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What theory of infinity should be taught and how?

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Abstract: *Recent educational studies in mathematics seek to justify a thesis that there is a conflict between students' intuitions regarding infinity and the standard theory of infinite numbers. On the contrary, we argue that students' intuitions do not match but to Cantor's theory, not to any theory of infinity. To this end, we sketch ways of measuring infinity developed at the turn of the 20th and 21st centuries that provide alternatives to Cantor's theory of cardinal and ordinal numbers. Some of them introduce new kinds of infinite numbers, others simply define new arithmetic for Cantor's infinite numbers. We also sketch a way how to introduce these new theories in students' courses. To do this the crucial is the concept of an ordered field, since we define the opposition finite vs infinite in terms of Archimedean and non-Archimedean fields.*

CANTORS'S PARADISE WITH THE ROTTEN ARITHMETIC

Cantor established two kinds of infinity: cardinal and ordinal numbers, each with its own arithmetic and its own relation greater than. In both cases, the set of natural numbers, $\mathbb{N}$, makes the yardstick of infinity, be it the cardinal number $\aleph_0$ or the ordinal number ω. In fact, the first infinite numbers are defined either as the cardinality of the set of natural numbers $\mathbb{N}$, or as the type of the well-ordered set $(\mathbb{N}, <)$, while the system of natural numbers models the arithmetic of finite numbers. Cantor's theory, thus, assumes that finite numbers are natural numbers, and it attempts to extend the system $(\mathbb{N}, +, \cdot, 0, 1, <)$ where the arithmetic is characterized by commutativity of sums and products and compatibility of the "natural order" (it is Cantor's phrase) with sums and products.

However, since natural numbers share characteristics of both cardinal and ordinal numbers, the extension of the system of finite numbers can be carried out in two directions. While Cantor's infinities are supposed to extend the system of finite numbers, they hardly mimic its arithmetic whatever direction we choose: sums and products of ordinal numbers are not commutative, the order of cardinal numbers is not compatible with their sums and products, e.g.

$$1 + \omega \neq \omega + 1,\ 2\omega \neq \omega 2, \quad or\ \ 2 + \aleph_0 = 3 + \aleph_0,\ \ although\ \ 2 < 3.$$

Cantor clearly realized these constrains, but he believed infinite numbers do not follow all the rules of arithmetic of finite numbers due to their very nature. He even proudly declared that infinite numbers do not comply with the Euclid's law *The whole is greater than the part*[7], since it is possible that a set and its subset have the same cardinality. In the 1947's paper *What is Cantor's Continuum Problem?*, Gцdel reinforced the belief that there is no alternative to Cantor's theory of infinite numbers. He claimed that when one adopts Cantor's infinite numbers, there is no room for an alternative arithmetic of these numbers. In this paper, Gцdel presents cardinal numbers as extending the system of natural numbers $(\mathbb{N}, +, \cdot, 0, 1, <)$ and seeks to show that "this extension can be effected in a uniquely determined manner".

Despite this, the first alternative to Cantor's theory has been developed already in 1906, and the last decades of the 20th century have brought another non-Cantorian theories. Yet, the no alternative attitude to Cantor's infinite numbers still prevails in the education of mathematics. Here is a sample view on supposed relation between students intuitions and mathematical infinity:

"There is the ideal, pure, final mathematical structure which is unquestionable as a logical construct. And there is the psychological reality of the same concept which may remain complex, contradictory, strongly related to intuitive difficulties. That is exactly the case with the concept of infinity. Accepting definitions, theorems and logical proofs is one thing. Using the concept of infinity in various real, psychological contexts in the process of thinking and interpreting, is another. It is fair to suppose that the main source of difficulties which accompany the concept of infinity is the deep contradiction between this concept and our intellectual schemes. Genuinely built on our practical, real life experience, these schemes are naturally adapted to finite objects and events" (Fischbein et al 1979, p. 3).

This paragraph clearly manifests the belief that there are neither alternative ways of measuring infinity, nor even alternative to Cantor's arithmetic of cardinal and ordinal numbers. As a result, it is believed, that when students seek to extend laws of arithmetic of finite numbers to the domain of infinity, their intuitions built on "practical life experience" are on a collision course with an iceberg of "ideal, pure, final mathematical structure which is unquestionable as a logical construct".

Although one may think that there is a kind of necessity linking mathematical theorems and definitions, in the case of infinite numbers, there is no such inherent necessity regarding definitions themselves. Therefore, when students seek to extend laws of arithmetic of finite numbers to the domain of infinity, their intuitions are in conflict but with Cantor's theory, not with

[7] See Euclid, *Elements*, Book I, Common Notions 5.

any possible theory of infinity. In fact, the crucial point is how do one defines a structure of finiteness. Cantor's attitude is clearly focused on finite sets. When teaching students, we can found our course on a concept of finite numbers exemplified by fractions or real numbers, rather then a finite set. Accordingly the arithmetic of finiteness is defined by axioms of an ordered field, rather then the arithmetic of natural numbers.

In what follows, we sketch some recent developments that shed a new light on intuitions concerning infinity. In section § 2, we show how to define a new arithmetic for Cantor's ordinal numbers, and the way how these numbers can be included in a bigger structure of an ordered field. In section § 3, we introduce basic definitions concerning an ordered field, as well as infinite numbers defined with no reference to sets. Then, in section § 4, we introduce a field of hyperreals which includes infinite numbers. Finally, in section § 5, we apply infinite hyperreal numbers to introduce a new measure of subsets of ℕ. These new measures, known as *numerosities*, provide an alternative to Cantor's theory of cardinal numbers.

NEW ARITHMETIC FOR ORDINAL NUMBERS

Commutative sums and products of infinite numbers

In 1897, Cantor proved the so-called the normal form theorem. It states: For every ordinal number α, there are ordinal numbers $\eta_1, \ldots, \eta_h$, and natural numbers $h, p_i \in \mathbb{N}$ such that

$$\alpha = \omega^{\eta_1} \cdot p_1 + \cdots + \omega^{\eta_h} \cdot p_h, \quad where \quad \eta_1 > \cdots > \eta_h.$$

This representation of α is unique. Moreover, it is finite, due to the assumption concerning the index h. Based on this theorem, in 1906, Hessenberg introduced the so-called normal sums and products of ordinal numbers. Namely, for

$$\alpha = \omega^{\eta_1} \cdot p_1 + \cdots + \omega^{\eta_h} \cdot p_h, \quad \beta = \omega^{\eta_1} \cdot q_1 + \cdots + \omega^{\eta_h} \cdot q_h,$$

their normal sum $+_n$ and normal product $\cdot_n$ is defined by[8]

$$0.5mm\alpha +_n \beta =_{df} \omega^{\eta_1} \cdot (p_1 + q_1) + \cdots + \omega^{\eta_h} \cdot (p_h + q_h),$$

$$\alpha \cdot_n \beta =_{df} \sum_{1\le i,j\le h} \omega^{\eta_i +_n \eta_j} \cdot p_i q_j.$$

Contrary to Cantor's sums and products, normal sums and products are commutative and compatible with the standard order of ordinal numbers, that is

$$\alpha +_n \beta = \beta +_n \alpha, \quad \alpha \cdot_n \beta = \beta \cdot_n \alpha,$$

[8] For the use of this definition, we assume that some p_i or q_i could equal 0.

$$\alpha < \beta \Rightarrow \alpha +_n \gamma < \beta +_n \gamma, \quad \alpha < \beta \Rightarrow \alpha \cdot_n \gamma < \beta \cdot_n \gamma.$$

Thus, the structure $(Ord, +_n, \cdot_n, 0, 1, <)$, where Ord stands for the class of ordinal numbers, is an abelian semigroup.

Hence, e.g. since $\omega = \omega \cdot 1 + 0$, and $1 = \omega \cdot 0 + 1$, we calculate the normal sums of $\omega +_n 1$ and $1 +_n \omega$ as follows,

$$1 +_n \omega = (\omega \cdot 0 + 1) +_n (\omega \cdot 1 + 0) = \omega \cdot (0 + 1) + 1 = \omega + 1,$$

$$\omega +_n 1 = (\omega \cdot 1 + 0) +_n (\omega \cdot 0 + 1) = \omega \cdot (1 + 0) + 1 = \omega + 1.$$

Similarly, we calculate

$$2 \cdot_n \omega = (\omega \cdot 0 + 2) \cdot_n (\omega \cdot 1 + 0) = \omega^2 \cdot 0 + \omega \cdot 2 + 0 = \omega \cdot 2,$$

$$\omega \cdot_n 2 = (\omega \cdot 1 + 0) \cdot_n (\omega \cdot 0 + 2) = \omega^2 \cdot 0 + \omega \cdot 2 + 0 = \omega \cdot 2.$$

As is well known, in Cantor's arithmetic the inequalities hold $1 + \omega < \omega + 1$, and $2 \cdot \omega < \omega \cdot 2$.

Cantor's ordinal numbers as elements of an ordered field

(Gonshor 1986) shows that ordered field of Conway numbers, as developed in (Conway 1976/2001), includes the structure $(Ord, +_n, \cdot_n, 0, 1, <)$; see (Biaszczyk & Fila, 2020). Hence, each Cantor's ordinal number is subject to ordered field operations, and all rules an ordered field can be applied to Cantor ordinal numbers. Therefore, in the field of Conway numbers, next to the number ω there are also numbers such as $-\omega$, $\omega/2$ or the inverse of ω, that is $1/\omega$. That is why, in the next section we introduce the concept of ordered field, as well as the idea of infinite elements of a field.

BASICS OF THE THEORY OF ORDERED FIELDS

The idea of an ordered field is a good starting point when we are to introduce new kinds of infinite numbers. It is is both simple, and well-known from the high-school courses, as fractions and real numbers provide model examples. Still, instead of constructions, we prefer an axiomatic account. Thus, our starting point is a simple mathematical idea introduced in an axiomatic fashion, rather then "real life experience", as suggested by Fischbein.

A commutative field $(\mathbb{F}, +, \cdot, 0, 1)$ together with a total order $<$ is an ordered field when the sums and products are compatible with the order, that is[9]

$$x < y \Rightarrow x + z < y + z, \quad x < y, 0 < z \Rightarrow xz < yz.$$

[9] Note, here $\mathbb{F}$ is supposed to be a set, but it can also be a proper class. Therefore, we can apply the concept of ordered field to Conway numbers.

In any ordered field we define in a usual way an absolute value, $|x|$, and a limit of sequence, $\lim_{n\to\infty} a_n$. Note, however, that while in real analysis the formula $\forall\varepsilon > 0$ stands for $\forall\varepsilon \in \mathbb{R}_+$, in an ordered field $(\mathbb{F}, +,\cdot, 0,1, <)$ it means $\forall\varepsilon \in \mathbb{F}_+$.

The term n is defined by

$$n =_{df} \underbrace{1 + 1 + \ldots + 1}_{n-tim},$$

while $\frac{n}{m} =_{df} n \cdot m^{-1}$. On this basis we assume that any ordered field includes natural numbers $\mathbb{N}$ and rational numbers $\mathbb{Q}$. In fact, the field of fractions $(\mathbb{Q}, +,\cdot, 0,1, <)$ is the *smallest* ordered field.

In every ordered field, we can define the following subsets of $\mathbb{F}$:

$$\mathbb{L} = \{x \in \mathbb{F}: (\exists n \in \mathbb{N})(|x| < n)\},$$

$$\mathbb{A} = \{x \in \mathbb{F}: (\exists n \in \mathbb{N})(\tfrac{1}{n} < |x| < n)\},$$

$$\Psi = \{x \in \mathbb{F}: (\forall n \in \mathbb{N})(|x| > n)\},$$

$$\Omega = \{x \in \mathbb{F}: (\forall n \in \mathbb{N})(|x| < \tfrac{1}{n})\}.$$

The elements of these sets are called limited, assignable, infinity, and infinitely small numbers respectively. Here are some obvious relationships between these kinds of elements, we will call them $\Omega\Psi$ rules,

1. $(\forall x, y \in \Omega)(x + y \in \Omega, xy \in \Omega)$,
2. $(\forall x \in \Omega)(\forall y \in \mathbb{L})(xy \in \Omega)$,
3. $(\forall x)(x \in \mathbb{A} \Rightarrow x^{-1} \in \mathbb{A})$,
4. $(\forall x \neq 0)(x \in \Omega \Leftrightarrow x^{-1} \in \Psi)$.

Archimedean axiom

When to the axioms of an ordered field we add the so-called Archimedean axiom, we obtain the class of Archimedean fields. Here are some equivalent forms of the Archimedean axiom:

1. $(\forall x, y \in \mathbb{F})(\exists n \in \mathbb{N})(0 < x < y \Rightarrow nx > y)$,
2. $(\forall x \in \mathbb{F})(\exists n \in \mathbb{N})(n > x)$,
3. $\lim_{n\to\infty} \frac{1}{n} = 0$,

4. $(\forall x, y \in \mathbb{F})(\exists q \in \mathbb{Q})(x < y \Rightarrow x < q < y)$,
5. For any Dedekind cut (A, B) of $(\mathbb{F}, <)$ obtains[10]

$$(\forall n \in \mathbb{N})(\exists a \in A)(\exists b \in B)(b - a < \tfrac{1}{n}),$$

6. $\Omega = \{0\}$.

Versions A1 and A2 are well-known, both in the mathematical as well as the historical context. A1 in the following form $(\forall x, y \in \mathbb{F})(\exists n \in \mathbb{N})(nx > y)$ originates from Euclid's *Elements*, Book V. It characterized the ancient Greek structure of magnitudes, specifically, line segments; see (Biaszczyk & Fila 2020). In modern times, it is included as an axiom of Euclidean geometry. In calculus courses, A3 is usually presented as a theorem rather than an axiom, however the Archmimedean axiom follows from some versions of the continuity of real numbers, or is explicitly included in other versions (see section 3.2. below). A6 reveals that in a non-Archimedean field the set of infinitesimals Ω contains at least one positive element, say ε. Then, by $\Omega\Psi$ rules, $\frac{\varepsilon}{n}$, as well as, $n \cdot \varepsilon$ are also infinitesimals.

Real numbers

The field of real numbers is defined as a commutative ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ in which every Dedekind cut (L, U) of $(\mathbb{F}, <)$ satisfies the following condition:

$$(\exists x \in \mathbb{F})(\forall y \in L)(\forall z \in U)(y \leq x \leq z). \qquad \text{(C1}$$

Here are some other equivalent forms of C1:

1. If $A \subset \mathbb{F}$ is a nonempty set which is bounded above, then there exists $a \in \mathbb{F}$ such that $a = \sup A$.

2. The field is Archimedean and every Cauchy (fundamental) sequence $(a_n) \subset \mathbb{F}$ has a limit in $\mathbb{F}$.

3. The field is Archimedean and if $\{A_n | \ n \in \mathbb{N}\ \} \subset \mathbb{F}$ is a family of descending, closed line segments, then $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

The above definition is based on the so called categoricity theorem which states that every two ordered fields that satisfy C1 are isomorphic. In that sense, the field of real numbers is the unique complete ordered field. Morever, the field of real numbers is the *biggest* Archimedean field, in the sense any Archimedean field is isomorphic to some subfield of real numbers. As a result,

[10] For the remainder, a pair (A, B) of non-empty sets is a Dedekind cut of a totally ordered set $(X, <)$ iff: (1) $A \cup B = X$, (2) $(\forall x \in A)(\forall y \in B)(x < y)$.

any field extension of real numbers is a non-Archimedean field, and includes infinitely small and infinite numbers. Based on these mathematical facts, we define a finite number is an element of the field of real numbers. We could also provide a historical motivation for such a definition. It would basically refer to the long-standing tradition of Euclidean geometry, and to the 18th century mathematics, first of all, to Euler's legacy; see (Biaszczyk & Fila 2020).

The field of hyperreals

In this section, we provide a specific field-extension of real numbers, namely the field of hyperreals (non-standard real numbers). Since (Robinson 1966), many different approaches to non-standard reals have been developed. The one presented below is based on the so-called ultrapower construction. The set $\mathbb{R}^*$ is defined as the quotient class of the set of sequences of real numbers, $\mathbb{R}^\mathbb{N}$, with respect to a specific relation defined on the set of indexes $\mathbb{N}$. We begin with that relation.

Ultrafilter on the set $\mathbb{N}$

We start with the definition of an ultrafilter on the set $\mathbb{N}$, and present some basic results concerning ultrafilters.

Definition 1 A family of sets $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ is an ultrafilter on $\mathbb{N}$ if (1) $\emptyset \notin \mathcal{U}$, (2) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, (3) if $A \in \mathcal{U}$ and $A \subset B$, then $B \in \mathcal{U}$, (4) for each $A \subset \mathbb{N}$, either A or its complement $\mathbb{N}\backslash A$ belongs to $\mathcal{U}$.

Take the family of sets with finite complements,

$$\{A \subset \mathbb{N} : \mathbb{N}\backslash A \quad is \;\; finite\}.$$

This family is usually called the Fréchet filter on $\mathbb{N}$. Indeed, it obviously satisfies conditions (1)–(3) listed in the Definition 1. Note, however, that neither the set of odd numbers nor the set of even numbers has a finite complement, hence, the Fréchet filter is not an ultrafilter. Still, by applying the Axiom of Choice it can be extended to an ultrafilter. In what follows, let $\mathcal{U}$ be a fix ultrafilter on $\mathbb{N}$ which extends the Fréchet filter.

Thus, we know that for every $k \in \mathbb{N}$, the family $\mathcal{U}$ includes the set

$$\mathbb{N}\backslash\{0,1,2,\dots,k\},$$

since sets of this kind belong to the Fréchet filter. Moreover, the set $\mathbb{N}$ also belongs to $\mathcal{U}$, since it belongs to the Fréchet filter. Next, due to the condition (4) of Definition (1), for any subset A of $\mathbb{N}$, either A, or $\mathbb{N}\backslash A$ belongs $\mathcal{U}$. We apply this fact to prove, e.g. an equivalence (1), as explained below. Finally, it can be shown that the following proposition holds.

Theorem 1 For any subsets $A_1, \dots, A_n$ of $\mathbb{N}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$. If $\bigcup_{i=1}^{n} A_i \in \mathcal{U}$, then $A_i \in \mathcal{U}$ for exactly one i such that $1 \leq i \leq n$.

By applying this proposition, one can show that relations $<^*$ defined on the set $\mathbb{R}^*$ and $\mathbb{N}^*$ are actually total orders.

Extending the field of real numbers

Here, we sketch how to extend the field of real numbers $(\mathbb{R}, +, \cdot, 0, 1, <)$ to a non-Archimedean field of the hyperreals.[11] The set $\mathbb{R}^*$ is defined as the quotient class of $\mathbb{R}^{\mathbb{N}}$ with respect to the following relation

$$(r_n) \equiv (s_n) \Leftrightarrow \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{U}.$$

Thus, $\mathbb{R}^* = \mathbb{R}^{\mathbb{N}}/_{\mathcal{U}}$.

New sums and products are defined pointwise, that is

$$[(r_n)] +^* [(s_n)] = [(r_n + s_n)], \quad [(r_n)] \cdot^* [(s_n)] = [(r_n \cdot s_n)].$$

New total order is defined by

$$[(r_n)] <^* [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{U}.$$

Hence, the product and sum of hyperreals $[(r_1, r_2, \dots)]$ and $[(s_1, s_2, \dots)]$ gives $[(r_1 \cdot s_1, r_2 \cdot s_2, \dots)]$, and $[(r_1 + s_1, r_2 + s_2, \dots)]$ respectively. The relation $[(r_1, r_2, \dots)] <^* [(s_1, s_2, \dots)]$ holds when, for example, the set $\{n \in \mathbb{N} : r_n < s_n\}$ equals $\mathbb{N}$ minus some finite set (though the definition of order $<^*$ also includes other cases).

Standard real number, $r \in \mathbb{R}$, is represented by the class $[(r, r, r, \dots)]$, i.e., the class of a constant sequence $(r, r, r, \dots)$. Note that the sequence representing standard real number, e.g. 1, can take the same value from some index on, for example

$$1 = [(1,1,1,1 \dots)] = [(0,0,1,1, \dots)].$$

Owing to the above definitions, we employ the same symbols for real numbers in the standard and non-standard context; we will also employ the same symbols for sums, products and order relation in the standard and non-standard context.

It follows from the notion of ultrafilter that the following relation holds

$$[(r_n)] \neq [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : \ r_n \neq s_n\} \in \mathcal{U}. \tag{1}$$

[11] For details, consult (Biaszczyk & Major 2014), (Biaszczyk 2016).

Due to this fact, we can control, e.g. an inequality such as this one $[(r_n)] \neq 0$. This fact, in turn, enables to show that the quotient structure is really an ordered field.

In the next section, we consider hyperreal numbers represented by sequences of natural numbers, that is $[(n_j)]$, where $(n_j) \subset \mathbb{N}$, for instance

$$\alpha = [(1,2,3, \ldots)] = [(n)]. \tag{2}$$

According to the definition of product, we have

$$\alpha^2 = [(1,2,3, \ldots)] \cdot [(1,2,3, \ldots)] = [(1^1, 2^2, 3^2, \ldots)] = [(n^2)].$$

Then, the hyperreal number $\frac{\alpha}{2}$ is determined by the following equalities

$$\frac{\alpha}{2} = [(1,2,3, \ldots)] \cdot [(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)] = [(\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots)] = [(\frac{n}{2})].$$

Similarly, that is point-wise, we define the hyperreal number $\sqrt{\alpha}$, namely

$$\sqrt{\alpha} = [(\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots)] = [(\sqrt{n})].$$

In a similar way, the floor function is defined, namely

$$\lfloor [(r_j)] \rfloor = [(\lfloor r_j \rfloor)].$$

Hence, hyperreal numbers such as

$$\lfloor \frac{\alpha}{2} \rfloor \quad and \quad \lfloor \sqrt{\alpha} \rfloor,$$

are represented by sequences of natural numbers, namely

$$\lfloor \frac{\alpha}{2} \rfloor = [(\lfloor \frac{n}{2} \rfloor)], \quad \lfloor \sqrt{\alpha} \rfloor = [(\lfloor \sqrt{n} \rfloor)].$$

More specifically,

$$\lfloor \frac{\alpha}{2} \rfloor = \lfloor [(\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots)] \rfloor = [(\lfloor \frac{n}{2} \rfloor)] = [(0,1,1,2,2,3,3, \ldots)].$$

Note that with natural numbers, the following equalities obtains $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = n$ or $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1 = n$, depending on whether n is even or odd. Similarly, $\lfloor \frac{\alpha}{2} \rfloor + \lfloor \frac{\alpha}{2} \rfloor = \alpha$ or $\lfloor \frac{\alpha}{2} \rfloor + \lfloor \frac{\alpha}{2} \rfloor + 1 = \alpha$, depending on whether the set of even numbers belongs or not to the ultrafilter $\mathcal{U}$. Yet, we skip a discussion needed to justify this claim.

Extending natural numbers

In this subsection, we apply the ultrapower construction, as explained above, to natural numbers $(\mathbb{N},+,\cdot,0,1,<)$. As a result, we obtain the nonstandard (and uncountable) model of Peano arithmetic $(\mathbb{N}^*,+,\cdot,0,1,<)$. Thus, the set $\mathbb{N}^*$ is the quotient class of $\mathbb{N}^{\mathbb{N}}$ with respect to the following relation

$$(n_j) \equiv (m_j) \Leftrightarrow \{j \in \mathbb{N}: n_j = m_j\} \in \mathcal{U}.$$

New sums and products are defined pointwise, new total order is defined by

$$[(n_j)] <^* [(m_j)] \Leftrightarrow \{j \in \mathbb{N}: n_j < m_j\} \in \mathcal{U}.$$

A standard natural number, $n \in \mathbb{N}$, is represented by the class $[(n,n,n,\dots)]$. Like in the case of hyperreals, we employ the same symbols for natural numbers, as well as for their sums, products and order, in the standard and non-standard context.

Again, from the fact that the Fréchet filter is the subset of $\mathcal{U}$, it follows that both the constant sequence $(2,2,2,\dots)$, and a sequence (n_j) which on a finite set of indexes A takes 0, and for other indexes takes 2, i.e.,

$$n_j = \begin{cases} 0, & for\ j \in A, \\ 2, & for\ j \in \mathbb{N} \setminus A, \end{cases}$$

represent number 2,

$$[(2,2,2,\dots)] = 2 = [(n_j)].$$

For the rest of our presentation, we call nonstandard natural numbers numerosities, and give a special role for the number α, as defined by formula (2): we will show that α is the numerosity of the set $\mathbb{N}$.

To unify developments of this and the previous sections, we can define nonstandard natural numbers as a subset of the set of hyperreals as follows

$$\mathbb{N}^* = \{[(n_j)] \in \mathbb{R}^* \mid \{j \in \mathbb{N} \mid n_j \in \mathbb{N}\} \in \mathcal{U}\}.$$

It is up to the reader to decide which option he/she finds easier to follow.

The Whole is Greater Than the Part

In section 2, we presented the alternative arithmetic to Cantor's ordinal numbers, in this section, we provide an alternative to Cantor's cardinal numbers.

Galileo is believed to be the first who identified the seemingly paradoxical fact that the set of even natural numbers, $2\mathbb{N}$ in short, is equinumerous with the set of all natural numbers, $\mathbb{N}$.

Viewed from the Cantorian perspective, it is simply because these two sets are of the same cardinality. Moreover, within the Cantorain framework, these sets are presented as a model counterexample to the law *The whole is greater than the part*, where *part* is interpreted as being a subset, while the relation *greater-than* refers to the cardinality of sets. Recently, however, Vieri Benci and Mauro Di Nasso developed a theory in which countable sets comply with the old Euclid's law interpreted in such a way that *part* means subset, and is *greater-than* refers to a new kind of infinite number called *numerosities*. In this approach, $numerosity(A) < numerosity(B)$, whenever $A \nsubseteq B$; see (Benci & Di Nasso, 2019).

Numerosities

We present a simplified version of the theory of numerosities, as developed in (Benci & Di Nasso, 2019). It considers subsets of ℕ only. Still it exemplifies an alternative to the Cantor's theory. Within Cantor system, every subset of ℕ is either a finite set or a set with the cardinality $\aleph_0$, that is, for every $A \subset \mathbb{N}$, either $A \sim \mathbb{N}$ or $A \sim n$, for some n. The theory developed by Benci and Di Nasso gives the same result regarding finite sets. Yet, infinite subsets of ℕ have a smaller numerosity than the numerosity of ℕ, while to the very set ℕ the nonstandard number α, as defined by the formula (2) above, is assigned.

How to measure subsets of ℕ by numerosities

The key role in Benci and Di Nasso's theory plays the way how numerosities are ascribed to subsets of ℕ. Here is this definition.

Let A be a subset of ℕ. We define a function $\varphi_A: \mathbb{N} \mapsto \mathbb{N}_0$, by

$$\varphi_A(n) = \overline{\overline{\{a \in A \mid a \leq n\}}}. \quad (3)$$

Usually, the symbol $\overline{\overline{X}}$ stands for the cardinal number of the set X. Here, yet, it stands for natural number, since for any n, the set $\{a \in A \mid a \leq n\}$ is finite. Thus, we may interpret the symbol $\overline{\overline{\{a \in A \mid a \leq n\}}}$ as follows *how many elements of the sets A are less or equal to n.*

Definition 2 The numerosity of the set A is the nonstandard natural number $\nu_\alpha(A)$ represented by the sequence $(\varphi_A(n))$, that is

$$\nu_\alpha(A) = [(\varphi_A(n))],$$
$$= [(\varphi_A(1), \varphi_A(2), \varphi_A(3), \dots)].$$

Here are some examples. 1) Let us start with finite sets, e.g. a two–elements set $A = \{k, l\}$, with $k < l$. We have,

$$\varphi_A(n) = \begin{cases} 0, & for\ n < k\,, \\ 1, & \quad for\ k \leq\ n < l\,, \\ 2, & for\ l \leq\ n\,. \end{cases}$$

Since for all but finite number of n we have $\varphi_A(n) = 2$, the numerosity of A equals 2, that is $\nu_\alpha(A) = 2$.

In a similar way, we obtain that numerosity of the set $A = \{a_1, \dots, a_k\}$ equals k.

2) Now, we assign a numerosisty to the set of natural numbers $\mathbb{N}$. To this end, observe that $\varphi_{\mathbb{N}}(n) = n$, for every n. Hence, the sequence $(\varphi_{\mathbb{N}}(n))$ is $(1,2,3,\dots)$, and

$$\nu_\alpha(\mathbb{N}) = [(1,2,3,\dots)] = \alpha.$$

This fact explains the role of the index α: it is the numerosity of the set $\mathbb{N}$ and other numerosities rely on this basic fact.

3) Now, let us calculate the numerosity of the set of even numbers $2\mathbb{N} = \{2,4,6,\dots\}$. One can easily figure our the first terms of the sequence $(\varphi_{2\mathbb{N}}(n))$. These are as follows

$$\varphi_{2\mathbb{N}}(1) = 0,\ \varphi_{2\mathbb{N}}(2) = 1,\ \varphi_{2\mathbb{N}}(3) = 1,\ \varphi_{2\mathbb{N}}(4) = 2,\ \varphi_{2\mathbb{N}}(5) = 2, \dots$$

Thus, $(\varphi_{2\mathbb{N}}(n)) = (0,1,1,2,2,3,\dots)$, and, finally

$$\varphi_\alpha(2\mathbb{N}) = [(0,1,1,2,2,3,\dots)] = \lfloor\tfrac{\alpha}{2}\rfloor.$$

4) Finally, by induction, we can prove the general rule

$$A \nsubseteq B \Rightarrow \nu_\alpha(A) < \nu_\alpha(B). \tag{4}$$

According to Benci and Di Nasso, it justifies the old law *The whole is greater than the part*, even when applied to infinite sets. In fact, rule (4) applies to subsets of the set $\mathbb{N}$. Still, the domain of numerosities can be extended to the subsets of real numbers.

SUMMARY We argued that although students intuitions may not fit to Cantor's theory, there are new theories that could match these intuitions. Generally, the conflict concerns rules of an ordered field and whether or not they can be extended on the realm of infinite numbers. While Cantor's arithmetic of ordinal numbers hardly mimics these rules, within a new framework, the arithmetic of ordinal numbers can be redefined in such a way that they comply with the rules of an ordered field. Moreover, numerositie, on the hand, provide an alternative to Cantor's cardinal numbers, on the other, as elements of the field of hyperreals, they are also subject to the rules of an ordered field.

References

[1] Benci,V., Di Nasso, M. (2019). *How to Measure the Infinite. Mathematics with Infinite and Infinitesimal Numbers*. World Scientific. New Jersey.

[2] Błaszczyk, P. (2016). A Purely Algebraic Proof of the Fundamental Theorem of Algebra. *Annales Universitatis Paedagogicae Cracoviensis Studia ad Didacticam Mathematicae Pertinentia* VIII, 6–22.

[3] Błaszczyk, P., Fila, M. (2020). Cantor on Infinitesimals. Historical and Modern Perspective. *Bulletin of the Section of Logic* 49(2), 149–179; https://www.researchgate.net/publication/342014176_Cantor_on_Infinitesimals_Historical_and_Modern_Perspective/stats

[4] Błaszczyk, P., Major, J. (2014). Calculus without the Concept of Limit. *Annales Universitatis Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia* VI, 19–38

[5] Cantor, G. (1932). *Gesammelte Abhandlungen mathematischen und philosophischenInhalts*. Springer. Berlin.

[6] Cantor, G. (1897). Beiträge zur Begründung der transfiniten Mengenlehre. In: (Cantor, 1932), 312–351.

[7] Conway, J.H. (1976). *On Numbers and Games*. Academic Press, London.

[8] Conway, J.H. (2001). *On Numbers and Games*. AK Peters. Natick.

[9] Fischbein, E., Tirosh, D., Hess, P. (1979). The Intuition of Infinity. *Educational Studies in Mathematics* 10, 3–40.

[10] Fiztpatrick, R. (2007). *Euclid's Elements of Geometry* translated by R. Fiztpatrick; http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

[11] Gonshor, H. (1986). *An Introduction to the Theory of Surreal Numbers*. Cambridge University Press, Cambridge.

[12] Gödel, K. (1947). What is Cantor's Continuum Problem? *The American Mathematical Monthly*, 54(9), 515–525.

[13] Robinson, A. (1966). *Non-standard Analysis*, North-Holland Publishing Company, Amsterdam.