Investigating Mathematics Pre-service Teachers' Knowledge for Teaching: Focus on Quadratic Equations

Jill Newton
Purdue University

Christina Alvey
Mount Saint Mary College

Rick Hudson
University of Southern Indiana

Received July 29, 2019 Accepted October 30 2022

A substantial body of research has documented that the types of knowledge that mathematics teachers draw upon during their practice often differs from the knowledge of individuals working in other fields. Drawing on the Mathematical Knowledge for Teaching (MKT) Framework and the School Mathematics Teaching Pedagogical Content Knowledge (SMTPCK) Framework, we investigated the knowledge that mathematics preservice secondary teachers (M-PSTs) used when solving quadratic equations and talking about teaching this topic during task-based interviews. Most of the M-PSTs were able to draw on their Common Content Knowledge (in MKT) and Content Knowledge in a Pedagogical Context (in SMTPCK) for procedures such as using the quadratic formula and completing the square. the M-PSTs, however, more often struggled and expressed uncertainty when asked to draw on their Specialised Content Knowledge (in MKT) and Clearly Pedagogical Content Knowledge (in SMTPCK) to, for example, provide multiple representations to support student learning. Our findings support persistent calls from professional organisations for a series of courses in secondary mathematics teacher education programs that provide opportunities for M-PSTs to engage with and investigate secondary mathematics content from an advanced perspective. Such experiences have the potential to enhance development of several domains of M-PSTs' MKT and SMTPCK. Similarities, differences, affordances, and limitations of MKT and SMTPCK frameworks are discussed.

Keywords · mathematics teacher education research · algebra · mathematics knowledge for teaching · mathematics pedagogical content knowledge · quadratic equations

Introduction

As documented broadly in research, the knowledge that mathematics teachers use in their work is quite different from the mathematical knowledge needed in other professions (e.g., Ball et al., 2008; Chick, 2007; Heid et al., 2015; Schifter, 2001; Shulman, 1986, 1987). Shulman (1986) proposed that

> teachers must not only be capable of defining for students the accepted truths in a domain [mathematics in this case]. They must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both with the discipline and without, both in theory and in practice (p. 9).

In an effort to elevate the profession of teaching, Shulman also highlighted the idea that content knowledge and pedagogical knowledge were not distinct domains and proposed Pedagogical Content Knowledge (PCK), a specialised form of professional knowledge of teachers situated at the intersection of these domains. He asked questions such as, "Where do teacher explanations come from? How do teachers decide what to teach, how to represent it, how to question students about it, and how to deal with problems of misunderstanding?" (Shulman, 1986, p. 8). Since Shulman's introduction of PCK to the education research community, many scholars have taken up this construct to describe the work of teachers (e.g., Depaepea et al., 2013; Kind, 2009; Koehler et al., 2014); this includes development and investigation of PCK in mathematics education (e.g., Ball et al., 2008; Chick, 2007; Depaepea et al., 2013).

Researchers have documented the development of specialised knowledge and skills for teaching elementary mathematics (e.g., Ball et al., 2009; Kazemi & Franke, 2004); however, some researchers have suggested that the proposed domains may not be discrete (e.g., Copur-Gencturk et al., 2019) and noted challenges as they applied these conceptualisations of mathematical knowledge to a secondary mathematics context (e.g., Asquith et al., 2007; Speer et al., 2015). For example, Speer et al. (2015) suggested that distinctions between Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK) described as part of the Mathematical Knowledge for Teaching (MKT) framework (Ball et al., 2008) may be less obvious at the secondary level than at the elementary level. They proposed that other ways of thinking about teachers' knowledge, such as attention to using students' mathematical thinking to make instructional decisions, might be a fruitful area for further research and development.

To build on work related to secondary mathematics teachers' knowledge, we conducted task-based interviews with mathematics pre-service teachers (M-PSTs) to investigate their algebraic thinking and how they would hypothetically engage their future students as they worked on algebra tasks. We investigated what M-PSTs knew and were able to do related to algebraic equations given this topic's prominent place in secondary mathematics curricula (e.g., International Association for the Evaluation of Educational Achievement [IEA], 2013; Organisation for Economic Co-operation and Development [OECD], 2019). In this paper, we report our findings from tasks focused on the teaching and learning of quadratic equations. This follows research conducted by the authors that focused on linear equations (Alvey et al., 2016). The specific questions under investigation in this study were: What explanations do M-PSTs provide as they solve quadratic equations? What types of mathematical knowledge do M-PSTs utilise when they solve quadratic equations and when they make conjectures about students' thinking related to solving quadratic equations?

Literature Review

Mathematics Teacher Education

The Conference Board of Mathematical Sciences (CBMS), a collaboration of 18 professional mathematical organisations in the United States, published *The Mathematical Education of Teachers II* (*MET II*) (CBMS, 2012) to explicate recommendations for mathematics teacher preparation. The authors suggested that to understand their future students' thinking, M-PSTs need not only mathematics education courses, but also mathematics courses designed specifically for them (hereafter, "Mathematics for Teachers" courses) that address big mathematical ideas, build connections within and beyond mathematics, and provide opportunities to study K–12 mathematics from an advanced standpoint. That is, these courses should "emphasise the inherent coherence of the mathematics of high school, the structure of mathematical ideas from which the high school syllabus is derived" (p. 63), thus providing opportunities for M-PSTs to experience mathematics as learners in tandem with a focus on their future students' thinking. The Association of Mathematics Teacher Educators (AMTE, 2017) in their *Standards for Preparing Teachers of Mathematics* echoed this call, stating the need for "the equivalent of an undergraduate major in mathematics that includes three courses with a primary focus on high school mathematics from an advanced viewpoint" (p. 136).

Murray et al. (2018) reached a similar conclusion about the need for Mathematics for Teachers courses, which they called "connecting" courses, using data from 17 countries in the Teacher Education and Development Study in Mathematics (TEDS-M). They found that "the correlations between content knowledge [CK], pedagogical content knowledge [PCK], and the opportunity to learn mathematics were modest and often low globally" (p. 19), which suggested that CK and PCK do not necessarily develop together in mathematics courses. That is, more opportunities to learn mathematics does not necessarily enhance PCK. Despite persistent calls for Mathematics for Teachers courses, Newton et al. (2014), in a survey of U.S. secondary mathematics teacher education programs, found that although most programs were aligned with the *MET II* recommendations for the majority of mathematics and methods courses, few programs met the course recommendations for Mathematics for Teachers courses. These courses, missing from many programs, may serve M-PSTs by enhancing their knowledge for teaching mathematics.

Mathematics Teacher Knowledge

The professional knowledge and practices of mathematics teachers are complex; therefore, it comes as no surprise that the work of exploring and theorising their knowledge and practices is also complex. Building on Shulman's (1986, 1987) notion of PCK, many ways of framing the knowledge and practices of mathematics teachers have emerged around the world, including: (a) Knowledge Quartet (e.g., Rowland et al., 2005); (b) Professional Competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy (COACTIV) (e.g., Kunter et al., 2013); (c) Mathematical Understanding for Secondary Teaching (MUST) (e.g., Heid et al., 2015); and (d) Mathematics Teacher's Specialised Knowledge (MTSK) model (e.g., Carrillo-Yañez et al., 2018). In this study, we utilised two such frameworks, the MKT framework developed by Ball and colleagues in the United States (e.g., Ball et al., 2008) and School Mathematics Teaching Pedagogical Content Knowledge (SMTPCK) developed by Chick and colleagues in Australia (e.g., Chick, 2007, Chick & Beswick, 2018).

Mathematical Knowledge for Teaching (MKT)

Ball et al. (2008) theorised the MKT framework with six domains—three domains in Subject Matter Knowledge and three domains in Pedagogical Content Knowledge (See Appendix A). Subject Matter Knowledge includes CCK, SCK, and Horizon Content Knowledge (HCK). CCK is described as "the mathematical knowledge and skill used in settings other than teaching" (p. 399); whereas the SCK domain distinguishes mathematical knowledge that is unique to the profession of teaching. Teachers also need HCK, knowledge of what mathematics content came before a topic and what content will come after, as they help students navigate the mathematical landscape. Pedagogical Content Knowledge includes Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). KCS involves teachers knowing information about their students that is relevant for teaching mathematics—what the students might already know, what they might struggle with, and what they might find interesting or motivating. Teachers must also be able to make instructional decisions based on their knowledge of mathematical content—how best to introduce a topic and what sequence of activities will help students understand the topic. This knowledge is referred to as KCT. Finally, KCC involves familiarity with a wide range of curricular materials. Since the introduction of the MKT framework, scholars have utilised, adapted, and measured these domains (e.g., Hill et al., 2005; Lai & Clark, 2018; McCrory et al., 2012). Given that much of the research that explores MKT has been conducted at the elementary level (e.g., Ball et al., 2008; Hill et al., 2005; Lai & Clark, 2018) and the ongoing debates about the application of the MKT domains at the secondary level (e.g., Speer et al., 2015), further investigation of MKT in a secondary mathematics context is warranted.

School Mathematics Teaching Pedagogical Content Knowledge (SMTPCK)

In a series of studies (e.g., Chick, 2007; Chick et al., 2006; Chick & Beswick, 2018), Chick and colleagues developed and utilised the SMTPCK Framework to categorise and describe aspects of the work of mathematics teachers (See Appendix B). SMTPCK includes a set of "component knowledge areas" presented in one of three points along a continuum indicating the blend of pedagogy and content knowledge: (a) Clearly PCK (CPCK) in the centre of the continuum, (b) Content Knowledge in a Pedagogical Context (CKiPC) on one end of the continuum, and (c) Pedagogical Knowledge in a Content Context (PKiCC) on the other end of the continuum. For example, CPCK would include teacher knowledge related to the cognitive demand of mathematical tasks and the use of representations of mathematics concepts. Aspects of CKiPC would be more mathematical in nature, including teachers' knowledge about deconstructing content to key components and mathematical structures and connections. Finally, PKiCC includes knowledge that is more pedagogical in nature, such as developing learning goals and assessment strategies (see Chick & Beswick, 2018, pp. 479–482 for the complete framework; also see Appendix B). Chick and colleagues have used this framework in a wide range of mathematics education contexts (e.g., decimal understanding with primary teachers, Chick et al., 2006; calculus with secondary teachers, Maher et al., 2015). The developers describe the framework as providing a set of filters/lenses through which to examine elements of teachers' PCK.

Besides being grounded in Shulman's work and used by researchers to describe the knowledge and practices of mathematics teachers, MKT and SMTPCK have much in common. The frameworks were both designed by observing and talking with teachers about their work and both have been used across multiple mathematics topics, with most early work focused on elementary teachers' knowledge and practices related to numbers and operations. The developers of both frameworks have explicitly acknowledged the complex nature of isolating and exploring aspects of teachers' knowledge, the limitations of categorising knowledge, and the challenging, if not impossible, task of measuring the knowledge. Although these frameworks both seek to describe mathematical knowledge for teaching, there are differences as well. For example, PCK is one of two major domains of MKT; the other is SMK. In contrast, SMTPCK only includes aspects of PCK. Whereas the domains of MKT are represented in an oval (often referred to as the "egg") (see Ball et al., 2008, p. 403) with three domains of SMT on one side and three domains of PCK on the other side, SMTPCK is described as a continuum with "Clearly PCK" in the centre, more content-focused PCK on one end of the continuum, and more pedagogy-focused PCK on the other end. Although developers of both frameworks have emphasised the non-discrete nature of the domains, the continuum model underscores this feature.

Algebra Teaching and Learning

Given algebra's role as a gatekeeper to both advanced mathematics courses and career opportunities, Moses and colleagues deemed access to algebra a civil right (e.g., Moses & Cobb, 2001). As a result of this perceived role, successful completion of at least one algebra course is now required of most students in the United States to earn a high school diploma (Teuscher et al., 2008). In a literature review of "what is known about early and universal algebra" in the United States, Stein et al. (2011) confirmed that increasing numbers of students are taking algebra and taking it earlier, pointing out inconsistencies for who has access to early algebra and mixed outcomes for students taking early algebra across the studies. Hoffer et al. (2007) administered a national survey to algebra teachers who also reported challenges related to students' poor preparation and lack of motivation to learn algebra. In 2007, Kieran synthesised recommendations (e.g., National Council of Teachers of Mathematics, 1989, 2000) and research (e.g., Arcavi, 2003; Stacey & McGregor, 1999) that promoted a broader vision of teaching algebra be taken (e.g., multiple representations, realistic problem settings, use of technological tools). Included was a study that focused on symbolic manipulation and formula memorisation. Another was related to providing opportunities for students to make meaning of algebraic concepts. These calls for moving beyond a "letter-symbolic and symbol-manipulation view" (Kieran, 2007, p. 747) of algebra were certainly not novel, as evidenced by Kieran's inclusion of research conducted in the early twentieth century; rather, she continued to build this case. Recommendations for expanding the conceptions of what it means to know and do algebra beyond symbolic manipulation continue, including calls for more robust understandings of algebraic procedures and flexible use of such procedures (e.g., Litke, 2020); early algebraic thinking focused on equivalence, use of variables, and generalization (e.g., Blanton, 2022); and experiences building expressions, functions, and equations to model situations (AMTE, 2017).

Although algebra's place in both professional and school mathematics is well established and, as described above, researchers have studied many aspects of teaching and learning algebra, critical areas remain understudied. For example, although a fundamental part of current secondary mathematics curricula (e.g., IEA, 2013; OECD, 2019), few research studies have explored solving quadratic equations and inequalities.

Quadratic Equations and Inequalities

A search for research and practitioner articles focused on quadratic equations and inequalities uncovered few results but included research published in *Mathematics Teacher*, in which non-traditional pedagogical strategies were recommended (e.g., Allaire & Bradley, 2001; Gunter, 2016). There were also, however, several studies in which the authors described challenges faced by either secondary mathematics students (Eraslan & Aspinwall, 2007; Vaiyavutjamai & Clements, 2006; Zakaria & Maat, 2010) or teachers (Huang & Kulm, 2012) in their knowledge and teaching of quadratics.

Vaiyavutjamai and Clements (2006) expressed concern about the lack of attention to the study of quadratic equations in extant mathematics education research. In their study of students in Thailand, they conducted interviews before and after teaching lessons that focused on solving quadratic equations using three methods (i.e., factorisation, completing the square, quadratic formula). Although the students performed better after the lessons, the authors noted that most gains in knowledge were "rote learned knowledge and skills" and little "relational" understanding was evident in the post-lesson interviews, particularly for the lower achieving students. Of concern for the authors were students' "misconceptions" related to variables and the fundamental question of "what quadratic equations actually are" (p. 73). They proposed a functions approach to teaching quadratic equations as a promising alternative to "traditional" teaching methods used widely in mathematics classrooms around the world at that time.

Zakaria and Maat (2010) conducted an error analysis of the work of secondary mathematics students in Indonesia as they solved quadratic equations using the three methods studied by Vaiyavutjamai and Clements (2006). They noted that most student errors were transformation or process skill errors indicating challenges selecting appropriate solution methods and correctly performing the methods once selected. In an analysis of a tenth-grade student's work on problems involving quadratics, Eraslan and Aspinwall (2007) noticed several challenges encountered by the student, including recognizing connections and translating between representations of quadratic functions. They proposed that teachers facilitate explicit discussions about the benefits and limitations of various forms of quadratic equations and the relationships between the forms to address the challenges. Huang and Kulm (2012), in a study that aimed to identify challenging algebra topics for middle school teachers, included several items that used multiple representations to investigate teachers' understanding related to solving quadratic equations and inequalities. They highlighted errors related to limited knowledge about (1) using algebraic or graphic representations with flexibility, (2) negotiating the use of various forms of the equation, and (3) following algebraic operation properties.

All articles, regardless of whether focused on student or teacher learning and whether they explicitly addressed pedagogy or research, noted the challenges inherent in the teaching and learning of quadratic equations and inequalities. Therefore, we continue this work to contribute to the limited knowledge base related to the teaching and learning of quadratic equations. In particular, we attended to explanations given by M-PSTs, the knowledge they explicitly drew upon when solving the tasks, and how they anticipated responding to students.

Methods

Participants

The participants for the study were 12 M-PSTs enrolled in a mathematics teacher preparation program at a large Midwestern university in the United States; an open invitation was presented to the secondary mathematics methods course and all volunteers participated. Four of the M-PSTs were in a post-Baccalaureate teacher licensure program and held an undergraduate degree in a STEM field, while the other eight were in their final year of a teacher education program in which they were earning an undergraduate degree in mathematics education. Eight participants were female and four were male; all names used in this article are pseudonyms.

Data Collection

To investigate M-PSTs' knowledge, we conducted semi-structured, task-based interviews (Maher & Sigley, 2014) focusing on the knowledge that teachers need to teach content related to solving algebraic equations and inequalities. We conducted this study with the recognition that knowledge is unique to each teacher; that is, what each of them knows and can do is informed by their own learning experiences and informs their own teaching practices. The mathematics tasks utilised were designed to elicit what M-PSTs know and can do, as well as how they might work with students on the tasks. Many of the tasks also included opportunities for M-PSTs to review and reflect on hypothetical student responses.

A semi-structured interview protocol was used to ensure that the interviews were conducted in a consistent manner (Merriam & Tisdell, 2016); this protocol was piloted with two M-PSTs who were not currently in methods. Following the pilot, we refined the follow-up questions that would be asked when PSTs said or did specific things (e.g., Follow-up Question 2: If PST mentions "taking the square root" of both sides, ask them how they know that they can do this.). In addition, because the pilot interviewees expressed some trepidation when they were not sure how to complete a problem, we added a statement explaining more about our purposes: "From time to time, I may ask you questions such as: 'What are you thinking? Why did you do that? What does that mean? etc.' My questions are not to indicate that you are doing anything right or wrong. I only want to understand more about what you are thinking in each circumstance."

Although the full interview protocol included 12 tasks related to solving equations and inequalities (e.g., solving linear inequalities with visual representations, solving quadratic equations by factoring, solving rational equations), we have limited the scope of the research described in this article to three tasks that addressed quadratic equations. These tasks address aspects of solving quadratic equations that often pose challenges for students to learn and teachers to teach, including completing the square (see Task 1), using the discriminant and quadratic formula (see Task 2), and understanding extraneous solutions (see Task 3). We recognise that M-PSTs may have known and been able to do things that they did not demonstrate during the interview. As Maher and Sigley (2014) suggested, task-based interviews are "intended to elicit in subjects estimates of their existing knowledge, growth in knowledge, and also their representations of particular mathematical ideas, structures, and ways of reasoning" (p. 579).

The task-based interviews were video recorded and the M-PSTs' written work on each task was collected. A member of the research team transcribed each interview, and another checked it for accuracy.

Data Analysis

In order to avoid a deficit perspective of teacher knowledge, the M-PSTs' responses were analysed first by asking the questions: What do the M-PSTs seem to know? What are they able to do? Initially, two members of the research team independently used constant comparison to develop a set of open codes for what the M-PSTs said and did as they completed the subtasks, creating research memos throughout the process that detailed their decision-making and emergent questions (Glaser & Strauss, 1967; Strauss, 1987; Teppo, 2015). Once the set of codes were established collaboratively, the researchers engaged in a second round of independent coding in which they reviewed the transcripts, coding each using the set of finalised codes, and extracted transcript excerpts that provided evidence for the codes. More notes were added to the research memos throughout this part of the process to further refine the codes and identify additional codes. From that analysis a "task summary" was developed for each subtask, which included: (a) codes that identified what M-PSTs seemed to know about the task, and (b) codes that identified what M-PSTs did in response to the task.

To illustrate the analytic process, we briefly describe the analyses for Task 1, which asked M-PSTs to solve the quadratic equation, $6x^2 + 7 = 9x^2 - 41$. Nine codes were identified in responses to the question, "What do the M-PSTs seem to know?" and five codes in responses to, "What do the M-PSTs seem to be able to do?" For each code, we note how many and which participants were identified with the code, and present samples of transcript excerpts in Table 1.

Table 1
Task 1 Sample Codes

	Number of M-PSTs (N = 12)	Sample data excerpt
What do the M-PSTs seem to know?		
Taking the square root of a variable produces two solutions.	11	"Since it's a square root, there's two answers, plus and negative."
Taking the square root is the same as raising to the one-half power.	4	"You would have to square root x, and another way of thinking of that is x to the one-half power."
What do the M-PSTs seem to be able to do?		
Use square root procedure for solving	10	"But when we take the square root of both sides ... or when we have an expression such as x squared is equal to the 16 as we did, there's two ways ... two numbers that we can square to get to 16. And these numbers are 4 times 4 and negative 4 times negative 4. And both of these equals 16."
Verify solutions	4	"And then we would check the other one [begins to plug in negative four to see if it works] as well and it'll be the same."

The tasks and responses were examined through the lenses of the MKT and SMTPCK frameworks, documenting the types of knowledge that may have been needed and/or that M-PSTs used to complete the questions in each task. We acknowledge the challenges in categorizing teacher knowledge in this way, as do the developers of the frameworks, as well as the possibility that certain knowledge or actions could be categorized in multiple ways. Nevertheless, the categories in both frameworks provide information about the knowledge that may need further development in teacher education programs. To date, no study has used these two frameworks, with similar foundations and developed in parallel timeframes on different sides of the globe, to explore mathematics teacher knowledge using the same data set. In addition to our primary analysis, we consider our data through the two lenses.

Findings

In this section, we first present each task and discuss the types of knowledge elicited by the task. Then, we report the explanations given and knowledge drawn on by the M-PSTs as they completed the task.

Task 1

Explain how you would solve the following equation: $6x^2 + 7 = 9x^2 - 41$.

Through the lens of the MKT framework, the task required M-PSTs to draw on their CCK to solve the equation and their SCK to explain their reasoning. In the SMTPCK framework, the task elicited *Procedural Knowledge* within CKiPC for solving and *Explanations* within CPCK for explaining the solution.

Eleven of the 12 M-PSTs successfully solved the quadratic equation in Task 1. Several strategies were used by the participants. Ten participants used a procedure involving square roots to solve the quadratic as their first strategy. For example, Faith used a square root procedure after originally solving the equation by factoring, and Cassidy solved the equation using the quadratic formula. Using the language of MKT, it could be said that the M-PSTs drew on their CCK to solve the equation. In relation to SMTPCK, there was evidence of two categories within the CKiPC domain: *Procedural Knowledge* and *Methods of Solution* as the M-PSTs solved the equation.

We identified both similarities and differences in the actions and reasoning of the 11 M-PSTs that resulted in solving the equation $x^2 = 16$. Adam, the one participant who did not solve Task 1 successfully, responded with "4" (see Figure 1). While solving the equation, Adam stated, "I'm gonna get *x* squared equals sixteen, I think, and *x* equals four. So, yes, that's what I'm gonna do. Is that right?" This final question suggested that Adam was uncertain about his result. When describing how he might explain his solution to high school students, Adam stated, "*x* equals square root of sixteen. So, there's this step between here where you can't skip with kids. Cause then you understand that taking this [exponent] away means putting this [square root] in."

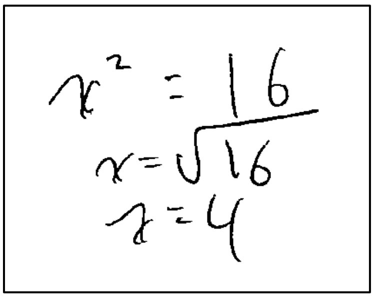


Figure 1. Adam's solution to $x^2 = 16$.

By acknowledging steps that you "can't skip with kids," Adam drew on his KCS in the MKT framework as he "anticipate[d] what students are likely to think and what they will find confusing" (Ball et al., 2008, p. 401). He demonstrated the use of two categories of knowledge in CPCK in the SMTPCK framework: *Cognitive Demand of Task* as he identified an aspect of the task that may be challenging and *Explanations* as he explained, albeit omitting one solution, his procedure.

As Isaac solved the equation (see Figure 2), he explained:

> The last thing we have here is *x* squared, which is the same as saying *x* times *x* ... So, taking the square root of both sides, we are left with 4 is equal to *x*. But when we take the square root of both sides, there are two ways, two numbers that we can square to get to 16. These numbers are 4 times 4 and negative 4 times negative 4 ... And both of these equals 16 ... And therefore, *x* is equal to 4 and *x* is equal to negative 4.

Like Adam, Isaac drew on his *Explanations* knowledge, but his explanation was enhanced as he described both how he solved it and why the procedure worked.

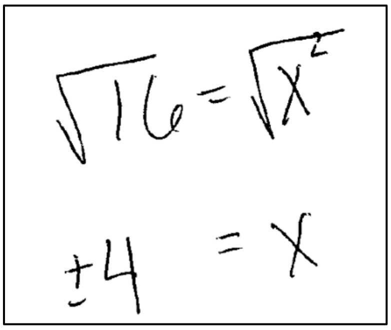


Figure 2. Isaac's solution to $x^2 = 16$.

When the interviewer prompted Isaac to describe how he knew he could take the square root of both sides, Isaac responded:

> You can take the square root because we know that this is *x* times *x* for whatever this *x*, … If we think of it in terms of exponents. It would be the same as taking each thing to the one half. So, let's see, 4, and when you raise an exponent to an exponent, you multiply exponents, so 2 times and then, to the one half, is *x* to the one, which is *x*.

Isaac continued to use his *Explanations* knowledge, as well as knowledge of *Structure and Connections* in CKiPC, to provide multiple ways to think about square roots. Four other M-PSTs demonstrated subtle variations of Isaac's work. For example, Faith did not include the square root symbol (see Figure 3).

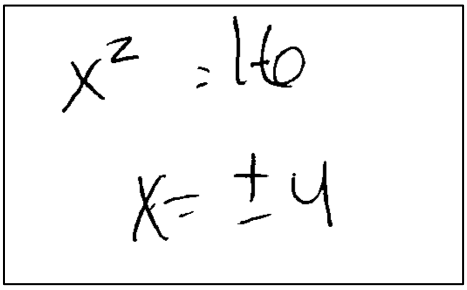


Figure 3. Faith's solution to $x^2 = 16$.

When Dakota was confronted with $16 = x^2$, the way she wrote and described her work was somewhat different (see Figure 4):

> So, we know that to undo a square, we do a square root because we always want to do the opposite. We divide to get rid of multiplication. We subtract to get rid of addition. We take square roots to get rid of squares. So, we have to do the same thing to both sides.... whenever we do take the square root, we have to add this plus or minus out front. So, square root of *x* squared takes away this squared, so it's just *x*. So, *x* can be a positive 4 or a negative 4...A lot of the college kids don't like this plus or minus, so, I would guess high school kids probably forget it, too.

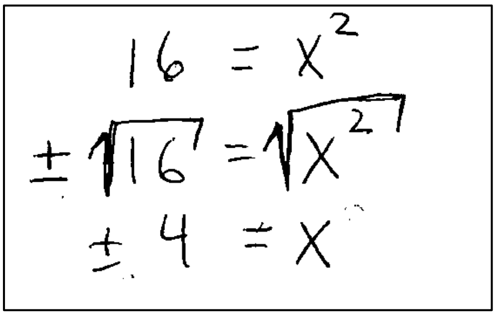


Figure 4. Dakota's solution to $x^2 = 16$.

Dakota compared taking the square root "to get rid of squares" to the relationships between other operations (e.g., subtracting to get rid of addition). She emphasised that one must add the plus or minus sign and drew on her KCS about college students to generalise this reasoning to high school students. In her work and description, Dakota demonstrated knowledge from four SMTPCK categories, three in CPCK (i.e., *Student Thinking, Student Affect, Explanations*) and one in CKiPC (i.e., *Structure and Connections*). Three other M-PSTs included notation like Dakota's, with some variation. For example, Kassidy wrote the square root symbol on the 16 only, as shown in Figure 5.

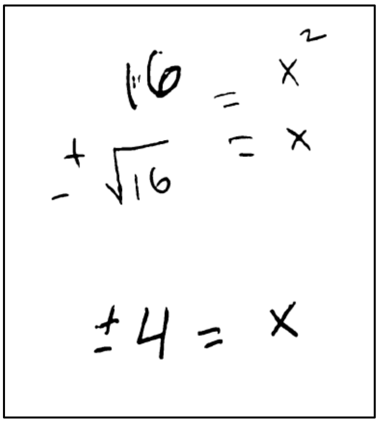


Figure 5. Kassidy's solution to $x^2 = 16$.

At the same point in his solution, Jackson showed the work in Figure 6 and responded,

> So, *x* equals plus minus four, and the reason it's plus minus four, cause if you square a number, it's always going to be positive. So, I guess the step before that actually, when you take the square root of an *x*-squared, you should get absolute value of *x* equals four. Which means the same thing, for students, to be plus or minus four.

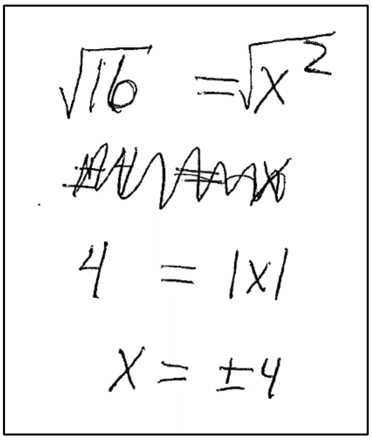


Figure 6. Jackson's solution to $x^2 = 16$.

Although he initially showed notation like Isaac's solution (scribbled out in Figure 6), Jackson noted that the square root of x^2 was equal to the absolute value of x; this justification was unique to Jackson among the 12 M-PSTs' solutions. He demonstrated knowledge of *Profound Understanding of Fundamental Content* and *Structure and Connections*, going beyond the square root symbols used by other participants to represent the solution using absolute value notation.

In summary, the M-PSTs drew on their CCK (in MKT) or CPCK (in SMTPCK) to solve the equation and use procedures for simplifying the square root. Several M-PSTs accessed their KCS as they discussed taking the square root as a means of "undoing" the exponent in a process similar to subtracting to "undo" addition, to make a connection for student learning about exponents; they were demonstrating their knowledge of *Structure and Connections* (in CKiPC). In determining that there were two solutions, the M-PSTs also included differing notation, with one M-PST introducing absolute value. Because Task 1 did not specify which method to use, the M-PSTs had flexibility to choose their approaches, which gave them the opportunity to discuss why they might use a particular method instead of another when solving the equation, thereby utilising their SCK and *Methods of Solution* (CKiPC).

Task 2

a) Sometimes we use the quadratic formula to solve quadratic equations. What do you remember about the quadratic formula?

b) How would you explain to someone why the quadratic formula works?

c) A lot of textbooks suggest using the discriminant to help make inferences about how many real roots a quadratic equation has. Your textbook includes the following table:

Number of Solutions to a Quadratic Equation

Value of the discriminant	Number of real solutions	Number of *x*-intercepts
$b^2 - 4ac > 0$	2	2
$b^2 - 4ac = 0$	1	1
$b^2 - 4ac < 0$	0	0

One of your students, Marcus, is trying to memorize the table, but keeps getting them mixed up.

What could you tell Marcus to help him understand why the discriminant provides information about the real roots of a quadratic equation?

As M-PSTs were asked to explain why the quadratic formula works and to make connections between the number of real roots of a quadratic equation and the value of the discriminant, the task required M-PSTs to draw upon their SCK and KCS. In the SMTPCK framework, we identified the categories of *Explanations* within CPCK and *Structure and Connections* within CKiPC for explaining why the quadratic formula works and building the connection with the discriminant, and *Student Thinking* within CPCK for explaining the table to a struggling student.

In response to Task 2, all 12 M-PSTs recalled using the quadratic formula. Eleven M-PSTs correctly stated the formula and four mentioned using a song to remember it. For example, Candice said, "I like to sing it to *Pop Goes the Weasel*, and that helps me remember it." Two M-PSTs discussed the discriminant and its effect on the quantity and nature of the roots, exhibiting their SCK not directly asked about in Task 2a. For instance, Kassidy said:

> And this is called, under here is called, the discriminant [pointed to b^2-4*ac*] and it tells us about what our answers are going to look like. And so, if this answer is a positive number, we will end up with *x* equaling two real answers. If it is zero, we will have *x* having a double root or just one real answer, and if it is negative, then we end up with two complex answers.

Similarly, Candice explained how she would draw on her knowledge in the classroom:

> You could talk about this inner term here [pointed to b^2-4*ac*] or the discriminant and talk about how whatever that value is will tell you what kinds of solutions you're gonna have … and talk about how they can sort of spot that before, by looking at the discriminant.

In response to Task 2b, four M-PSTs described the relationship between the quadratic formula and the general form of a quadratic equation, exhibiting their knowledge of *Structure and Connections*, as well as *Methods of Solution*. For example, Belinda stated:

> This is just a generalisation of solutions when you have *a x*-squared plus *b x* plus *c* equals zero, where you're solving for *x*. And then it's kind of like an application of completing the square, where you break up your terms and you make it fit … based on your general coefficients here, so instead of like numbers, the generalised completing the square will give you the quadratic formula.

Whereas most M-PSTs indicated that the quadratic formula was something to be memorised, three M-PSTs, like Belinda, mentioned that the formula can be derived by completing the square, utilising their SCK to recognise that the quadratic formula is an extension of completing the square. Gabe was uncertain, at first, how to explain the derivation of the formula:

> I don't think in my entire school career I've ever derived why that works. We've always been like, oh, this is what you do when you have quadratic equations that don't factor nicely. You put them in this, and you figure it out from there.

However, Gabe eventually derived the formula for himself during the interview:

> And then you complete the square on this side so you can get an even factoring. So, that you can make it a nice square root number. So, I would take, divide this by two, so *x*-squared plus and then square it, and there's the quadratic formula.

Three M-PSTs also discussed the relationship between the quadratic formula and the expression for the *x*-coordinate of the vertex. For example, Isaac said:

> I was thinking, we know this is a quadratic ... And therefore, it's going to be in the form of some polynomial ... or of a parabola. And the vertex of a parabola could be found by *b* divided by 2*a* ... but that doesn't tell us anything about why it works. Um, I don't know.

Belinda mentioned the vertex in a different way, by describing the graph of the parabola and the solutions that the quadratic formula represented. She stated:

> All I know is negative *b* over 2*a* is the vertex of a parabola ... but I don't know how that would solve. Like if you have a graph, that gives you this point, [drew parabola and darkened vertex], but we're trying to find these points. So, I actually don't know why it works. I never learned that or asked. I just believed them.

Like Isaac, Belinda described the vertex of the parabola, but stopped short of recognising its relationship to the quadratic formula. Further, both M-PSTs acknowledged the difference between knowing and understanding this relationship.

In response to Task 2c, nine M-PSTs mentioned that the discriminant is part of the quadratic formula. Emma stated, "So, yes this doesn't have a square root but in the quadratic formula it does." Ten M-PSTs indicated that the discriminant has three cases, because it is under a square root. For instance, Hollie explained:

> Just think of it as a square root. I mean, can you have a square root of a negative number? Well, technically, not right now ... So that's why it's 0 ... And then what is the square root of 0? Well, that's 0. So that's only one solution. And think of a square root of some positive number ... that's greater than 0, it can go to at most two, plus or minus.

Three M-PSTs related the discriminant to the graph of the equation, further demonstrating their knowledge of *Structure and Connections* in CKiPC and *Representations of Concepts* in CPCK. Leslie stated (see Figure 7), "Like drawing pictures helps a lot of students because they can't just memorize the formula. But having a graph of what it's supposed to look like helps represent like, just somehow makes that connection."

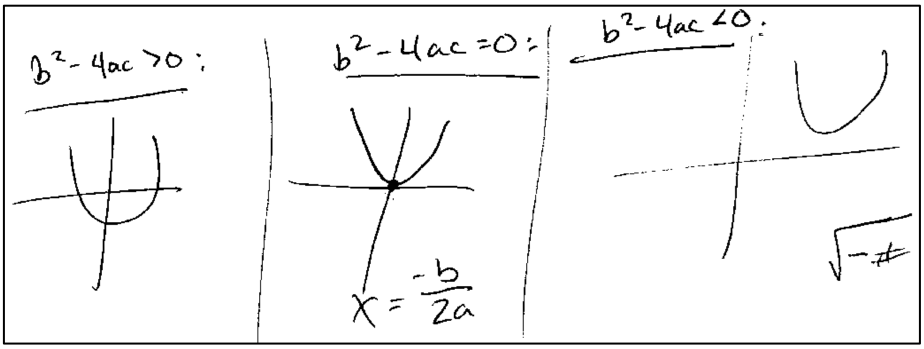


Figure 7. Leslie's explanation of Task 2c.

Leslie drew upon her SCK, suggesting that a visual representation can help students make connections to enhance their understanding, aside from the simple memorisation of a formula.

In Task 2, the M-PSTs discussed the quadratic formula and identified ways to help students remember the formula and how and why it works. Before responding to Task 2c, several M-PSTs had already discussed connections to the graph of the quadratic equation, indicating the relationship between the number of *x*-intercepts and the discriminant. Although much of the teacher knowledge utilised in Task 2 would be considered SCK in MKT, SMTPCK's categories delineate specific types of CPCK and CKiPC. For instance, Leslie's identification of three outcomes of the discriminant and their corresponding graphs revealed her knowledge of CPCK, as she demonstrated *Representations of Concepts* and *Knowledge of Examples*. Throughout the course of the interview, Gabe seemed to enhance his CKiPC as he explained how to derive the quadratic formula. He drew on his knowledge of *Structure and Connections* to develop profound understanding of the relationship between completing the square and the quadratic formula. Similarly, Kassidy and Candice used their CKiPC and CPCK in descriptions of the discriminant to determine the solution types, demonstrating their *Procedural Knowledge,* knowledge of *Structure and Connections*, and knowledge of *Representations of Concepts*.

Task 3

a. One of the sections in your mathematics textbook focuses on completing the square. Complete the square to solve the following equation: $3x^2 + 18x - 16 = 5$
b. One of your students, Ava, can usually make sense of mathematics if she can visualize why the method works. She's really struggling to understand the method of completing the square. How would you draw a visual representation to help Ava?

In terms of MKT, Task 3a may appear to only be eliciting M-PSTs' CCK; however, because they were asked to use a particular method (arguably one that is not the most efficient and likely not often used in other professions) suggests the need for SCK. In fact, a case could be made for KCT as well, given that this domain includes the expectation for teachers to understand multiple approaches to problems and their instructional affordances. Task 3b elicits KCT and KCS as the M-PST needs knowledge of a range of possible representations (i.e., KCT) to select an appropriate one that will be most helpful to students (i.e., KCS). Given the content-pedagogy continuum structure of SMTPCK, the knowledge elicited in Task 3 is situated in CKiPC and CPCK. In particular, the M-PSTs are asked to draw on *Methods of Solutions* in CKiPC (Task 3a) and *Representations of Concepts* in CPCK (Task 3b).

When asked to complete the square in Task 3a, 10 M-PSTs first simplified the equation by dividing by three. For example, Dakota said, "So the first thing you always want to do is have the number in front of *x* be 1. So, to do that, we can divide both sides by 3." Dakota provided an example of *Explanations* (CPCK) as she went beyond saying what she was doing to share her goal for a particular procedure; this statement also demonstrates Dakota's SCK. Others, like Adam, expressed uncertainty. After subtracting five from both sides, Adam stated,

> I mean, these are all divisible by three; that seems like a clue. If I were to divide everything by three, would that give me something? Let's try it. I'm gonna divide everything by three. Am I allowed to do that? I don't even know if I'm allowed to do that.

Seven M-PSTs correctly completed the square to solve the equation. For instance, Dakota said:

> So, this gives us *x* plus 3 equals 4 for the plus 4, and *x* plus 3 equals negative 4 for the minus. And so, then this is just an equation that we probably recognize how to solve. By subtracting 3 from both sides, we get *x* equals 1 and *x* equals negative 7.

Other M-PSTs remembered something about the procedures involved but expressed uncertainty. For example, Hollie stated, "It has something to do with taking half of *b* and either adding it or subtracting it." Several M-PSTs who correctly solved the equation also shared doubts, like Belinda, "So, *x* equals 3 plus or minus 4. So, *x* equals 7 or -1. I'm not even sure if that's right."

Although not explicitly asked in the question, two M-PSTs drew on their KCT (in MKT) or *Representations of Concepts* (in CKiPC) as they described the solutions of the equation as being *x*-intercepts of the graph. Adam stated, "See where the *x*-axis, where the line crosses the *x*-axis. That's probably how I would have solved it on that test." Two M-PSTs also described why completing the square may be useful. For example, Dakota said, "And then it's just an alternate way of the quadratic equation to solve it when we can't factor easily."

In response to Task 3b, three M-PSTs drew on their KCT and *Representations of Concepts*, as they recollected using a diagram of a square to demonstrate completing the square visually. However, it was difficult for them to remember the corresponding explanation. Candice stated (see Figure 8),

> I recently saw, sort of a visual representation of this, let's see if I can reproduce it. If you really think of completing the square, you have your first value here [drew a large square and darkened left side]. ... And then you have some section and another section [drew smaller rectangles near the bottom of the square and darkened in the top side of the larger square], where you have your first terms, like this. And then when you multiply them together, that's great, but then you want to fill out this space, where this is missing. And you have to add that on, in order to complete the whole square or the whole picture that you have here. And you'd have an actual representation where this is something like three *x*-squared here [wrote $3x^2$ along the left side], and then your other piece like the thing in the parentheses. Such that when you're multiplying them together, you get this.

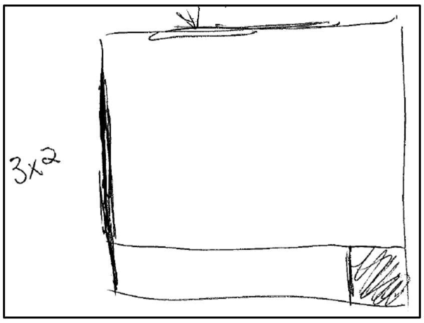


Figure 8. Candice's work for Task 3b.

Isaac mentioned learning this strategy in a university course that included a teaching seminar that supported instruction in a college algebra course, but he could not remember the explanation (see Figure 9), "What we're doing, we can think of it as a box here. And it was, how did we do that? I specifically learned this, and I can't remember the reasoning."

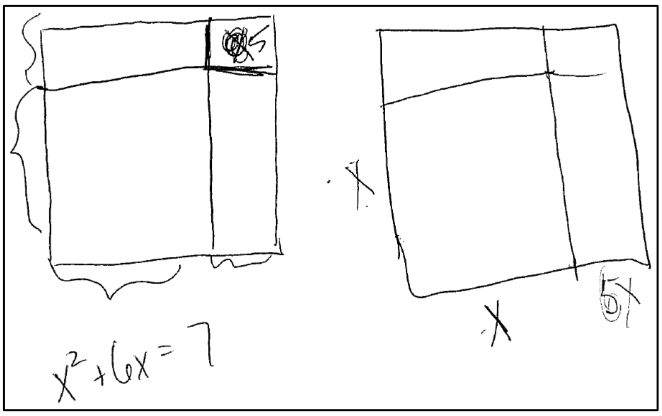


Figure 9. *Isaac's work for Task 3b.*

Although several M-PSTs, like Candice and Isaac, attempted to create an area model to represent the process, they exhibited difficulty and uncertainty in explaining how their model could be interpreted to foster mathematical connections between the symbolic and the visual representations. Whether we contend that SCK is a type of teacher knowledge that involves creating and interpreting multiple representations of mathematical phenomena, or if we assert that this representational fluency is CPCK in SMTPCK, the M-PSTs in this study struggled to create a meaningful representation to model the process of completing the square. The majority of the M-PSTs were able to draw on their CCK in 3a to complete the square. This would be considered *Procedural Knowledge*, CKiPC in SMTPCK. The M-PSTs did not completely create and explain a visual representation in 3b, either because they had never had the opportunity to learn about such a representation or because they could not recall how to reproduce it.

In Task 3, M-PSTs were asked to both solve a quadratic equation by completing the square and to create a visual representation to represent this process. Most of the M-PSTs were able to correctly solve the equation; however, although several M-PSTs recalled seeing a visual representation related to completing the square, none of them provided a complete representation or were able to explain it correctly. Several M-PSTs expressed doubt about their solutions for both 3a and 3b, calling into question their ability to draw on the KCT and CPCK required to complete this task.

Discussion

In this study, we investigated what 12 M-PSTs seemed to know and be able to do related to solving quadratic equations on three tasks. In addition to the descriptive summaries provided, we considered their responses through the lenses of the knowledge domains described in MKT and SMTPCK, two frameworks designed to illustrate knowledge for teaching mathematics. Viewing the data through the lenses of both frameworks afforded us the opportunity to better understand the data and consider each framework's affordances and limitations. Here, we summarise our findings, discuss the use of the two frameworks, and suggest implications for research and teacher education.

Summary of Findings

Across the three tasks, we asked M-PSTs to investigate problems that involved solving quadratic equations using various methods that required knowledge of the discriminant and completing the square. This set of tasks elicited and provided opportunities for M-PSTs to demonstrate their CCK, SCK, KCS, and KCT (in MKT). In addition, knowledge from both CPCK (e.g., *Cognitive Demand of Task*, *Representations of Concepts*) and CKiPC (e.g., *Structure and Connections*, *Methods of Solutions*) in the SMTPCK framework, was elicited and demonstrated. We noted that many of the M-PSTs were able to draw on their CCK (in MKT) and CKiPC (in SMTPCK) for solving quadratic equations, stating the quadratic formula, discussing how the discriminant was related to the number of real roots of the equation, and completing the square. However, the M-PSTs more often struggled and expressed uncertainty when asked to draw on their SCK (in MKT) and CPCK (in SMTPCK).

Investigations of MKT and SMTPCK

Exploration of M-PSTs' task responses through the lenses of MKT and SMTPCK provided insights into the frameworks. As described earlier, these two frameworks have much in common, including their foundations in Shulman's (1986) notion of PCK, their multi-dimensional structure, and their goal of describing the knowledge needed to teach mathematics. There are also differences between the frameworks, most notably the way that the domains are organised, and the number of categories detailed within the domains. As demonstrated in our findings, both frameworks were productive structures for exploring M-PST's knowledge, allowing us to highlight what M-PSTs knew and were able to do as well as knowledge and skills that they had less experience with and, therefore, found more challenging.

Through the lens of MKT, we found that M-PSTs demonstrated their CCK, SCK, KCS, and KCT; however, our findings raised questions about using the MKT framework at the secondary level. Like Speer et al. (2015), at times we struggled to differentiate between CCK, SCK, and KCS in the secondary mathematics context explored in this study. This was true both in the tasks as presented to the M-PSTs as well as their responses to the tasks. For example, in Task 2, several M-PSTs connected the solutions of a quadratic equation to the graphical representation without being prompted by the written tasks or the interviewer. It seems that a case could be made that this knowledge is CCK (i.e., many professionals who are not teachers would know this), SCK (i.e., this is specialised knowledge within the mathematical domain that is unique to teaching), or KCS (i.e., this knowledge is useful when helping students make sense of complex mathematical concepts). Ball et al. (2008) described a similar dilemma involving teaching fractions; however, as Speer et al. (2015) suggested, these boundaries are even more blurry in the secondary context.

We were also left with questions about how the evolution of curriculum standards might impact how we think about CCK and SCK. As standards are now often written to require a deeper understanding of mathematical concepts than previous standards (e.g., students are often asked to justify their answers), will this change what is considered CCK or SCK? That is, if secondary students are expected to graduate from high school with knowledge, for example, about how multiple representations are related to one another, does this SCK then become CCK? Related to this, can procedural fluency ever be considered SCK, and can conceptual understanding ever be considered CCK? It could be argued that these distinctions are not important and perhaps the fuzziness of these domains is to be expected. However, to make the constructs useful for research, more work is likely needed to establish boundaries between or perhaps sub-domains within them. Lai and Clark (2018) have taken up this work, proposing a model for three constructs within SCK: Justification, Explanation, and Representation. This creation of categories in the domains (beyond the descriptors provided in the original framework) moves the MKT framework toward the level of detail provided in the most recent version of the SMTPCK framework (i.e., Chick & Beswick, 2018, pp. 479–482; also see Appendix B).

The SCK constructs included in Lai and Clark's (2018) framework (justification, explanation, and representation; see p. 81 of their article) were highlighted in our findings. For example, the quadratic equation in Task 1 was solved correctly by 11 of the 12 M-PSTs (i.e., justification). Adam and Isaac both provided explanations for their result (see Figures 1 and 2 and accompanying text); however, Adam's explanation was *procedural* (i.e., how to execute the algorithm) whereas Isaac's explanations were *conceptual* (i.e., addressed the underlying mathematical reasoning). In Task 2c in which M-PSTs were asked about the discriminant, several non-visual representations were provided as the students described how the discriminant was related to the graph; however, Leslie drew a visual representation (see Figure 7) to illustrate this relationship. Lai and Clark's SCK constructs are all addressed within CPCK of the SMTPCK framework.

M-PSTs demonstrated knowledge of 10 categories within two domains of the SMTPCK framework: in particular, seven categories within CPCK and three categories within CKiPC. Given the mathematical nature of the tasks used in our interviews, evidence of PKiCC was absent. Interestingly, the most common SMTPCK categories demonstrated were *Procedural Knowledge*, *Explanations*, *Methods of Solution*, and *Representations of Concepts*, which were closely related to Lai and Clark's (2018) proposed SCK constructs. As when using the MKT framework, it was not always clear how to categorise a M-PST's response. However, the "Evident when a teacher…" and "Example" sections of Chick & Beswick's most recent version of the framework were helpful for making these decisions. Still, there is much overlap between categories; this non-discrete aspect of categories of teacher knowledge is present in both frameworks, recognised by the developers, and likely unavoidable. We found the idea of a continuum between CKiPC (in which mathematics is foregrounded) and PKiCC (in which pedagogy is foregrounded), and with CPCK (in which the mathematics and pedagogy are "inextricably linked") in the centre of the continuum, helpful for our investigation of M-PSTs' knowledge. This representation of SMTPCK as a continuum accommodates the blurring of boundaries between domains and categories, which is not facilitated by the area model with discrete boundaries used to represent MKT.

Implications for Research and Teacher Education

Our study was possibly the first to simultaneously explore the use of the two frameworks, MKT and SMTPCK, to investigate teacher knowledge utilising the same data set; there is much more to be learned about these frameworks and their connections and possibilities associated with those connections through further research. As mentioned earlier, Lai and Clark (2018) have begun to develop constructs within SCK, one of the MKT domains, moving it in the direction of the categories outlined in SMTPCK. Our initial analysis revealed that MKT's CCK, SCK, KCS, and KCT domains often intersect with categories within the CPCK and CKiPC domains in SMTPCK. Additional analyses are needed to continue to understand these intersections and how best to categorise this knowledge in ways that are useful to researchers and mathematics teacher educators. As previous research has suggested, further consideration of secondary mathematics content in both frameworks is needed given the frameworks' origins and use in elementary mathematics contexts.

Given the current requirement for nearly all students to take at least one algebra course in secondary school and algebra's persistent role as a gatekeeper to postsecondary career and college opportunities, the knowledge and preparation of M-PSTs to teach algebraic topics, in this case solving quadratic equations, is worthy of investigation. Our findings suggest that using tasks that are embedded in student thinking has the potential to promote discussions with M-PSTs about developing procedural fluency, conceptual understanding, and mathematical reasoning in algebra classrooms. For example, Marcus' (the fictitious student in Task 2c) need for further explanation provided an opportunity for M-PSTs to consider strategies to develop conceptual understanding of the discriminant. Such discussions offer opportunities for M-PSTs to both use and likely enhance various domains of knowledge needed for teaching mathematics. The uncertainty reported, at times, by the M-PSTs seem to support calls for "connecting" courses that provide opportunities for mathematics teachers to engage with and reflect on secondary mathematics to understand it more deeply and, thus, to be better prepared to teach it to their future students (e.g., CBMS, 2012; Murray et al., 2018).

Given the rich discussions that our research team engaged in while using the MKT and SMTPCK frameworks to investigate teacher knowledge, it is worth asking if such questions and investigations would be interesting to and educative for M-PSTs. Perhaps, regardless of the framework under consideration, the professional discussions about the knowledge domains are as important as the knowledge in the domains themselves. In fact, it is likely that explicit discussions of the domains within the context of mathematical tasks would serve to develop knowledge in the domains. Perhaps instead of seeking ways to measure teacher knowledge, time would be better spent engaging M-PSTs and practicing teachers in reflection on their own knowledge (including domains in which they feel comfortable and domains in which they wish they knew more) through the MKT or SMTPCK framework and designing activities and tasks to further develop aspects of teacher knowledge that they identify in their investigation.

Conclusion

Considering the movement for "algebra-for-all" continues and algebraic reasoning is required for mathematical success in secondary and postsecondary education (e.g., Stein et al., 2011), M-PSTs need opportunities to develop their knowledge for teaching algebra. It is imperative that teachers have the knowledge and skills related to solving quadratic equations and other algebraic topics to serve their future students. Our study found that M-PSTs were often able to draw on their CCK (in MKT) and their *Procedural Knowledge* (in SMTPCK) to solve equations and recall formulas but struggled with being able to draw on more conceptual knowledge to explain why procedures worked and to express their understanding of the equations. In fact, the M-PSTs expressed frustration that they had not had opportunities to develop this knowledge. Although Mathematics for Teachers courses have become commonplace in elementary teacher education programs, they have yet to become standard in secondary programs (Newton et al., 2014). Perhaps some assume that M-PSTs can develop such knowledge on their own as they take standard mathematics courses (e.g., Linear Algebra, Discrete Mathematics) along with mathematics methods courses. However, the M-PSTs in this study often expressed doubt about their algebraic knowledge and skills when asked questions that went beyond standard algorithms for solving quadratic equations.

It seems difficult to imagine that new teachers who have not had rich learning experiences in secondary-level mathematics content will be able to provide rich experiences for their own students without interventions during their teacher education program. Some textbook writers have attempted to provide curricula for courses to offer rich learning experiences for M-PSTs related to secondary curriculum (e.g., Cuoco & Rotman, 2013; Sultan & Artzt, 2010); however, the specific content for the courses described in CBMS (2012) remains uncertain. What specific types of teacher knowledge and competencies might become a focus for such courses? What types of activities could instructors utilise to support the development of such knowledge and competencies? For example, in the context of algebra, the samples of solving the equation $x^2 = 16$ provided in Figures 1 through 6 could be used in a course for teachers to consider the diverse thinking about the Square Root Property; M-PSTs could analyse the mathematical notation, conceptions evidenced, and potential reasoning. Our findings suggest that secondary mathematics teacher education programs need to provide such courses, including the challenging task of creating space in programs before progress can be made toward the goals of M-PSTs developing a deeper understanding of secondary mathematics topics and how their students approach these topics. The development of such courses should take seriously the work of the developers of the MKT and SMTPCK frameworks. How, in the set of courses in our secondary mathematics teacher education programs, are we addressing the various domains of teacher knowledge? How can we use these frameworks in course and program design, both in terms of attention to each category and domain, but also how can we provide opportunities for M-PSTs to reflect on their own knowledge in these domains? The findings here suggest such attention is critically important as we seek to prepare M-PSTs to teach algebra in more conceptual, engaging, and meaningful ways.

References

Allaire, P. R., & Bradley, R. E. (2001). Geometric approaches to quadratic equations from other times and places. *Mathematics Teacher*, *94*(4), 308–315. https://www.jstor.org/stable/20870675

Alvey, C., Hudson, R. A., Newton, J. & Males, L. M. (2016). Secondary pre-service teachers' algebraic reasoning about linear equation solving. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal, 1*. https://files.eric.ed.gov/fulltext/EJ1115162.pdf

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215–241. https://doi.org/10.1023/A:1024312321077

Asquith, P., Stephens, A. C., & Knuth, E. J., & Alibali, M. W. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. *Mathematical Thinking and Learning*, *9*(3), 249–272. https://doi.org/10.1007/BF02655899

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. https://amte.net/standards

Ball, D. L., Sleep, L., Boerst, T. A., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *The Elementary School Journal*, *109*(5), 458–474. https://www.jstor.org/stable/10.1086/596996

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407. https://doi.org/10.1177/0022487108324554

Blanton, M. (2022). The role of learning progressions in "democratizing" students' access to algebra. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Middle Tennessee State University (pp. 30-42).

Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, A., Ribeiro, M., & Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialised knowledge (MTSK) model. *Research in Mathematics Education*, *20*(3), 236–253. https://doi.org/10.1080/14794802.2018.1479981

Chick, H. L. (2007). Teaching and learning by example. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice (Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 3–21). MERGA. https://files.eric.ed.gov/fulltext/ED503746.pdf

Chick, H., & Baker, M., Pham, T., & Cheng, H. (2006, July). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th Conference of* *the International Group for the Psychology of Mathematics Education* (pp. 297–304). PME. http://www.igpme.org/publications/current-proceedings/

Chick, H., & Beswick, K. (2018). Teaching teachers to teach Boris: A framework for mathematics teacher educator pedagogical content knowledge. *Journal of Mathematics Teacher Education*, *21*(5), 475–499. https://doi.org/10.1007/s10857-016-9362-y

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. American Mathematical Society and Mathematical Association of America. https://www.cbmsweb.org/archive/MET2/met2.pdf

Copur-Gencturk, Y., Tolar, T., Jacobson, E., & Fan, W. (2019). An empirical study of the dimensionality of the mathematical knowledge for teaching construct. *Journal of Teacher Education*, *70*(5), 485–497. https://doi.org/10.1177/0022487118761860

Cuoco, A. & Rotman, J. J. (2013). *Learning modern algebra: From early attempts to prove Fermat's last theorem*. Mathematical Association of America.

Depaepea, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and Teacher Education, 34,* 12-25. https://doi.org/10.1016/j.tate.2013.03.001

Eraslan, A., & Aspinwall, L. (2007). Quadratic functions: Students' graphic and analytic representations. *Mathematics Teacher*, *101*(3), 233–237. https://www.jstor.org/stable/20876091

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* Aldine de Gruyter.

Gunter, D. (2016). Jumping to quadratic models. *Mathematics Teacher, 110*(3), 174–180. https://www.jstor.org/stable/10.5951/mathteacher.110.3.0174

Heid, M. K., Wilson, P. S., & Blume, G. W. (Ed.). (2015). *Mathematical understanding for secondary teaching: A framework and classroom-based situations*. Information Age.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers’ mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371–406. https://doi.org/10.3102/00028312042002371

Hoffer, T. B., Venkataraman, L., Hedberg, E. C., & Shagle, S. (2007). *Final report on the National Survey of Algebra Teachers for the National Math Panel*. http://www2.ed.gov/about/bdscomm/list/mathpanel/final-report-algebra-teachers.pdf

Huang, R., & Kulm, G. (2012). Prospective middle grade mathematics teachers’ knowledge of algebra for teaching. *Journal of Mathematical Behavior, 31*(4), 417–430. https://doi.org/10.1016/j.jmathb.2012.06.001

International Association for the Evaluation of Educational Achievement. (2013). *TIMSS 2015 assessment frameworks*. TIMSS & PIRLS International Study Center, Lynch School of Education, Boston College.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203–235. https://doi.org/10.1023/B:JMTE.0000033084.26326.19

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Information Age.

Kind, V. (2009). Pedagogical content knowledge in science education: Perspectives and potential for progress. *Studies in Science Education, 45*(2), 169-204. https://doi.org/10.1080/03057260903142285

Koehler, M. J., Mishra, P., Kereluik, K., Seob Shin, T., & Graham, C. R. (2014). The Technological Pedagogical Content Knowledge Framework. In J. M. Spector, M. D. Merrill, J. Elan, & M. J. Bishop (Eds.), *Handbook of research on educational communications and technology* (pp. 101-111). Springer.

Kunter, M., Klusmann, U., Baumert, J., Richter, D., Voss, T., & Hachfeld, A. (2013). Professional competence of teachers: Effects on instructional quality and student development. *Journal of Educational Psychology*, *105*(3), 805. https://doi.org/10.1037/a0032583

Lai, M. Y., & Clark, J. (2018). Extending the notion of Specialized Content Knowledge: Proposing constructs for SCK. *Mathematics Teacher Education and Development*, *20*(2), 75–95. https://mted.merga.net.au/index.php/mted/article/view/403

Litke, E. (2020). The nature and quality of algebra instruction: Using a content-focused observation tool as a lens for understanding and improving instructional practice. *Cognition and Instruction, 38*(1), 57-86. https://doi.org/10.1080/07370008.2019.1616740

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers’ understanding of fundamental mathematics in China and the United States*. Lawrence Erlbaum.

Maher, C. A., & Sigley, R., & S. (2014). Task-based interviews in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 579–582). https://doi.org/10.1007/978-94-007-4978-8

Maher, N., Muir, T., & Chick, H. (2015). Examining PCK in a senior secondary mathematics lesson. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics Education in the Margins* (Proceedings of the 38th Annual Conference of the Mathematics Education Research Group of Australasia) (pp. 389–396). MERGA. https://files.eric.ed.gov/fulltext/ED572479.pdf

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M. D., & Senk, S. L. (2012). Knowledge of algebra for teaching: A framework of knowledge and practices. *Journal for Research in Mathematics Education, 43*(5), 584–615. https://www.jstor.org/stable/10.5951/jresematheduc.43.5.0584

Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative research: A guide to design and implementation.* Jossey-Bass.

Moses, R. P., & Cobb, C. E. (2001). *Radical equations*. Beacon.

Murray, E., Durkin, K., Chao, T., Star, J. R., & Vig, R. (2018). Exploring connections between content knowledge, pedagogical content knowledge, and the opportunities to learn mathematics: Findings from the TEDS-M dataset. *Mathematics Teacher Education and Development, 20*(1), 4–22. https://mted.merga.net.au/index.php/mted/article/view/310

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Author.

Newton, J., Maeda, Y., Alexander, V., & Senk, S. (2014). How well are secondary mathematics teacher education programs aligned with the recommendations made in *MET II*? *Notices of the American Mathematical Society*, *61*(3), 292-295. https://www.ams.org/journals/notices/201403/201403-full-issue.pdf

Organisation for Economic Co-operation and Development. (2019). PISA *2018 assessment and analytical framework*. OECD. https://doi.org/10.1787/19963777

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255–281. https://doi.org/10.1007/s10857-005-0853-5

Schifter, D. (2001). Learning to see the invisible: What skills and knowledge are needed to engage with students' mathematical ideas? In T. Wood, B. S. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 109–134). Erlbaum.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://www.jstor.org/stable/24636916

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22. https://doi.org/10.17763/haer.57.1.j463w79r56455411

Speer, N. M., King, K. D., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*, *18*, 105–122. https://doi.org/10.1007/s10857-014-9277-4

Stacey, K., & McGregor, M. (1999). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior, 18*, 149–167. https://doi.org/10.1016/S0732-3123(99)00026-7

Stein, M. K., Kaufman, J. H., Sherman, M., & Hillen, A. F. (2011). Algebra: A challenge at the
crossroads of policy and practice. *Review of Educational Research*, *81*(4), 453–492. https://doi.org/10.3102/0034654311423025

Strauss, A. L. (1987). *Qualitative analysis for social scientists*. Cambridge University.

Sultan, A., & Artzt, A. F. (2010). *The mathematics that every secondary school math teacher needs to know.* Routledge.

Teppo, A. R. (2015). Grounded theory methods. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (pp. 3-21). Springer.

Teuscher, D., Dingman, S. W., Nevels, N. N., & Reys, B. J. (2008). Curriculum standards, course requirements, and mandated assessments for high school mathematics. *Journal of Mathematics Education Leadership, 10*(2), 49–55.

Vaiyavutjamai, P., & Clements, M. A. (2006). Effects of classroom instruction on students' understanding of quadratic equations. *Mathematics Education Research Journal, 18*(1), 47–77. https://doi.org/10.1007/BF03217429

Zakaria, E., & Maat, S. M. (2010). Analysis of students' error in learning of quadratic equations. *International Education Studies*, *3*(3), 105–110. https://doi.org/10.5539/ies.v3n3p105

Authors

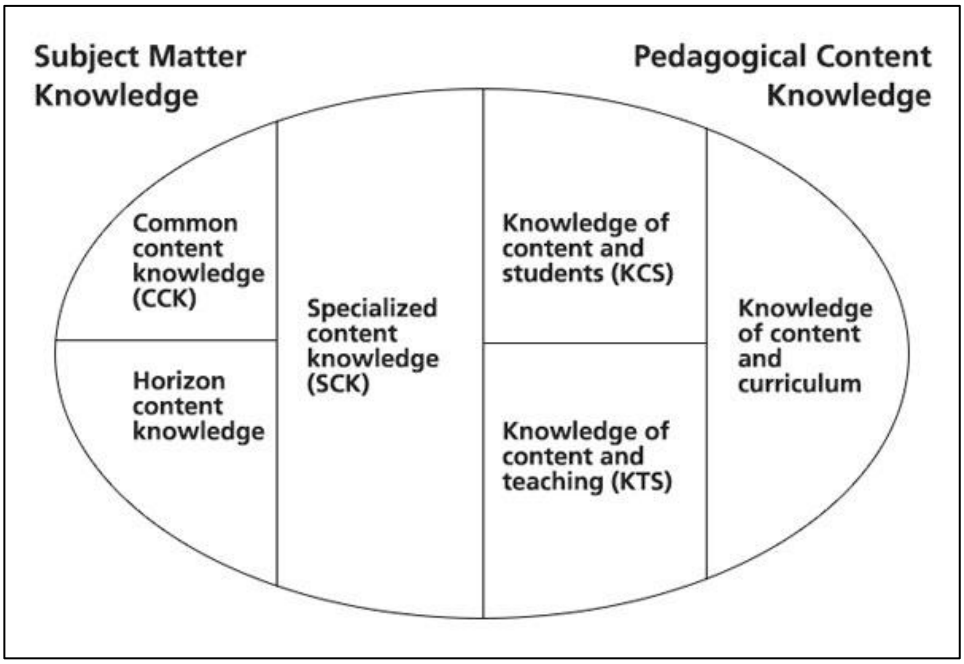
Jill Newton
Purdue University
100 N. University St.
BRNG 4136
West Lafayette, IN 47907
janewton@purdue.edu

Christina Alvey
Mount Saint Mary College
330 Powell Ave.
Aquinas 50C
Newburgh, NY 12550
Christina.Alvey@msmc.edu

Rick Hudson
University of Southern Indiana
Science Center 3273
8600 University Blvd.
Evansville, IN 47712
rhudson@usi.edu

Appendix A

Mathematical Knowledge for Teaching (MKT) Framework



Ball et al. (2008, p. 403)

This MKT Framework summary was produced using quotes from Ball et al., 2008. The authors did not provide acronyms for Horizon Content Knowledge or Knowledge of Content and Curriculum; however, for convenience we refer to these as HCK and KCC.

Subject Matter Knowledge

Common Content Knowledge (CCK)
"the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399).

Specialized Content Knowledge (SCK)
"the mathematical knowledge and skill unique to teaching. This is the domain in which we have become particularly interested. Close examination reveals that SCK is mathematical knowledge not typically needed for purposes other than teaching...teachers have to do a kind of mathematical work that others do not. This work involves an uncanny kind of unpacking of mathematics that is not needed—or even desirable—in settings other than teaching. Many of the everyday tasks of teaching are distinctive to this special work" (Ball et al., 2008, p. 400).

Horizon Content Knowledge (HCK) - "provisional" domain in Ball et al. (2008)
"an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403).

Pedagogical Content Knowledge

Knowledge of Content and Students (KCS)

"knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and what they will find confusing. When choosing an example, teachers need to predict what students will find interesting and motivating. When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. Teachers must also be able to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language. Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking" (Ball et al., 2008, p. 401).

Knowledge of Content and Teaching (KCT)

"combines knowing about teaching and knowing about mathematics. Many of the mathematical tasks of teaching require a mathematical knowledge of the design of instruction. Teachers sequence particular content for instruction. They choose which examples to start with and which examples to use to take students deeper into the content. Teachers evaluate the instructional advantages and disadvantages of representations used to teach a specific idea and identify what different methods and procedures afford instructionally. Each of these tasks requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning" (Ball et al., 2008, p. 401).

Knowledge of Content and Curriculum (KCC) - "provisional" domain in Ball et al. (2008)

Ball et al. (2008) referred directly to Shulman (1986) when describing this domain: "*curricular knowledge* is represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (Shulman, 1986, p. 10). In addition, Shulman pointed to two other dimensions of curricular knowledge that are important for teaching, aspects that he labeled *lateral curriculum knowledge* and *vertical curriculum knowledge*. Lateral knowledge relates knowledge of the curriculum being taught to the curriculum that students are learning in other classes (in other subject areas). Vertical knowledge includes "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them (Shulman,1986, p. 10)" (Ball et al., 2008, p. 391).

Appendix B

School Mathematics Teaching Pedagogical Content Knowledge (SMTPCK)

(Chick & Beswick, 2018, pp. 479-482)

PCK Category	Evident when the teacher…	Example
Clearly PCK (CPCK)		
Teaching strategies	Discusses or uses general or specific strategies or approaches for teaching a mathematical concept or skill	Uses concrete materials to demonstrate a concept
Student thinking	Discusses or addresses student ways of thinking about a concept, or recognizes typical levels of understanding	Identifies that a student doesn't recognise the equivalence of equivalent fractions
Student thinking—misconceptions	Discusses or addresses student misconceptions about a concept	Recognises that students often think ''multiplying makes bigger''
Student affect (in relation to content)	Discusses or addresses students' affective responses to particular mathematics topics	Recognises that adolescent students may have negative emotional reactions to the prospect of learning algebra
Cognitive demand of task	Identifies aspects of the (SMT) task that affect its complexity	Recognises 627–359 is more difficult to model than 687–321
Representations of concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)	Uses MAB to model subtraction
Explanations	Explains a topic, concept or procedure	Explains why we can write a 0 on the end of a whole number when multiplying by 10
Knowledge of examples	Uses an example that highlights a concept or procedure	Uses the 5-12-13 Pythagorean triangle to model how to solve a right-angled triangle problem
Knowledge of resources	Discusses/uses resources available to support teaching	Identifies and uses a mathematics website that is useful for students
Curriculum knowledge	Discusses how topics fit into the curriculum	Recognises that multiplication should be understood by Year 4
Purpose of content knowledge	Discusses reasons for content being included in the curriculum or how it might be used	Knows that knowledge of rounding is needed for money transactions
Content Knowledge in a Pedagogical Context (CKiPC)		
(Beliefs about) The nature of content	Expresses an appreciation of the nature of mathematics that goes beyond the school curriculum and aligns with mathematicians' view of the discipline	Compares the aesthetic qualities of two solution methods
Profound understanding of fundamental content	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics (i.e., Profound Understanding of	Understands why we ''invert and multiply'' when dividing fractions

PCK Category	Evident when the teacher…	Example
	Fundamental Mathematics [Ma, 1999])	
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept	Refers to the importance of the distributive law in the long multiplication algorithm
Structure and connections	Makes connections between mathematical concepts and topics, including interdependence of concepts	Links percentages with decimals and the base 10 system
Procedural knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)	Can apply the long division algorithm
Methods of solution	Demonstrates a method for solving a mathematical problem	Demonstrates a method for solving a mathematical problem
Pedagogical Knowledge in a Content Context (PKiCC)		
Assessment approaches	Discusses or designs tasks, activities or interactions that assess learning outcomes	Designs a multiple-choice quiz with appropriate distractors
Goals for learning	Describes a goal for students' learning	Justifies an activity as developing understanding of long-term probability
Getting and maintaining student focus	Discusses or uses strategies for engaging students	Designs a puzzle that is solved by answering some routine exercises
Classroom techniques	Discusses or uses generic classroom practices	Talks about grouping students according to ability levels
Student affect (general)	Describes how student affect influences pedagogical approach	Knows a particular student will respond to negatively to being asked for an answer in a large group session