Spontaneous and reasoned approaches to the algebraic tiles' factorization problem

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Abstract: *In this paper, we present the spontaneous and reasoned approaches by a Physicist (B.C.) and a Mathematician (T.B.). Section 1 is by B.C and section 2 is by T.B.*

The events described below shows the interaction between two teacher-researchers during their class preparation. At the same it represents the first stage of the teaching experiment in the effectiveness of algebra tile method for understanding factorization through geometric pattern arrangement. During the next stage which will take place next semester they will introduce the method into instruction of intermediate algebra course. There is a striking difference between their methods which in general can be characterized as the inductive (BC) and deductive (TB) methods of reasoning.

Section 1 Aha!Moment

The problem found in one of the resource[3] materials is stated below:

$2x^2 + 3x + \square$ Fill in the blanks by finding the largest and smallest integers that will make the quadratic expression factorable.

I did experience a gentle that is of low intensity Aha! Moment, when suddenly I felt empowered to go ahead; that is, to generalize from the separate concrete examples I have investigated into the general case, which helped me to solve the problem.

I found one case without the hint but did not know the meaning here of looking for maximum and minimum integers that solve it, and how to look for them. By trial and error, I was able to find one case. The case I found was $2x^2 + 3x + 1$ and, following the method of instruction for such a case that is to multiply "a" by "c" ($ax^2 + bx + c$) and to find its two factors whose sum is b, I found the integer +1 that solves the problem. I didn't know what to do next, so I looked at the Hint to the problem, which suggested to use the visual model of the algebra tiles.

[3] OpenMiddle.com

Hint

How would you represent this problem using visual model (such as algebra tiles)?

Then I recalled the method (Gningue, 2000) of the tiles on the example $x^2 + 3x + 2 = (x+1)(x+2)$; next, I modeled on it my own first solution, which was $2x^2 + 3x + 1$,

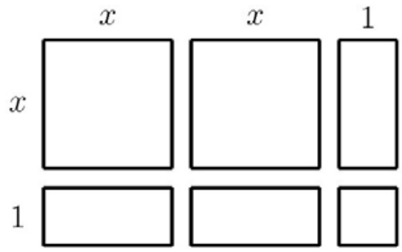


Figure 0.1 Algebra tile model for $2x^2 + 3x + 1$

and then I started trying different possibilities, of which I could not find any by changing for different positive numbers a,b,c, even daring to employ fractions.

Only when I tried $(2x-1)(x+2)=2x^2 + 3x - 2$ to make the connection between its tiles model and the corresponding polynomial, and realized how it works, I had a gentle Aha! Moment, definitely noticeable on the cognitive level as generalization and on the affective level as empowerment. It allowed me to formulate the general condition of $(2x + a)(x + b)$ as the hidden analogy between examples I did, with the condition that $a+2b = 3$. Since $c = ab$, we get $c(b) = -2b^2 + 3b$, the analysis of this quadratic function helps to get the polynomials that work by assuming b and getting c(b), showing that b=1 is the maximum integer that solves it, while there is no minimum (Czarnocha and Baker 2020; Chapter 4)

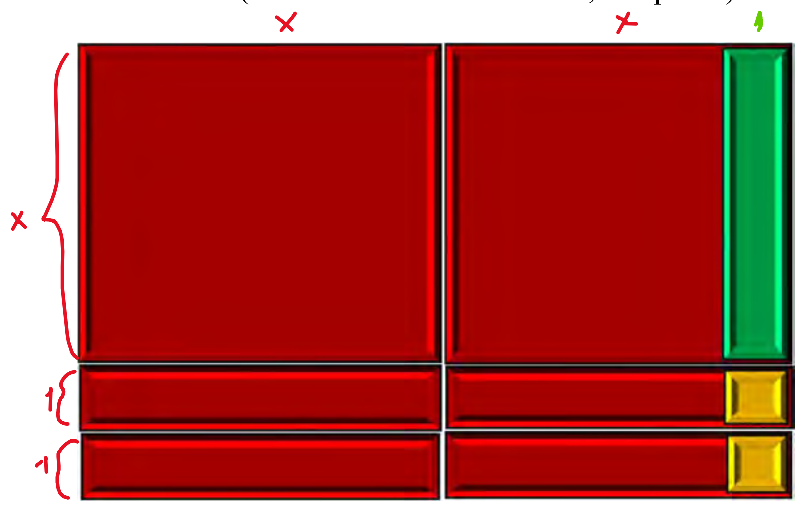


Figure 0.2 Algebra tile model for $2x^2 + 3x - 2$

Very proud of myself to have figure out how to deal with subtraction in the environment of tiles, and to deal in such a way that prompted generalization needed to solve the problem completely, I carelessly handed the bag of color tiles to my office roommate, a mathematician. The next section describes what happened after my careless act.

Section 2 Some interesting ideas about factoring a trinomial using algebra tiles.

In this section we show what you need to know about algebra tiles and why and how you use the tiles to factor a polynomial. Using the geometric meaning of a rectangle, we show unusual ways to factor a trinomial based on the titles. I gave my colleagues eight algebra tiles consisting of one big red square that represents x^2, three green rectangles whose length represents minus three x and four small yellow squares that represents negative four, i.e., $x^2 - 3x - 4$. I asked them to form a rectangle from these eight tiles. I also told them the red side of the big square represents positive x^2 and the black side represents negative x^2. The length of the red side of the rectangle represents positive x, the width positive one. The length of the green side of the rectangle represents negative x, the width negative one. The red side of the small square represents one, the yellow side represents negative one. Note: the actual dimensions are: big square 5cm x 5cm, rectangle 5cm x 1cm, and the small square 1cm x 1cm. Before looking at the examples my colleagues did, we showed the box method for multiplying $(x-4)(x+1)$ which is shown in figure 2.1

$x + 1$

x	x^2	x
-4	$-4x$	-4

Figure 2.1

Then $x^2 + x - 4x - 4 = x^2 - 3x - 4$

A problem that everyone realized was, you cannot make a rectangle with these eight tiles. You are allowed to use more tiles if you follow condition 1.

Condition 1: You can add as many tiles to the original number of tiles as long as the number of positive tiles added of the same size is the same number of negative tiles added of the same size, that is you are adding zero.

I then asked them to multiply their left side with the top side. The examples we look at have a corresponding matrix that consists of $+$ and $-$ signs.

Our first example is Figure 2.2

Figure 2.2

The left column is $x - 1$ at the top is $x - 2$ (one red and one green rectangle cancel each other out). This represents $(x-1) \quad (x-2)$, The related matrix has $A_{11} = +$, $A_{12} = -$,$A_{13} = -$,$A_{14} = -$,$A_{15} = +$,$A_{21} = -$,$A_{22} = -$,$A_{23} = -$,$A_{24} = -$and $A_{25} = -$, that is $A = \begin{pmatrix} + & - & - & - & + \\ - & - & - & - & - \end{pmatrix}$. $A_{22} = A_{21}\ x\ A_{12} = (-)\ (-) = +$, but $A_{22} = -$. We have a contradiction, that is that the condition for multiplying sign numbers fails. This example required one more positive x and one more negative x. The "longer" side is the top which uses five tiles that has length $x - 4$, the "shorter" side is on the left which uses two tiles whose length is $x + 1$, physically the top is the length, and the left side is the width, while mathematically the top is the width and the left side is the length.

Our second example is Figure 2.3

Figure 2.3

represents $(x+1)\ (x-4)$, the related matrix is A= $\begin{pmatrix} + & - & - & - & - \\ + & - & - & - & - \end{pmatrix}$ When we check, A_{22}, A_{23}, A_{24}, and A_{25} all follow the conditions for multiplying sign numbers. For example, $A_{23} = A_{21}\ x\ A_{13} = (+)\ (-) = -$ and A_{23} is negative. We have our first candidate for the solution of the problem.

Our third example is figure 2.4

Figure 2.4

This represents $(x+2)\ (x-5)$. The related matrix A= $\begin{pmatrix} + & - & - & - & - & - \\ + & - & - & - & - & - \\ + & - & - & + & + & + \end{pmatrix}$

When we look at $A_{34} = A_{31}\ x\ \ A_{14} = (+)\ (-) = -$, but $A_{34} = +$, a contradiction again.

Our fourth example is figure 2.5

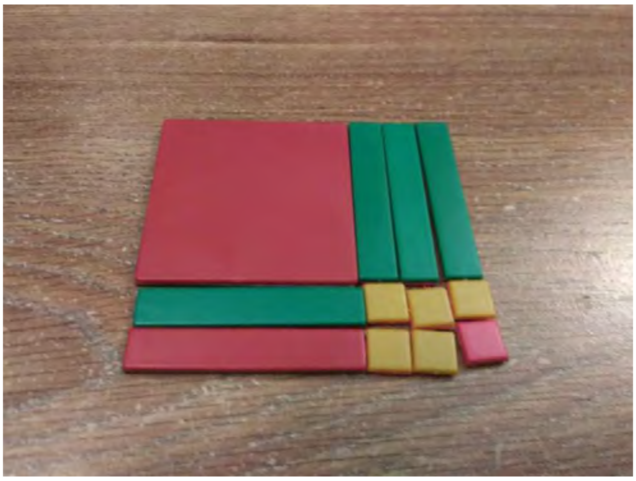


Figure 2.5

This represents x $(x-3)$ with the related matrix A = $\begin{pmatrix} + & - & - & - \\ - & - & - & - \\ + & - & - & + \end{pmatrix}$

As in example 1, we have the same contradiction, that is $A_{22} = A_{21}\, x\ A_{12} = (-)\,(-) = +$, but $A_{22} = -$.

Our fifth example is figure 2.6

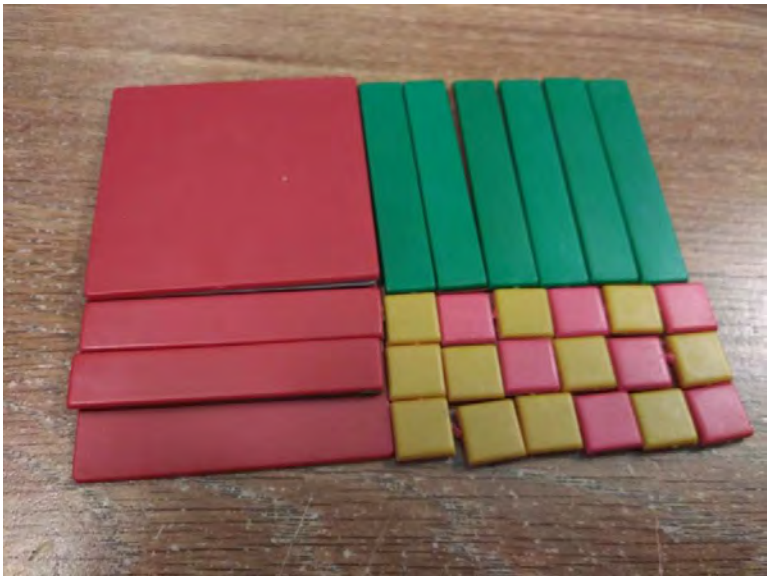


Figure 2.6

This represents $(x+3)\,(x-6)$ with the related matrix

$$A = \begin{pmatrix} + & - & - & - & - & - & - \\ + & - & + & - & + & - & + \\ + & - & - & + & - & + & - \\ + & - & - & - & + & - & + \end{pmatrix}$$

We have the same contradiction as examples 1 and 4.

Our sixth example is figure 2.7

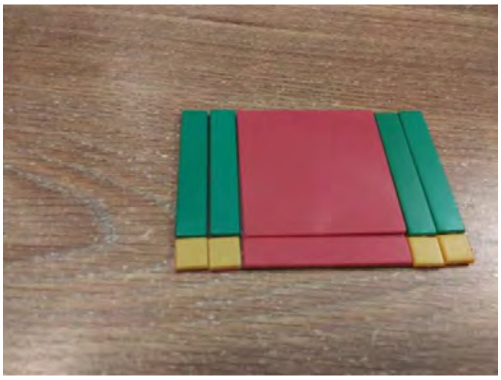


Figure 2.7

This represents $(-x\ -1)\,(-2+x-2) = (-x-1)\,(x-4)$

The related matrix is $A = \begin{pmatrix} - & - & + & - & - \\ - & - & + & - & - \end{pmatrix}$ Again we have the same contradiction as examples 1, 4 and 5.

Our seventh example is figure 2.8.

Figure2.8

This represents $(x + 1)$ $(x - 4)$ The related matrix is A = $\begin{pmatrix} + & - & - & - & - & - & + \\ + & - & - & - & - & - & + \end{pmatrix}$

When we check, A_{22}, A_{23}, A_{24}, A_{25}, A_{26}, and A_{27}, all follow the conditions for multiplying sign numbers. For example, $A_{23} = A_{21}\ x\ A_{13} = (+)\ (-) = -$ and A_{23} is negative. We have our second candidate.

Our last example is Figure 2.9

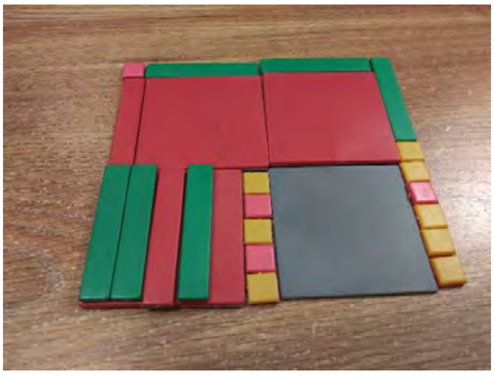


Figure 2.9

This represents x $(2\ x)$ A black rectangle is multiplied by a white rectangle, is this a black square? This example has even stranger problems.

It is clear we need another condition.

Condition 2: the conditions for multiplying sign numbers must be followed.

We then have two possible answers, example 2 (figure 2.3) and example 7 (figure 2.8)

Looking more closely at example 2 (figure 2.3), we see that we used one x^2, four negative x's, one positive x and four negative ones, that is $x^2 - 4x + x - 4$, which is factoring $x^2 - 3x - 4$ by grouping. Also, in example 2, the top is $x - 4$, the bottom is $x - 4$, the left side is $x + 1$ and the right side is $-x - 1$. Example 2 physically is a rectangle, but geometrically it is not a rectangle since the left side and the right side are not the same, i.e. $x + 1 \neq -x - 1$. Looking more closely at example 7 (figure 2.8), we see that we used one x^2, five negative x's, two positive x's and five negative one's, that is $x^2 - 5x + 2x - 5 + 1$, Also, in example 7 (figure 2.8), the top is $x - 4$, the bottom is $x - 4$, the left side is $x + 1$, and the right side is $x + 1$. Example 7 is the only one that is a physical rectangle and geometric rectangle. We example 7 is the correct answer, so we add another condition

Condition 3: the physical rectangle must be a geometric rectangle.

Our answer is

Figure 2.8

Example seven has one positive x^2, five negative x's, two positive x's, five negative ones and one positive one, that is $x^2 -5x+2x -5+1$.The top of the rectangle is $x -4$ and the left side is $x +1$. How does one use $x^2 -5x + 2x - 5 + 1$ to factor?
First, we use the top side $x -4$ in the factoring
$x^2 - 3x - 4$

$= x^2 - 5x + 2x - 5 + 1$

We rewrite $x^2 -5x$ as three terms where the first two terms have $x -4$ as a factor

$= x(x - 4) - x + 2x - 5 + 1$

We rearrange $-x + 2x - 5 + 1$ into four terms $ax + b + cx + d$ where $x - 4$ is a factor of

$ax + b$ and $x - 4$ is a factor of $cx + d$

$= x(x - 4) + 2x - 8 - x + 4$

$= x(x - 4) + 2(x - 4) - (x - 4)$

$= (x - 4)(x + 2 - 1)$

$= (x - 4)(x + 1).$

Next, we use the left side $x + 1$ in the factoring

$= x^2 + 2x - 5x - 5 + 1$

We rewrite $x^2 + 2x$ as three terms where the first two terms have $x + 1$ as a factor

$= x^2 + x + x - 5x - 5 + 1$

We rearrange $x - 5x - 5 + 1$ into four terms $ax + b + cx + d$ where $x + 1$ is a factor of $ax + b$ and $x + 1$ Is a factor of $cx + d$

$= x(x + 1) - 5x - 5 + x + 1$

$= x(x + 1) - 5(x + 1) + (x + 1)$

$= (x + 1)(x - 5 + 1)$

$= (x + 1)(x + 4)$.

Note that $x - 5 + 1$ is what is on the top and bottom of the rectangle\

We include more examples that satisfy conditions 1-3.

Example 9

Figure 2.10

The tiles used are one positive x^2, five negative x's, two positive x's, five negative one's and seven positive one's that is $x^2 - 3x + 2$ The top side is $x - 2$ and the left side is $x - 1$

First, we use the top side $x - 2$ in the factoring

$x^2 - 3x + 2$

$= x^2 - 5x + 2x - 5 + 7$

We rewrite $x^2 - 5x$ as three terms where the first two terms have $x - 2$ as a factor

$= x^2 - 2x - 3x + 2x - 5 + 7$

We rearrange $-3x + 2x - 5 + 7$ into four terms $ax + b + cx + d$ where $x - 2$ is a factor of $ax + b$ and $x - 2$ Is a factor of $cx + d$.

$= x(x - 2) + 2(x - 2) - 3x + 6$

$= x(x - 2) + 2(x - 2) - 3(x - 2)$

$= (x - 2)(x + 2 - 3)$

$= (x - 2)(x - 1)$

Next, we use the left side $x - 1$ in the factoring

$= x^2 - 5x + 2x - 5 + 7$

We rewrite $x^2 - 5x$ as three terms where the first two terms can be factored by

$x - 1$

$= x^2 - x - 4x + 2x - 5 + 7$

We rearrange $-4x + 2x - 5 + 7$ into four terms $ax + b + cx + d$ where $x - 1$ is a factor of $ax + b$ and $x - 1$ Is a factor of $cx + d$.

$= x(x - 1) + 2x - 2 - 4x + 4$

$= x(x - 1) + 2(x - 1) - 4(x - 4)$

$(x - 4 + 2)(x - 1)$

$=(x-1)\ (x-2)$.

Note that $x - 4 + 2$ is what is on the top and bottom of the rectangle

Example 10

Figure 2.11

The tiles used are one positive x^2, six negative x's, three positive x's, twelve negative one's and two one's. That is $x^2-6x+3x-12+2$.

$x^2 -3x+10$

First, we use the top side $x-5$ in the factoring

$=x^2-6x+3x-12+2$

We rewrite the first two terms (x^2-6x) as three terms where the first two terms can be factored by $x-5$

$=x(x-5)-x+3x-12+2$

We rearrange $-x+3x-12+2$ into four terms $ax+b+cx+d$ where $x-5$ is a factor of $ax+b$ and $x-5$ is a factor of $cx+d$.

$=x(x-5)+3x-15-x+5$

$=x(x-5)+3(x-5)-(x-5)$

$=(x-5)(x+3-1)$

$=(x-5)(x+2)$

Next, we use the left side $x+2$ in the factoring

$x^2-3x-10$

$=x^2+3x-6x-12+2$

We rewrite the first two terms x^2+3x as three terms where the first two terms can be factored by $x+2$

$=x^2+2x+x-6x-12+2$

We rearrange x $-6x$ -12 $+2$ into four terms $ax + b + cx + d$ where $x + 2$ is a factor of $ax + b$ and $x + 2$ is a factor of $cx + d$.

$=x(x + 2) -6x -12+x +2$

$=x(x + 2) -6(x +2)+x +2$

$=(x +2) (x -6+1)$

$=(x +2) (x -5)$

Note that $x - 6 + 1$ is what is on the top and bottom of the rectangle

The next two examples we leave it to the reader to verify the tiles.

Example 11: $x^2 - 3x -18$

The tiles used are: One positive x^2, seven negative x's, four positive x's, twenty-one negative one's , and three one's. That is $x^2 -7x+4x -21 +3$.

First, we use the top side $x -5$ in the factoring

$x^2 - 3x -18$

$=x^2 - 7x + 4x -21 +3$

We rewrite the first two terms $x^2 -7x$ as three terms where the first two terms can be factored by $x - 6$

$=x^2 -6x - x + 4x -21+3$

We rearrange $-x + 4x - 21 + 3$ into four terms $ax + b + cx + d$ where $x - 6$ is a factor of $ax + b$ and $x - 6$ is a factor of $cx + d$.

$=x(x-6) + 4x - 24 - x + 6$

$=x(x-6) + 4(x-6) - x + 6$

$=x(x-6) + 4(x-6) - (x-6)$

$=(x-6)(x+4-1)$

$=(x-6)(x+3)$

Next, we use left side $x + 3$ in the factoring

$x^2 - 3x - 18$

$=x^2 + 4x - 7x - 21 + 3$

We rewrite the first two terms $x^2 + 4x$ as three terms where the first two terms can be factored by $x + 3$

$=x(x+3) + x - 7x - 21 + 3$

We rearrange $x - 7x - 21 + 3$ into four terms $ax + b + cx + d$ where $x + 3$ is a factor of $ax + b$ and $x + 3$ Is a factor of $cx + d$.

$=x(x+3) - 7x - 21 + x + 3$

$=x(x+3) - 7(x+3) + x + 3$

=$(x+3)\ (x-7+1)$

=$(x+3)\ (\mathrm{x}-6)$

Note that $x-7+1$ is what is on the top and bottom of the rectangle

Example 12: $6x^2 - x - 15$

The tiles used are: Six positive x^2, twelve negative x's, eleven positive x's, eighteen negative ones and three one's.

$6x^2 + 11x - 12x - 18 + 3$

One way is to rewrite the first two terms $(6x^2 + 11x)$ as three terms where the first two terms can be factored by $2x+3$, were $2x+3$ is the top side

=$3x(2x+3) - 6(2x+3) + 2x + 3$

=$(2x+3)\ (3x-6+1)$

=$(2x+3)\ (3x\text{-}5)$

We again look at $6x^2 - x - 15$

=$6x^2 - 12x + 11x - 18 + 3$

The other way is to rewrite the first two terms $6x^2 - 12x$ as three terms where the first two terms can be factored by $3x-5$, where $3x-5$ is the left side

=$6x^2 - 10x - 2x + 11x - 18 + 3$

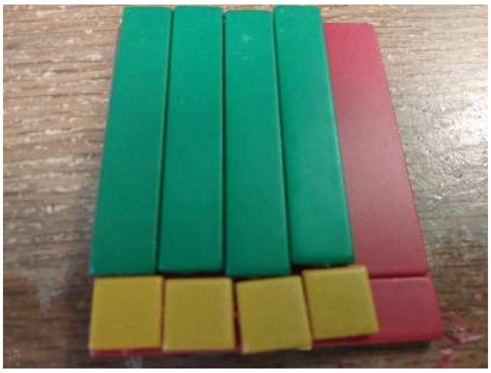
$= 2x(3x-5)+3(3x-5)$

= ($3x$-5) ($2x$ +3).

We look at $x^2 + ax + c$. If a and b are positive integers, the rectangle will follow conditions 1-3. If a and b are negative integers, the rectangle will not follow condition 3, since the left side $\neq$ rightside, or top$\neq$ bottom If a is negative and b are positive integers, the rectangle will not follow condition 3, since the left side $\neq$ rightside and top $\neq$ bottom.

Section 3 Comparison of the Two Methods.
B.C. thinks the difference shows that there are two different ways of mathematical thinking. T.B.'s approach shows the beauty and elegance of the mathematical problem solving by reasoning, the problem being how many rectangles are needed to factorize the trinomial expression, while B.C. (a physicist) presents a direct approach to the stated problem with the help of Aha! Moment insight. The direct approach yielded a different type of tile model solutions presented than those presented by T.B. in the second section. In order to convey the correct tile model for a given trinomial, B.C. utilized the third dimension in the representation of x – 1 by putting tiles on top of tiles. In order to convey the correct tile model for the given trinomial, T.B. utilized the second dimension in the representation of x – 1 by not putting tiles on top of tiles but by adding them in the same 2-dimensional plane. This "spatial" difference between the two reflects the difference between x – 1 and
x + (-1). Whereas the latter algebraic expression symbolizes addition of the negative on the plane, x – 1 symbolizes taking away 1 unit from the side x.

T.B. also thinks the difference is how each view the problem. He sees B.C. as a two-layer problem and his own as a one- layer problem. Both B.C. and T.B. used
$x^2 - 3x - 4$ and the algebra tiles to form their rectangles. Both used the number of squares, positive x's, negative x's, positive one's and negative one's from their rectangle to factor $x^2 - 3x - 4$. The difference is in how they made their rectangles. Figure 3.1 is B.C. two-layer version.



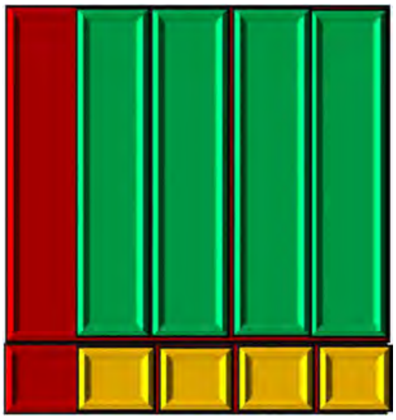


Figure 3.1 A

Figure 3.2 B

His answer is the lower level that is not covered. This rectangle is both a physical and geometric rectangle. He used "four terms" $x^2 - 4\,x - \; + x - 4$ and this is factoring by grouping. T.B. saw this same problem as a one-layer problem. To get his rectangle to be both a physical and geometric rectangle, he needed more tiles than B.C.

Figure 3.1 is T.B. one-layer version

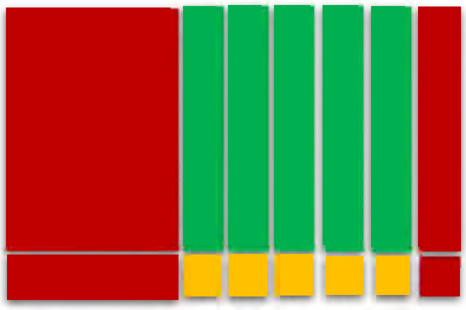


Figure 3.2A

Figure 3.2B

From his rectangle he used "five terms" $x^2 - 5x + 2x - 5 + 1$. He then developed a way to factor $x^2 - 5x + 2x - 5 + 1$ which is unusual. What is unusual is one factors that he gets is: $x - 5 + 1$, which is the top and the bottom rows of his rectangle. He used "five terms" and changed them into "six terms". With the "six terms", he used the factoring by grouping twice. Comparing both methods with the available resources on line(e.g.teachers.desmos.com) we find out no two layer approach and no new approach to factorization of $ax + b + cx + d$ presented in the text.

References

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