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Research Article

Active learning strategies for an effective mathematics teaching and learning

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ABSTRACT

Learning is an active enterprise, where three dimensions stand out, cognitive, social, and physical, and, in addition, not all students learn in the same way. Grounded on these ideas, this article reports a study that aims to understand and characterize the performance of pre-service teachers when experiencing active learning strategies during their mathematics classes. The participants were 48 future teachers of primary education (3-12 years old) that experienced paper folding, a gallery walk, and a math trail as active learning strategies. We followed a qualitative methodology, collecting data though observations, written productions, and photographic records. The analysis involved a qualitative and inductive approach resorting to content analysis. The results of the study show that the participants valued these experiences, due to their potential in the development of a diversity of mathematical concepts and abilities, and throughout them showed traits of cognitive, social, and physical engagement. Active learning provided collaborative work and mathematical communication enabling the emergence of different strategies to solve the proposed tasks. The participants were able to reflect and be aware of their ideas, mistakes, and difficulties, as well as of others, in a non-threatening environment, where movement was highlighted for not being a popular dimension in mathematics classes. Although more research is needed, the results encourage the use of active learning strategies as a valuable approach to teaching and learning.

Keywords: active learning strategies, visualization, problem solving, challenging tasks, teacher education

INTRODUCTION

Mathematics learning is largely shaped by the teacher and the tasks proposed, as well as the strategies used to convey them. Thus, the teacher must contribute to the development of students' mathematical understanding, creating opportunities for them to be challenged and engaged in high-level thinking, through a thoughtful choice of the strategies and tasks to use. This implies working inside and outside the classroom, in an active learning environment (Prince, 2004; Vale & Barbosa, 2020a), that addresses different types of thinking displayed by the students, engaging them in challenging tasks, desirably with multiple solutions, privileging hands-on activities, that motivate them to learn mathematics and work collaboratively. This paper presents a study carried out with pre-service teachers of primary education (3-12 years) that aims to understand and characterize their performance when experiencing active learning strategies in the teaching and learning of mathematics. Based on this problem, the following two research questions were posed:

RQ1. Which aspects of the three dimensions of active learning (intellectual, social and physical) can we identify in the participants?

RQ2. How can we characterize the participants' reactions to the strategies used?

THEORETICAL FRAMEWORK

Active Learning

Organizations such as the National Council of Teachers of Mathematics (NCTM, 2014) have long advocated the use of teaching strategies that require cognitive engagement in the construction of new knowledge, highlighting in particular the importance of problem solving in mathematics. However, in addition to strategies of intellectual nature, those involving social and physical activity are also important to help promote active learning, which is usually defined as an instructional method that involves learners in the learning process (Prince, 2004), requiring students to think about what they are doing. These principles contrast with what happens in a so-called traditional classroom, where the teacher uses tasks to introduce a new concept, or procedure associated with a certain concept, and then the students' practice, using similar tasks, passively acquiring the information exposed by the teacher (Vale & Barbosa, 2020a). Active learning has its roots in the socio-constructivist learning theory (Vygotsky, 1996), and translates into a classroom practice that engages students in activities such as talking, listening, reading, writing, discussing, reflecting, conjecturing, arguing about the contents, through problem solving, in small groups, involving experimentation or other activities. These are situations that require the students to apply what they have learned, invoking higher-order thinking abilities (Braun et al., 2017; Meyers & Jones, 1993; Stein & Smith, 1998).

There is a common agreement that the most lasting learning outcomes emerge from a direct interaction between intellectual, social and physical environments (Edwards, 2015; Edwards et al., 2014; Nesin, 2012), which leads us to emphasize importance of these three strands in the context of active learning (**Figure 1**).

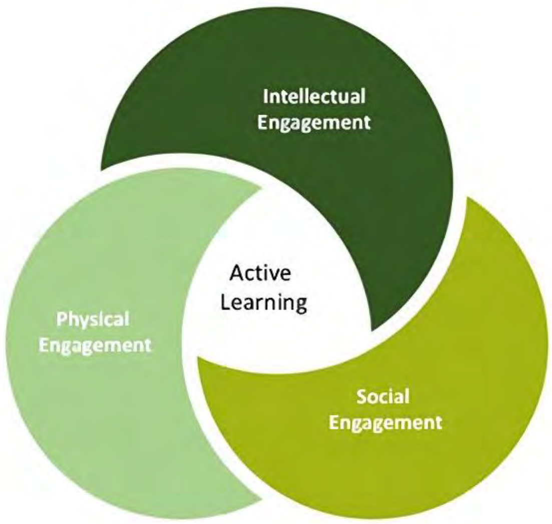

Figure 1. Dimensions of active learning (Vale & Barbosa, 2020b, adapted from Edwards, 2015)

Active learning strategies that simultaneously integrate intellectual, social and physical engagement are most likely to provide a more pleasant experience for students, while working on important contents that they need to learn (Edwards, 2015). In this sense, the adoption of an active learning approach tends to increase students' performance and interest in STEM disciplines (Lucke et al., 2017; Prince, 2004; Vale & Barbosa, 2020a), raising their expectations/motivations regarding learning (of mathematics). To better understand principles underlying the intellectual, social and physical dimensions we will discuss each strand individually.

It is part of the teacher's role to ensure that students become intellectually engaged with the contents to address, showing intrinsic motivation to establish relationships, develop conceptual understanding and use critical thinking, which allow them to go beyond memorization or the acquisition of a more limited comprehension (Edwards, 2015). In the particular case of mathematics, it is through problem-solving tasks, which foster reasoning and communication, that the teacher is able to challenge students, helping them establish connections and reach a deeper understanding (Edwards, 2015; Prince, 2004; Vale & Barbosa, 2020a). Beyond these principles we have to consider that different individuals usually have different preferences regarding the way they think and communicate, that is why Gardner's (1983) work about multiple intelligences is so important in the educational context. It is common, specifically in more traditional teaching practices, to expose students to the same contents in the same way, disregarding their needs and individuality. To have intellectually engaged students and promote understanding, teachers must be aware of different learning styles, beyond the logical-verbal (Borromeo-Ferri, 2012). According to Krutetskii (1976), there are three learning styles related to reasoning, which emerge when students solve problems:

(1) analytic, those who prefer the use of non-visual solution methods, choosing logical-verbal methods, involving algebraic, numeric and verbal representations,

(2) visual (or geometric), those who prefer the use of visual solution methods, choosing visual-pictorial schemes, involving graphic representations (figures, diagrams, and images), and

(3) harmonic (or integrated), those who have no specific preference for logical-verbal or visual-pictorial representations, combining both styles.

This implies the use of problem solving tasks that meet the different ways of thinking displayed by the students, building on the knowledge they bring to the classroom (Nesin, 2012; Stein & Smith, 1998). In this sense, multiple-solution tasks comply with this requirement, encouraging students' flexibility and creativity (Leikin, 2016; Vale & Barbosa, 2015), by allowing them to pursue diversified solution strategies, of analytical, visual or mixed nature, which also increases students' chances for success. The contact with different approaches/solution methods, increases the students' repertoire of strategies, making them more aware of other forms of thinking and more able to critically choose the best path to the solution.

Students' success in mathematics does not depend only on their intellectual engagement. We must not neglect the role of discourse, more generally, mathematical communication during the orchestration of the interactions and discussions that occur in the classroom among the different actors, teacher and student(s), and students themselves. In an active learning context, one of the good practices in the mathematics classroom should also consider the social engagement of students, encouraging collaboration, either through small group activities or whole class discussions (NCTM, 2014; Nesin, 2012). This type of collaboration facilitates the sharing and development of mathematical concepts and meanings however the teacher is responsible for the development of a favorable context, making students feel safe and confident to express their ideas and take risks encouraging them to share their actions with their peers and with the teacher.

Reinforcing these principles, we can also argue that an individual learns more effectively as a result of an interaction with more knowledgeable peers or the teacher (Vygotsky, 1996). The contact with the activity of others (e.g., strategies, discourse, and ideas) creates opportunities for reflection, that does not happen so frequently through traditional approaches, which optimizes and ultimately benefits students' learning.

According to Hannaford (2005), thinking and learning are not just in the mind, quite the contrary, the body plays a decisive role in the entire intellectual/cognitive process, since the very beginning of our lives. Intelligence, which is generally thought of as a purely analytical ability, measured and evaluated in terms of IQ, depends more on the body than we normally realize. Students who move, and particularly move around the classroom, can learn more effectively than those who attend typically sedentary classrooms, regardless of the activity. On the other hand, creating opportunities for students to move during lessons allows them to be more involved, improving attention levels and hence their comprehension (Edwards, 2015). Several studies have highlighted the existence of positive correlations between movement and learning (Jensen, 2011; Ratey, 2008). In this context, the field of neuroscience has made a fundamental contribution to the understanding of the relationship between the body and the brain, and it can be assumed, in general terms, that the body is an external extension of the brain. It is through the body that each individual experiences the world around them and it is up to the brain to give meaning to these experiences. Through movement, more blood and oxygen are sent to the brain, and since we use our brain to learn, this increase in oxygen and blood flow increases brain activity, which can improve learning. In this state, students are more likely to discover new ideas and establish connections with existing ones (Jensen, 2011).

The integration of movement in educational contexts must be rethought, going beyond recess or physical education classes, being incorporated into the classroom itself (Webster et al., 2015), in any curricular area.

Considering that movement is an external stimulus, when students are involved in a certain task, we have an indicator that their attention is focused on what they are supposed to learn. If the teacher chooses a traditional approach, giving students a passive role, then the continuous learning process is likely to be ignored, as the mind tends to wander. In cases where attention is strictly mental, the task becomes difficult to sustain, especially for long periods of time, because the nervous and muscular systems are inactive (Shoval, 2011). On the other hand, by incorporating tasks that entail movement, students are essentially forced to engage in the learning process, unless they choose not to, a fact that makes engagement noticeable (Shoval, 2011). Gardner (1983) also advocates action and activity, as "the brain learns best and retains more when the organism is actively involved in exploring physical sites and materials, asking questions to which it really craves answers" (p. 82). The use of kinesthetic teaching and learning strategies, which include hands-on tasks or walking around the surrounding space (inside/outside the classroom) are, therefore, more effective in improving students' attention and engagement levels, especially younger students, breaking with the routine of sitting in the classroom just listening to the teacher. So, movement can assume many different forms in educational settings, in terms of active learning, but overall students are expected to construct, modify and integrate ideas, interacting with the physical world, materials and their peers (Nesin, 2012).

Many educational organizations (Center for Curriculum Redesign [CCR], 2015; Ontario Ministry of Education [OME], 2016; World Economic Forum [WEF], 2016) agree that learning requires an active and reflective endeavor, focusing not only on the acquisition of content knowledge, but also on the development of transversal skills, in line with the demands of 21st century problems. In this scope students should master four skills, generically emphasized as essential to tackle the challenges of a rapid and dynamic world development, designated as the 4Cs: critical thinking, including problem solving–make informed decisions or judgements, to achieve the best solution; communication–understand and share ideas, thoughts and solutions with others; collaboration–provide opportunities for working together to make decisions in favor of a common goal; creativity–provide opportunities for new and efficient approaches. Active learning calls for the use of the 4Cs through its core dimensions. The shift from a traditional approach to one in which the students assume an active role in the learning process implies a series of actions that demand: solving challenging problems, thinking about possible strategies to reach a solution; social interactions and sharing ideas/thoughts with others, establishing different forms of communication; working with their peers; connecting ideas in a flexible way. Active learning environments are grounded on the experiences and interactions of the learners, interconnecting intellectual, social and physical engagement (Nesin, 2012; Prince, 2004), and are especially suited for the application of the 4Cs, though the orchestration of productive discussions; the creation of opportunities to solve problems, communicate and reason, be creative, think critically, make decisions and understand nuclear ideas (Vale & Barbosa, 2020a).

Active Learning Strategies in the Mathematics Classroom

In the previous section, we generally discussed ideas about the meaning of active learning and the relevance of implementing this approach in current educational settings. It is clear that active learning encompasses a diversity of student-centered instructional practices, but what does it mean for mathematics education presently? In the case of mathematics, it is fundamental that the teacher uses different strategies that allow students to understand the aesthetics and usefulness of this discipline, simultaneously aiming at the development of higher-order abilities. Students must be faced with challenging tasks (Leikin, 2016; Vale & Barbosa, 2015; Vale & Barbosa, 2021) that trigger positive attitudes towards mathematics (Hannula, 2001). It should also be clear that the way tasks are implemented can also have an impact on students' reactions. An effective mathematics teaching must include active learning strategies, aiming to involve students intellectually, socially and physically, in solving tasks that lead them to think, make decisions, solve problems and be critical. In an active learning context, activities are learner's initiated; based on the natural curiosity of the students and on real-life related or meaningful issues, that trigger mathematics learning, which makes tasks and mathematical concepts socially and culturally appropriate (National Association for the Education of Young Children [NAEYC], 2009).

Depending on the strategies used, students can be intellectually active at the same time that they are socially and/or physically active. It is possible for an instructional practice to fit into one dimension (Edwards, 2015), however, in this study, we will privilege strategies that involve and articulate the three dimensions mentioned in **Figure 1**. This work focuses on the implementation of different active learning strategies in the context of mathematics education, related to the use of hands-on activities or movement around the surrounding space (inside the classroom, in a limited space, or the outdoors), namely: paper folding (PF), gallery walk (GW), and math trail (MT).

The main principle of hands-on activity-based strategies is learning by doing or manipulating. A popular trend of hands-on learning in mathematics is the use of manipulatives. The concepts are visualized, and students more easily get insights about mathematical ideas (Clements & McMillen, 1996; NCTM, 2000). The teacher can use different resources for this purpose, as long as they are tangible and allow manipulation to trigger reflection and discussion, also promoting a minds-on perspective. Students can rely on existing structured or daily-use materials or solve tasks that imply the construction of a model, falling within the scope of hands-made activities. In this paper we highlight PF tasks, as a useful teaching tool that enables a diversity of skills (e.g., communication, problem solving, connections, creativity, and proof), with specific impact on the understanding of geometry, particularly spatial orientation (Blanco et al., 2019; Boakes, 2009), but also number concepts (e.g., fractions). The actions involved in folding a paper allow its transformation into different shapes, two or three-dimensional, opening the opportunity to discover relationships of different nature. It allows students to perceive abstract mathematical concepts more easily in a concrete manner, by manipulating a sheet of paper, and improve their mathematical ideas and thoughts (DeYoung, 2009). In this way, PF can be a hands-on, dynamic, creative and challenging strategy to approach several concepts in mathematics, facilitating visualization and problem solving (Vale et al., 2020). It is undoubtedly an active learning strategy, as it involves students intellectually, in the sense of the cognitive challenges it provides, and physically, because it requires auditory abilities and visual stimuli, and it is through these actions that it also involves spatial skills, promoting the construction of meanings and ideas (Blanco et al., 2019; Vale et al., 2020).

GW is a teaching and learning strategy, adapted from Fosnot and Jacob (2010), also compatible with the principles underlying active learning, that resembles the perspective used by artists when they expose their work in a gallery (Francek, 2006; Vale & Barbosa, 2019, 2020a). The GW has a series of procedures that can be divided into six steps (Vale & Barbosa, 2019):

1. Problem solving–students, in groups, solve the proposed task.
2. Construction of posters–students discuss among themselves how to display their solutions in a poster.
3. Presentation and observation of posters–posters are fixed in the walls of the classroom or in another space.
4. Analysis and elaboration of comments–each student, individually, goes through the different posters to analyze the presented solutions and, after evaluating them, writes, in post-its, personal comments, doubts, questions, possible errors, etc. While students discuss their colleagues' solutions, the teacher circulates around the classroom, evaluating the observations and discussions.
5. Group discussion–after this round, the students take their own poster and analyze the contents of the post-its, making a small report.
6. Collective discussion–with all the posters fixed again in the wall, the groups orally present the solutions and respond to the comments of their peers.

This moment is adequate for the teacher to highlight some of the expressed ideas, making connections between the different approaches, making syntheses, clarifying doubts and errors and provoking reflection. The last step is also an excellent opportunity for the teacher to give feedback to the content of each poster, grounded on the work displayed, commented and discussed.

The dynamics of a GW takes students out of their seats and actively engages them with the mathematical ideas of their colleagues, being especially attractive to the more kinesthetic learners (Fosnot & Jacob, 2010; Vale & Barbosa, 2020a). It also provides the opportunity for students to contact with different ideas and/or solutions, getting written and oral feedback, which can improve their learning.

Kenderov et al. (2009) highlight that many students do not get the opportunity to fully appreciate mathematics, being deprived of significant learning experiences, fact that can contribute to the construction of a negative image of mathematics. According to Kenderov et al. (2009), outdoors education is as a means to complement the work developed inside the classroom. Among various possible experiences and strategies, we find MTs: a sequence of tasks along a pre-planned route (with beginning and end), composed of a set of stops in which students solve mathematical tasks in the environment that surrounds us (Barbosa & Vale, 2018, adapted from Cross, 1997). Richardson (2004) proposes a series of steps to prepare a mathematical trail:

1. The first is the selection of the site. It can be anywhere as long as it is rich in mathematics. The teacher must observe the elements of the chosen site and look for patterns, shapes, objects to measure, count, or represent.
2. Take pictures of each location/object to later use in the task design.
3. Once the photographs have been selected, a map must be created on which the locations of the tasks must be identified, to verify the distribution of the stops and the distance of the trail.
4. Create the different tasks and the instructions to transition from one to the next. These tasks must have different levels of cognitive demand and involve different mathematical contents. The tasks must be solved applying the knowledge previously acquired in the classroom.
5. Whenever possible, it is interesting to establish connections between mathematics and other curricular areas through the tasks created.

Richardson (2004) also points out the importance of asking good questions, questions that arouse students' curiosity and motivates them to have a closer look at the surrounding environment in order to achieve a successful and meaningful solution.

The participants of a MT are challenged to solve mathematical tasks, in a real context, applying acquired knowledge, developing skills such as problem solving, communication, and the establishment of connections. Due to the atmosphere of discovery implied, MTs are stimulating learning situations, that make tasks more meaningful and challenging for the participants, contributing to the improvement of mathematical knowledge, in an active learning perspective. This is a strategy that facilitates group work, through collaboration in the multiple requirements of the trail (writing, measuring, and discussing); involves movement, walking around a certain site, observing specific features of elements in the surroundings; and engages students in problem solving, thinking about adequate strategies to reach a solution (Barbosa & Vale, 2021; Vale et al., 2019).

Recently, an app was created with the purpose of experiencing MTs through mobile devices, MathCityMap. It aims to combine the dimensions of outdoors mathematics and mobile learning. The user accesses the trail guide with the tasks to solve, through a smartphone or tablet, that are located with the help of the GPS functionality and introduces the solution directly in the device receiving feedback on the correctness of the answer. This digital technology has proven to be useful for outdoors mathematics supporting teachers and students in the teaching and learning process, including the affective-motivational aspects (Barbosa & Vale, 2021; Cahyono & Ludwig, 2019; Ludwig & Jablonski, 2021). The use of this app enhances the principles of active learning already underlying MT itself, increasing students' engagement, which led us to use it in this study.

Looking back at the strategies described, although having different features and highlighting more profusely one or two of the dimensions of active learning, they require students to be intellectually, socially, and physically engaged, with the purpose of empowering them to discover information on their own using a variety of resources, to deal with new information until it makes sense, and to create new ideas using the information they have learned (Edwards, 2015). In the scope of active learning and teacher education we were not able to find studies that use this symbiose approach combining the three chosen strategies, which led us to this exploratory study.

METHODOLOGY

Research Design and Participants

This paper presents an exploratory study carried out with pre-service teachers of primary education that aims to understand and characterize their performance in solving some mathematical tasks when they experience active learning strategies during their didactics of mathematics classes. We opted for the use of PF, GW, and MT, three strategies that these participants did not know and designed a set of tasks that fitted those strategies in the scope of the math contents of the course. In accordance with the main goal of this study we adopted a qualitative and interpretative approach (Erickson, 1993). This choice of paradigm was sustained by the fact that the main goal was to understand the perspective and reactions of the participants to a particular situation.

We conducted this research with 48 pre-service teachers that were attending the second semester of the 3rd year of an undergraduate course of elementary education, with a three-year duration (six semesters), that precedes the frequency of a master's course qualifying them for the teaching of pre-school and primary education (3-12 years old students). This course has different subjects connected to the areas of didactics, general education, content knowledge, and practice in formal and non-formal educational contexts. This group of future teachers were enrolled in a curricular unit of didactics of mathematics, that was the setting of the study, and included teaching modules related to traditional content strands (numbers and operations, geometry, algebra, and data analysis and probability). These contents were explored emphasizing the role of tasks, instructional strategies and resources that meet the current needs of mathematics teaching and learning for the elementary levels, with a special focus on mathematical connections and problem solving. Particularly, this curricular unit highlighted the principles of active learning, giving the student teachers the opportunity to contact with and experience some strategies that they could henceforward use in their own practice, namely the use of PF, a GW, and an MT. We were also concerned that the tasks explored, through these strategies, contributed to the development of the future teachers' 4Cs skills (CCR, 2015; OME, 2016; WEF, 2016). The two researchers were the teachers of this unit course, which enabled the implementation of the study as well as the contact with the participants, that worked in groups of three/four elements during all classes.

Data Collection and Analysis

Data was collected during the classes in a holistic and interpretative manner, through participant observation (reactions to the experiences); notes of the participants' reactions and interactions during the different experiences; written productions (solutions of the tasks proposed to better understand their reasoning); written reports (perceptions about the experiences); and photographic records. The researchers accompanied the pre-service teachers during the experiences, observing in loco the work developed, during three classes of three hours each, recording free-flowing notes, focusing on the future teachers' reactions, interactions, conversations, discussions and interpretations, having the problem in mind. Additionally to the notes, the written productions and reports, we highlight the importance of the photographs, a visual method used in a qualitative research (Barbour, 2014) that evokes complementary information. The support of the photos during the transcriptions helped us to consider relevant non-verbal information in the descriptive analysis such as non-verbal actions, including the participants' interactions and gestures with the materials and their peers. Beyond acting a complementary method of data collection, the photos also served as a means to visually illustrate specific moments to the reader.

Data was analyzed using an inductive approach, resorting to content analysis, according to the ideas of Miles and Huberman (1994). After repeatedly reading the information collected with the different methods, data was categorized to systematize the information regarding the performance and perceptions/reactions evidenced by the participants, crossed with the evidence emerged from the observation. In this process we reached categories grounded on the problem, and the guiding questions, framed by the theoretical framework and the data collected: intellectual/cognitive dimension (task performance); social dimension (mathematical communication, collective discussions); and physical dimension (actions/movement). To ensure the quality of the study we used an iterative process of writing, reading, reviewing, rewriting and consulting the collected data, that led to a refinement of the information. The triangulation associated to the used of multiple methods and data sources also contribute for the quality of the study allowing a more comprehensive understanding of the problem.

RESULTS & DISCUSSION: EXPERIENCES IN PRE-SERVICE TEACHER TRAINING

The pre-service teachers solved several tasks during this curricular unit, related to the mathematics content strands. In this paper we will present three tasks used in each of the three active learning strategies implemented.

Paper Folding

The example presented in **Figure 2** is a task proposed to the participants, to solve in small groups, that had two main goals: find the optimal solution and use spatial abilities to transform a 2D figure into a 3D figure.

Use a square sheet of paper with 10 cm side length, to draw the net of a cube with the maximum volume. The image presents an example of a cube net. Does this net make it possible to build the cube with the largest volume?

Construct the cube by folding that net.

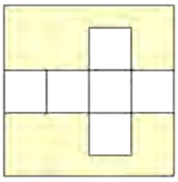

Figure 2. Paper folding cube task (Vale et al., 2020)

The student teachers began to discuss how to dispose the net into the paper. The most common solutions were solutions 1, 2, and 3, respectively presented in **Figure 3**. The authors of solution 1 accepted the given suggestion, considering the disposition of the net as the solution, and they only arranged the net using the sides of the square paper. They divided the square paper into 16 squares with a side length of 2.50 cm and got the volume, 15.625 cm^3. In solution 2, they only disposed the squares of the net of the cube into different orientations and got the same volume as in solution 1. As for solution 3, other participants took advantage of the diagonal of the square. They first tried the previous solutions but found that they could have a net with a greater length side if they disposed the "cross" net in a way that the diagonal of the square paper was a symmetry axis of the square and the net. A difficulty emerged as they determined the side length of one of the squares of the net by trial and error, finding a near value of 2.82 cm and a volume of 22.430 cm^3. This last solution (solution 3) was the best approach to the correct answer.

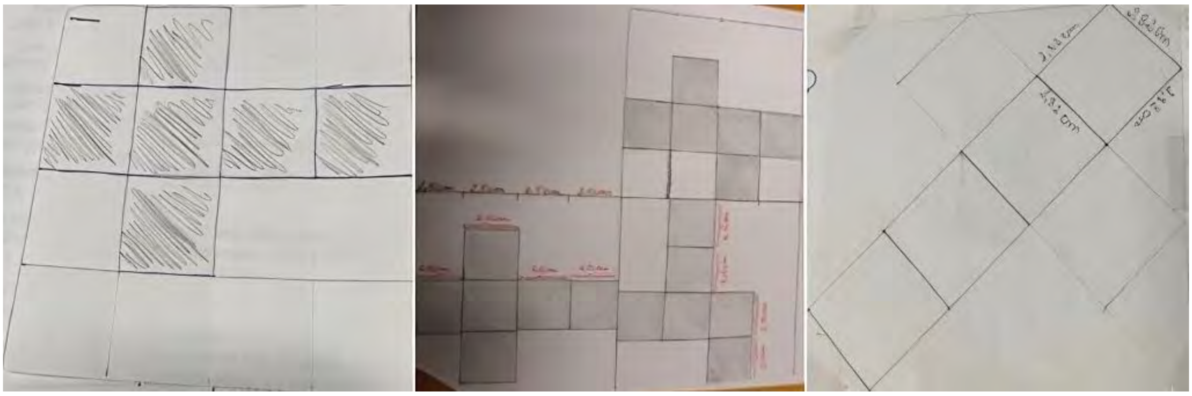

Figure 3. Incorrect solutions presented by the students (Source: Authors)

However, after many net trials, calculations and group discussions, the correct answer was discovered. Through an analytical approach, they noticed that their nets did not have the largest volume. **Figure 4** shows one of the analytical productions where the participants compared the volume of two nets. The first solution (solution 1) was an already discovered net (this time with adequate calculations) and the other (solution 2) represents a correct approach. This task involves the use of geometric and spatial reasoning as well as orientation. There are many ways to draw the nets on the square sheet, but only one complies with the given conditions. This problem has some complexity, mainly for the elementary levels that still use basic mathematical contents/procedures. Despite having made different trial designs of possible traditional nets, none of them led the students to the expected outcome, because they did not achieve the largest volume. Most of these pre-service teachers attacked the problem, exploring the most obvious and traditional nets in which the segments representing the edges in the net were parallel to the sides of the square. The solution for this problem requires the solver to have a sound knowledge of mathematics, and also a visual intuition of the different nets of a cube. Furthermore, the exploration required divergent thinking, thinking outside of the box, in order to imagine a whole different net from the traditional ones.

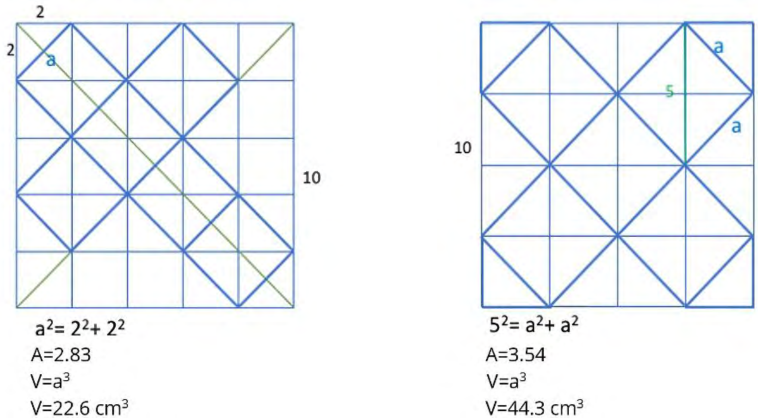

Figure 4. Analytical solutions presented by the students (Source: Authors)

The last part of the problem was to construct the cube by folding the net with the maximum volume. So, after having the correct net, the biggest challenge was to fold the square paper to get the cube, without cutting. We realized that these student teachers had no familiarity with PF activities, which made more difficult the manipulation of the paper. So, to help them, it was suggested that they looked at the paper and imagined how to fold it, starting from the center square, to get the cube. Then, the participants used trial and error, doing the folds on the square paper, to come up with more positive results (**Figure 5**).

Figure 5. Attempts to get the cube (Source: Authors)

Many attempts were made to get the cube, but not all the participants achieved the solution autonomously. Clearly, visualization was a needed ability to solve this task, as well as PF abilities. A possible short sequence of the folds is presented in **Figure 6**.

Figure 6. Correct solution and paper folding cube (Source: Authors)

Despite the level of complexity of this task, particularly with regard to its application to elementary levels, the participants felt challenged and engaged. Although none of the groups achieved the optimal solution for this task, they were committed and willing to overcome the obstacles, managing to evolve during the solution process itself. The emerging discussions allowed a better understanding of some more obscure aspects and of the importance of the use of different approaches, analytical and visual, to solve a mathematical situation, as well as of the concepts involved. In addition to the cognitive perspective, the PF strategy also promoted affective and behavioral engagement, expressed by the participants' interest and persistence. The movement associated to the actions performed on the paper, the hands-on, that includes also minds-on activity, allowed them to reflect on the existing relationships and/or experiment different possibilities until they reached a solution. These results were very similar to those obtained in a previous study with other participants mainly regarding the engagement and the productive discussions (Vale et al., 2020).

Gallery Walk

The GW focused on a multiple solution task (**Figure 7**), which the future teachers were asked to solve in two different ways, working through the six phases described previously.

The figure represents a square with 4 cm^2 of area, where P and Q are midpoints of two opposite sides. Given the conditions of the figure, what is the area of the shaded part?
Solve the task in two different ways.

Figure 7. Shaded square problem (Vale & Barbosa, 2020a)

The participants started by solving the task individually, and when they came to a solution or had doubts, they got involved in discussions in each of their groups, looking at the different solution processes, trying to identify which could be the most effective. Working collaboratively facilitated the exchange of ideas and decision making. After solving the task, the participants decided, as a group, the best way to structure the poster with the different solutions, the way of presenting them or the clarity of the text. Then, the walk began going through the different posters. Afterwards, with all the posters exhibited around the classroom, all the participants analyzed and commented with post-its, followed by the group discussion about the relevance of the feedback received through those post-its. Ultimately, the posters were discussed with the whole class, where each group summed up their work, answered questions, clarified doubts and comments of their peers, and the teacher highlighted the main ideas of the discussion. **Figure 8** illustrates the actions of the students throughout the different phases of GW.

Figure 8. Students & teachers executing the different phases of GW (Source: Authors)

Analyzing the solutions, and as expected, there were no noteworthy difficulties. Different approaches were used: only visual; only analytical; or visual solutions complemented with analytical ones. Most of the groups presented analytical solutions using formulas to find the area of the square and the triangle. Fundamentally, they chose more traditional approaches with which they felt more comfortable, as showed in **Figure 9**.

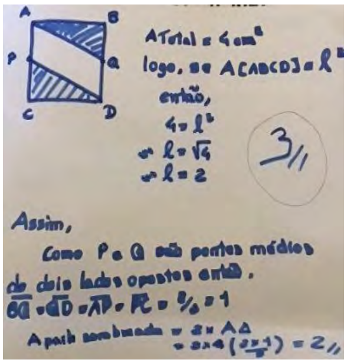

Figure 9. Analytical solution (Source: Authors)

However, two of the groups presented two visual solutions using the idea of part-whole (**Figure 10**).

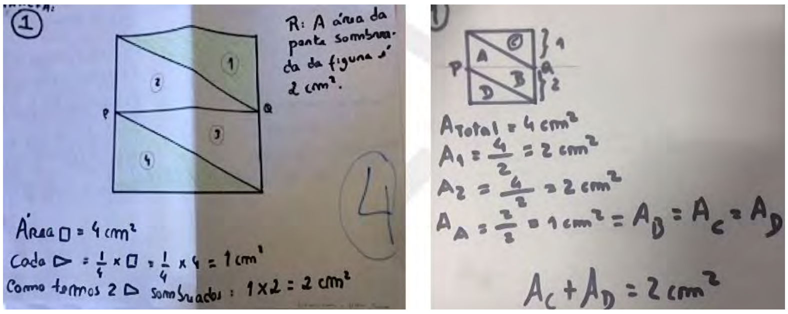

Figure 10. Visual solutions (Source: Authors)

These visual solutions were surprising to the rest of the colleagues who found them simpler and more elegant. In general, the future teachers showed some resistance to the use of visual solutions, because they had more difficulty in looking for a visual approach and felt more comfortable with the use of analytical methods. The comments on the posters exhibiting these solutions can be summarized in two main ideas: "it's a simple strategy and easy to understand" [in visual solution] or "you did not need to use a square root" [in a non-visual, analytical solution]; which reveals the recognition of the potential of strategies of visual nature.

In general, the participants reacted positively to this experience, showing interest and motivation. They valued the contribution of the GW to their own learning, as future teachers, but also to the mathematics learning of their own pupils. This evidence was identified in the comments made by the participants throughout their experience and in the report they wrote.

Here are some of the expressed ideas: "a more enriching approach than the usual strategy of solving a task individually and only one or two students showing how they thought"; "helps to share ideas"; "gives the more timid and less confident students the opportunity to participate without fear of reprisals"; "allows you to observe other strategies and other reasoning paths and question colleagues about aspects that were not so clear, helping to develop critical thinking"; "it is a good strategy to motivate students to participate in solving mathematical tasks, improving their performance"; "makes students realize that it is sometimes difficult to get ideas down on paper clearly"; "break with monotonous routines"; "promotes free movement".

The opportunity to contact and be engaged with other problem solving strategies, promoted by the GW dynamics, contributed to the extension of the repertoire of strategies. Besides the intellectual domain, the GW allowed us to identify other types of engagement on the part of the participants, like social (in the interactions in small groups and whole group) and physical (free movement in the classroom), which confirms the potential of a GW as an active learning strategy.

Math Trail

The last example describes an MT experienced by the future teachers in the surrounding environment of their institution, using MathCityMap app. MT had 10 tasks that address a diversity of thematic contents of elementary mathematics (e.g., patterns, rational numbers, measurement, probabilities, and symmetries). **Figure 11** illustrates the dynamics of MT implementation where the participants solved the proposed tasks.

Figure 11. Student teachers during a math trail using MathCityMap app (Source: Authors)

Among the various tasks that integrated MT implemented, we illustrate this experience of the students through one of the tasks (**Figure 12**)–*Combatentes da Grande Guerra Avenue*–that addresses as main topics patterns and measurement.

Figure 12. *Combatentes da Grande Guerra Avenue* task (Source: Authors)

> *Combatentes da Grande Guerra Avenue* is the main artery of the city of Viana do Castelo and was built between 1917 and 1920. It starts at the Railroad Station and ends at Liberty Square. Observe the lighting of the Avenue. It has lamps of two types, according to a repetition pattern. Using this pattern as reference, discover an approximate value of the length of the Avenue in meters.

This task involved a series of contents and illustrates the use of a real-life context in problem solving. As well as other tasks in the trail, this one also has multiple solutions, allowing the participants to choose different strategies to tackle the problem. MathCityMap app does not allow the user to introduce the solution process, only the answer. However, through observations and conversations with the students, we could conclude that, after reading the task they began to discuss among themselves how to start. Many ideas emerged through brainstorming, and, after some discussion, they began working collaboratively, solved the task in a successful way. The two main strategies used by these student teachers were: begin by calculating the distance between a two-globe lamp and a one-globe lamp, then determine the length between two lamps with two globes, and after estimating the number of lamps in the avenue and using the pattern of the two-globe lamp, calculate the length of the avenue; the other strategy used as reference the AB pattern involving consecutive two-globe lamps and one-globe lamps.

To clarify the dynamics of the trail implementation and the approach to the presented task, **Figure 13** illustrates the importance of collaborative work between the pre-service teachers in aspects like discussing ideas, making the decisions about the best strategy to use, the execution of certain procedures.

Figure 13. Student teachers solving *Combatentes da Grande Guerra Avenue* task (Source: Authors)

Through this experience we emphasize the positive outcomes concerning affective features, like the reduction of stress and anxiety towards mathematics or enhanced motivation and engagement with the tasks. The persistence in solving the tasks, fueled by cooperative learning, was crucial in the continuous attempt to find a solution, which implied also deepening the knowledge of the topics involved. This is a pathway to introduce meaningful and challenging tasks following the principles of active learning, allowing students to be intellectually, socially, and physically involved.

CONCLUDING REMARKS

The future teachers involved in this study contacted, throughout the semester, with active learning strategies, solving challenging mathematical tasks. This paper focuses on three instructional strategies of this nature, PF, GW, and MT, and aims to identify traits of cognitive, social and physical engagement as well as the participant's reactions.

Addressing the first research question, we concluded from the analysis of the results that the experiences conducted with the pre-service teachers revealed the presence and articulation of the three dimensions of engagement underlying active learning (Nesin, 2012; Vale & Barbosa, 2020b). At a cognitive level, we can say that all the student teachers benefited through each of the three experiences. All the strategies were compatible with the proposal of challenging tasks (e.g., Leikin, 2016; Stein & Smith, 1998; Vale & Barbosa, 2015), which was also a fundamental aspect to keep the participants engaged and willing to reach a solution, despite some emerging difficulties. We noticed the prevalence of analytical approaches over visual ones, explained by the previous school experiences, which in the case of the paper folding task may explain some of the incorrections emerged while trying to get a solution. At a social level, the experiences conducted helped the participants improve their mathematical communication, both oral and written, benefiting from the peer interactions, through collaborative work, either by comparing and contacting with different problem solving strategies or by getting feedback about their own work (NCTM, 2014; Vygotsky, 1996). The three experiences were seen as non-threatening contexts that favored more willing participation, instigating interactions in a natural way. This trait of active learning promotes the establishment of communities of practice and more effective teaching and learning practices. At a physical level, movement was always present and crucial to trigger mathematical activity, leading the pre-service teachers to reflect about their actions and find relations (e.g., folding a paper) or to distance themselves from the traditional educational context, moving around/outside the classroom, to contact with mathematical ideas in a more motivating way, maintaining their levels of attention and the intrinsic motivation.

Reviewing the second research question, we noticed that, despite some difficulties identified, mainly in the PF task, the future teachers recognized the potential of the active learning strategies and the tasks proposed, reacting positively to all the experiences, expressing interest and additional engagement (NAEYC, 2009; Vale et al., 2020). They assumed that these experiences also made them aware that they could improve mathematical oral and written communication, increase their repertoire of strategies and deepen their knowledge about the underlying topics. These experiences were seen as an opportunity to simultaneously develop abilities such as problem solving, communication, creativity, cooperative work, but also to establish mathematical connections (Barbosa & Vale, 2021; Cross, 1997; Fosnot & Jacob, 2010; Vale & Barbosa, 2021).

As future teachers, they concluded that PF, a GW, and an MT can act as dynamic and meaningful paths to simultaneously assess and improve their future students' mathematical abilities and knowledge.

As previously stated, and reinforced by the results, the three dimensions of active learning have an important contribution in students' engagement, however we would like to draw attention to the physical component, through movement, as being the dimension less used or valued in the classroom at any level, despite of its' relevance in the learning process. As Hannaford (2005) states, students of any age only have to benefit if they have the opportunity to move during classes, collaborating and discussing with colleagues, energizing the brain.

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