piña-Aguirre and Farfán (2022). **Table 1** summarizes significant elements of this conceptualization; furthermore, it presents ways in which these elements were addressed by the author.

When conceptualizing complex quantities as points in the plane, the search for internal consistency of the mathematical apparatus refers to the recollection of results of the mathematical apparatus as well as to the justification and reduction of hypotheses of the theorem. Working with this conceptualization requires the recognition that the incorporation of points and curves in the plane does not depend on their counterpart in the algebraic setting; figures are accepted as having an epistemic

Table 1. Symbolic-formal conceptualization of complex quantities ($z = x + iy$)

Elements of this conceptualization	How was the element addressed?
Search for internal consistency of the mathematical apparatus	Proposal of alternative methods to compute the value of integrals (Cauchy, 1814) as well as to give meaning to integrals comprised between complex numbers (Cauchy, 1825).
Generalization	Equations expressed in the real variable are extended through the introduction of complex quantities of the form $x + iy$.
Work by analogy	Justification of results in CA based on known results in RA.

Table 2. Graphical-descriptive conceptualization of complex quantities (points in a plane)

Elements of this conceptualization	How was the element addressed?
Iconic work that regulates symbolic work	Description and/or explicit presentation of points and curves in the plane that precede and guide the algebraic work.
Search for internal consistency of the mathematical apparatus	Compilation of results of the mathematical apparatus (Briot & Bouquet, 1875), justification of hypotheses of the theorem (Goursat, 1900) and reduction of hypotheses (Moore, 1900).
Reductio ad absurdum	Method through which infinite iterative processes allows the construction of nested regions in the plane (Goursat, 1900; Moore, 1900).

value that is far from serving as means of representation of equations and functional relations. It is important to note that the 1900 Goursat and Moore's intellectual productions do not present figures per se; however, the narrative expressions used by the authors refer to the construction of nested regions by means of an infinite iterative process. **Table 2** presents significant elements of this conceptualization; in addition, it presents ways in which these elements were addressed by the authors.

On the Changes in the Techniques and Fields of Problems That Emerged in the Development of Cauchy's Integral Theorem

What has been unveiled so far is the following. In Cauchy's (1825) memoire, the author extends the definition of definite integral comprise between real numbers by working with complex quantities in their symbolic-formal conceptualization $x + iy$. 50 years later, Briot and Bouquet (1875) structured a plethora of results in the field of mathematics that took Cauchy's ideas about complex quantities as a starting point; however, these authors attend to them from the graphical-descriptive conceptualization. Thanks to this latter conceptualization, in 1900 Goursat justifies one of the hypotheses of the theorem given in his article published in 1884. Furthermore, Moore in 1900 presents an alternative proof that does not require the lemma proven by Goursat in that same year, which corresponds to a reduction of hypotheses for the establishment of the theorem.

The authors who conceptualize complex quantities as points in the plane prove the theorem with the use of geometric formulations (figures presented by the authors, or those that can be constructed from narrative expressions) that precede and support the mathematical work with algebraic formulations (equations, functional relations, literals that represent constants or variables). This type of mathematical activity differs from the types of mathematical work that allowed Cauchy to establish the theorem.

Hereinafter, we are concerned with the transition between both conceptualizations, which is problematized through the notion of confrontation, made evident from *the changes in the techniques* used in the transition between both conceptualizations and the *fields of problems that emerged* from the incorporation of the graphical-descriptive conceptualization.

According to Smithies (1997), in Cauchy's memoire of 1825 the author "suddenly introduces, without any prior warning […] some geometrical language in the description of his results. Before this time he had carefully avoided the use of any such language" (p. 98), which is closely related to what we characterize as the use of geometric formulations. To exemplify this use of geometric language, in the following excerpt (translated from the original work) the narrative expressions that allude to the use of geometric formulations are highlighted in italics.

> 9. If, among the equations (7), we eliminate t, we will obtain another one of the form
>
> $$(65)\ F(x, y) = 0$$
>
> And, assuming that x and y represent *rectangular coordinates*, equation (65) will represent *a curve drawn in the* x, y *plane* between the two points (x_0, y_0), (X, Y). Let us further contemplate that the functions $x = \varphi(t), y = \chi(t)$ […] increase or decrease from $t = t_0$ to $t = T$. Under this hypothesis, the value of y, taken from equation (65) and determined as a function of x, will generally increase from $x = x_0$ to $x = X$, and the curve (65) *will be included in a rectangle formed by four straight lines* parallel to the axes, namely, those having the equations:
>
> $$(66)\ \begin{cases} x = x_0, x = X, \\ y = y_0, y = Y. \end{cases}$$
>
> Thus, each particular form of the function $F(x, y)$ will provide *a particular curve* and a corresponding value of the integral (4). It should even be noted that this function can change its nature by varying x, and that consequently the curve $F(x, y) = 0$ can be transformed into *a system of straight lines or curves*, starting from the point (x_0, y_0) and ending at the point (X, Y). Then each of the lines in question corresponds to an integral […] in which *the extreme values of* x *and* y *represent the coordinates of the two ends of this line* […] In general, *we conceive that one of the lines drawn* between point (x_0, y_0) and point (X, Y) extends from point (ε_0, η_0) to point (ε, η). *If this line* has the equation:

$$(74)\ y = \psi(x)$$

the corresponding integral can be presented in the form:

$$(75)\ \int_{\varepsilon_0}^{\varepsilon} \left[1 + \psi'(x)\sqrt{-1}\right] f\left[x + \psi(x)\sqrt{-1}\right] dx.$$

If, however, *this line* has the equation:

$$(76)\ x = \psi(y),$$

the integral (75) must be replaced by the following:

$(77)\ \int_{\eta_0}^{\eta} \left[\psi'(y) + \sqrt{-1}\right] f\left[\psi(y) + y\sqrt{-1}\right] dy$ (Cauchy, 1825, p. 55-57).

The purpose of presenting this translation of Cauchy's original work is to show evidence of *a sudden change in the technique* used by the author to communicate his results. It should be noted that at the beginning of his memoire–as well as in his previous mathematical works concerning CA–Cauchy relies on the symbolic-formal conceptualization of complex quantities, but all of a sudden he communicates his results using narrative expressions that allude to geometric formulations that are always accompanied by algebraic formulations that describe them symbolically, i.e., the geometric formulations are not detached from their counterpart in the algebraic setting.

To present evidence of the *fields of problems that emerged* from the recognition of the graphical-descriptive conceptualization of complex quantities, we offer some results from Bottazzini and Gray (2013), Bottazzini and Tazzioli (1995), Papadopoulos (2017), and Weil (1979) concerning several original works by Cauchy, Puiseux, and Riemann.

According to Bottazzini and Gray (2013), Cauchy recognizes a geometric setting for complex quantities in his memoire *Sur les quantités géométriques* (Cauchy, 1849). In this memoire, the author defines geometric quantities as radius vectors and explains how to do operations–add, subtract, multiply and divide–with them. Cauchy bases his rules for operations through generalization, insofar as the notion of geometric quantity includes the notion of algebraic quantity as a special case, which in turn is an extension of the notion of arithmetic quantity. Bottazzini and Gray (2013) show that this geometric setting allowed Cauchy to establish results that he had previously obtained from the algebraic setting, as is the case of the definition of function in his *Mémoire sur les fonctions des quantités géométriques* (Cauchy, 1853a), and the definition of a continuous function in his work *Mémoire sur les fonctions continues de quantités algébriques ou géométriques* (Cauchy, 1853b).

Drawing on Cauchy's ideas about complex integrals, Puiseux's mathematical works *Recherches sur les fonctions algébriques* (Puiseux, 1850) and *Nouvelles recherches sur les fonctions algébriques* (Puiseux, 1851) attend to the problem of uniformization of algebraic functions (Papadopoulos, 2017). According to Papadopoulos (2017), the problem of uniformization of algebraic functions refers to finding methods for converting algebraic functions u, of the complex variable z, defined by polynomial equations of the form $f(u,z) = 0$. In simple terms, the problem of uniformization lies in finding methods to transform multivalued functions of a complex variable into single valued ones. Papadopoulos (2017) shows evidence that Puiseux (1850) proposes a solution by considering that the value of the single valued functions can be obtained via the path of integration that variable z traverses in the complex plane. In Puiseux (1850) words:

> Let us now conceive that z varies in an arbitrary manner starting from the value c, and reaches another value k […] Thus, there will be one, which is first equal to b, which will pass by infinitely small steps to a determinate value h, which it will attain for $z = k$ (Puiseux, 1850, quoted by Papadopoulos, 2017, p. 213).

It is noteworthy that the above quotation describes the movement of variable z by means of geometric formulations and not by the explicit use of algebraic formulations. In the same publication, Papadopoulos (2017) relates a solution to the problem of uniformization via the Riemann surface, a mathematical object that was first introduced by Riemann in his doctoral dissertation (Riemann, 1851), which had a tremendous influence on the development of the theory of complex functions.

As shown by Bottazzini and Tazzioli (1995), the general mathematical work of Bernhard Riemann attends to the search of a general mathematical formulation of the laws underlying all natural phenomena. For instance, Riemann's mathematical work related to two-dimensional potential theory relies on the theory of complex functions via the Riemann surface, a domain that opened "the way to the study of complex functions independently of their analytical expressions" (Riemann, 1851, p. 35, quoted by Bottazzini & Tazzioli, 1995, p. 7), which points to the fact that at the heart of the mathematization of potential theory lies a conception of the space of complex variables by means of geometric formulations.

The mathematical work that proliferated with the aid of geometric formulations, detached from their analytical conceptualizations, encompasses the production of Riemann's mathematical knowledge regarding *Analysis Situs* (now known as topology). Weil (1979) suggests that Riemann made use of geometric formulations to communicate his ideas in this area of mathematics. As evidence he presents a couple of letters written in 1896 by Betti, described by Bottazzini and Gray (2013) as Riemann's most faithful friend. In said letters the author's mathematical work is imbued with narrative expressions that refer to the description of surfaces as well as their cuts and deformations. A section of one of these letters is presented as an example of the geometric formulations (highlighted in italics) conceived by the author. The complete letter can be found in Weil (1979):

> I have newly talked with Riemann about the connectivity of spaces and have formed an accurate idea of the matter […] *The interior of an ellipsoid* is an SC space. *The space bounded by two concentric spheres* is not SC, since *a third concentric sphere*, comprised between those two, is not the complete *boundary of part of the sphere*, while it is closed and wholly contained in it. In that space, however*, every closed line* may be regarded as *the entire boundary of a surface contained in the space*. This space can be reduced to an SC space by a 1-cut, *viz., by a line going from the outer to the inner sphere*. *After the cut is made*, *its points* have to be regarded as being *outside the space*, and the *concentric spheres between the two given ones are no more contained in the space*, since they *intersect the cut* (p. 94).

The previous results have a two-fold objective. In the first place it allows us to propose a confrontation in the production of mathematical knowledge in CA via the following reconstruction:

The production of mathematical knowledge regarding Cauchy's mathematical activity required the author to build knowledge *relying* on a purely symbolic setting in which the symbolic-formal conceptualization of complex quantities allowed the generalization of results from RA. The sudden change of technique employed by Cauchy in his 1825 memoire suggest that he required a graphical setting–that accompanied his use of algebraic formulations–to further develop his results in CA. Nevertheless, incorporating a graphical setting detached from algebraic formulations indicates that the authors who solely used geometric formulations had to *go against* the predominant algebraic setting (in which the symbolic-formal conceptualization played a significant role) for the development of the theory of complex functions, the mathematization of natural phenomena, and the development of other fields within mathematics. Therefore, we claim that the symbolic-formal conceptualization of complex quantities represented a confrontation in the production of mathematical knowledge in CA, insofar as it required to be complemented by means of a graphical-descriptive conceptualization that recognizes the use of geometric formulations independent of their algebraic formulations.

In second place, these results, along with the vertical and horizontal analysis of the intellectual productions, enable us to categorize different ways in which geometric formulations and algebraic formulations come into play for the communication and establishment of mathematical knowledge in CA. These different ways of mathematical work are described below under three categories.

The category *geometric formulations as means of representation* refers to the way in which Cauchy's symbolic expressions evoke the construction of curves and points in the plane as a means of an alternative representation of what is described in the symbolic setting; however, in the production of mathematical knowledge, this type of graphic representations can be disregarded because the algebraic formulations subsist without the need of other means of representation.

The category *geometric formulations as means of construction* refers to Cauchy's change of technique to communicate his results in which geometric formulations come into play to obtain results in the symbolic setting. It is worthwhile mentioning that this category is characterized by the recognition of points and curves in the plane as indispensable to communicate results (hence, the reason for Cauchy to suddenly introduce them even though they had never previously appeared as a means of justification), which cannot subsist without the use of their analogous representation in the symbolic setting.

Finally, the category *geometric formulations with epistemic value* refers to the use of geometric formulations to produce mathematical knowledge. This category is characterized by the development of mathematical activity through geometric formulations that do not require symbolic representations to be accepted as valid; the geometric formulations themselves enable the production of mathematical knowledge.

It is important to highlight that the category *geometric formulations as means of representation* was identified in the process of understanding, via geometric representations, the purely symbolic mathematical procedure of Cauchy's original work. On the other hand, the category *geometric formulations as means of construction* was configured by recognizing the change of technique used by Cauchy to communicate his results in his 1825 memoire. Finally, the category *geometric formulations with epistemic value* was configured by the recognition of the fields of problems undertaken by Riemann for the development of CA, by the development of *analysis situs*, by the mathematization of natural phenomena, as well as by the mathematical works by Briot and Bouquet (1875), Goursat (1884), and Moore (1900).

DISCUSSION: THE USE OF GEOMETRIC FORMULATIONS IN MATHEMATICS EDUCATION RESEARCH IN COMPLEX ANALYSIS

In Mathematics Education, the studies that have problematized the production of mathematical knowledge in the origins of CA (Cantoral & Farfán, 2004; Cantoral et al., 1987; Martínez, 2017) seek to provide an algebraic and arithmetic meaning to the logarithm of negative numbers. Although, this type of research has raised awareness of the production of mathematical knowledge as a fully human action, these studies do not incorporate any of the categories herein proposed to deal with geometric formulations to produce mathematical knowledge in the theory of complex functions.

Additionally, in the discipline there are studies such as those by Hanke (2020) and Soto-Johnson et al. (2016) that aim to understand how professional mathematicians perceive and communicate different mathematical tools and concepts of CA to their peers. This kind of research has revealed that attributing geometric meanings to purely algebraic expressions is not a trivial task. However, studies such as that of Soto-Johnson (2014) recognize that the arithmetic operations of complex numbers can be endowed with a geometric representation as follows: the addition and subtraction of complex numbers corresponds to translations of the complex plane; the multiplication of complex numbers corresponds to a dilation and rotation of the complex plane; the division of complex numbers can be interpreted as a reflection and dilation of the complex plane, and finally, the conjugation of complex numbers corresponds to a reflection on the real axis in the complex plane.

The geometric interpretation proposed by Soto-Johnson (2014) is what we recognize as the epistemological basis that allowed other research studies to provide a geometric meaning to different mathematical objects of CA. For example, Soto-Johnson and Troup (2014) report that university students working with these geometric representations were able to "better understand [...] algebraic equations, their components, and the processes used to justify equations algebraically" (p. 124). With the aid of this geometric representation, Dittman et al. (2016) sought to develop the geometric knowledge of pre-service secondary mathematics teachers by exploring the geometric behavior of complex functions. In Soto-Johnson and Hancock (2019) and Troup et al. (2017), this epistemological basis provided a geometric meaning to the concept of derivative of a complex function through the notion of Amplitwist (Needham, 1997), a notion that describes how a small circle dilates and rotates around a point. Finally, this epistemological basis allowed Oehrtman et al. (2019) and Soto-Johnson and Oehrtman (2022) to address the concept of complex integral.

It should be noted that Soto-Johnson's (2014) proposal recognizes what we have called *geometric formulations as means of representation* and *geometric formulations as means of construction* insofar as the geometric representation of the arithmetic of complex numbers endows symbolic expressions (equations and algebraic functions) with geometric meaning, in which every representation in the complex plane (for example the concept of derivative and the integral concept) is accompanied by algebraic formulations by incorporating the degree of rotation and dilation that the complex plane undergoes. Although, our analysis has revealed that both categories favored the production of mathematical knowledge in CA, these forms of geometric work were not the only ones used by Briot and Bouquet (1875), Goursat (1884), Moore (1900), and Riemann (1851).

The category *geometric formulations with epistemic value* refers to the introduction of geometric formulations that precede, guide and structure the symbolic mathematical work without involving algebraic formulations, which means that the geometric formulations constitute as evidence for mathematical justification. This category is consistent, to some extent, with the type of mathematical work described by De Toffoli and Giardino (2015) regarding low dimensional topology; however, we believe that studies in Mathematics Education, regarding CA, need to further develop this approach.

From the above, we recognize that our epistemological hypothesis can be the starting point for the configuration of didactic interventions that recognize the different ways of mathematical knowledge production that our hypothesis envisages. In this way, our epistemological hypothesis can serve as an epistemological model (Gascón, 2014), in the sense that it serves as a reference system for constructing didactic phenomena and for formulating and addressing didactic problems related to them.

CONCLUSIONS: AN EPISTEMOLOGICAL HYPOTHESIS OF COMPLEX ANALYSIS

The vertical and horizontal analysis of the original works showed that the production of mathematical knowledge, regarding CIT, requires the symbolic-formal and the graphical-descriptive conceptualization of complex quantities; in which both conceptualizations are used to provide internal consistency to the mathematical apparatus.

Interested in the bifurcation identified through the *changes of techniques* used in the transition between both conceptualizations as well as the *fields of problems that emerged* from the recognition of the graphical-descriptive conceptualization of complex quantities, we were lead to the conclusion that the conceptualization of complex quantities via the symbolic-formal conceptualization represents a confrontation in the construction of mathematical knowledge should they not be complemented by the graphical-descriptive conceptualization from their epistemic (*geometric formulations with epistemic value*) and constructive (*geometric formulations as means of construction*) aspects, which distance themselves from considering geometric formulations only as a means of representation (*geometric formulations as means of representation*).

The conjugation of both visions leads to the following epistemological hypothesis:

> The production of mathematical knowledge regarding CIT was enabled by the search for internal consistency of the mathematical apparatus via two conceptualizations of complex quantities. The introduction of complex quantities from its symbolic-formal conceptualization attends to the search for internal consistency of the mathematical apparatus through the generalization of results from RA, as well as by the validation of the results derived from the generalization process via a work by analogy in the field of real numbers. However, this conceptualization of complex quantities presents itself as a confrontation in the construction of mathematical knowledge if it is not complemented by a graphical-descriptive conceptualization, which provides internal consistency of the mathematical apparatus through the justification and reduction of hypotheses of the theorem. This second conceptualization is characterized by conceiving complex quantities as points in a plane, which allows the use of geometric formulations from their epistemic and constructive aspects that distance themselves from considering geometric formulations as ways of conviction (via representation) of what is sustained by algebraic formulations.

The epistemological hypothesis implies that the symbolic-formal conceptualization of complex quantities constitutes a confrontation in the production of mathematical knowledge related to CIT. However, it is important to recognize that this conceptualization is the one that makes it possible to generalize results from RA, as well as the one that favors intrinsic forms of validation of the mathematical apparatus. Therefore, it is not being claimed that the symbolic-formal conceptualization must be eradicated for an accurate production of mathematical knowledge in CA; rather we recognize that it has its limitations insofar as it did not allow the complete development of the theory of complex functions.

As further work, it is expected that the epistemological hypothesis will be enriched via empirical data, obtained through a learning situation that does not overlook the conjecture. The enrichment of the epistemological hypothesis will allow us to provide elements to subvert, by incorporating innate forms of mathematical production in the theory of complex functions, the current mathematical school discourse regarding CA.

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