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Examining the pedagogical content knowledge of in-service mathematics teachers on the permutations and combinations in the context of student mistakes

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Abstract

Permutations and combinations are generally taught by requiring students to memorize formulas and solve problems using the appropriate formula. Students who learn these topics may succeed in gaining high scores on end-of-chapter exams in textbooks, while lacking the conceptual understanding required to deal with problems in the real world. Therefore, this study aimed to examine in-service mathematics teachers' pedagogical content knowledge (PCK) to determine students' mistakes in solving permutations and combinations problem and their teaching strategies to eliminate these errors. Data were collected by distributing vignettes, CoRe, and PaP-eRs to thirteen mathematics teachers from ten provinces in Indonesia after they finished an online professional teacher education program to determine their PCK in teaching permutations and combinations. The data collected were analyzed qualitatively using a content analysis approach to obtain categories inductively. The result showed that PCK of in-service mathematics in teaching permutations and combinations was observed by identifying student mistakes conceptually and procedurally, even though some could not determine their mistakes in permutations. On the other hand, the knowledge of instructional strategies can engage all students in active learning, but most of them only give general answers. Furthermore, an in-depth understanding of permutations and combinations topic is needed to support the development of teachers' pedagogic competencies sustainably. The contribution of this research will be of interest to curriculum development and mathematics educators.

Keywords: In-Service Mathematics Teachers, Pedagogical Content Knowledge (PCK), Permutations and Combinations, Qualitative Content Analysis, Students Mistakes

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Solving permutation and combination problems is a numeracy for developing mathematical reasoning needed in advanced mathematics (Lockwood et al., 2015). Additionally, it encourages students to interpret, represent, structure, conjecture, justify, and generalize mathematical formulas to promote desirable mathematical thinking strategies (Soto et al., 2022). Several studies (Hubeňáková & Semanišinová, 2020; Semanišinová, 2021; Soto et al., 2022) have been carried out by PCK pre-service and in-service mathematics teachers to determine the combinatorics concepts involved in solving permutations and combinations. Hubeňáková and Semanišinová (2020) stated that the knowledge needed by prospective teachers to teach combinatorics using the Mathematical Knowledge for Teaching

Model is still inadequate. Semanišinová (2021) also assessed knowledge of combinatoric topics using a multiple-solution task instrument. This is in addition to the research by Soto et al. (2022), which focused on the teacher's knowledge of how students conceptualize and perceive combinatorial problems. However, there has not been a learning intervention to increase teachers content knowledge on students' thinking and their development in permutations and combinations topic, which are an important part of the Indonesian Mathematics curriculum at high school and post-secondary education levels. It also offers many chances for students to engage in deep mathematical reasoning while valuing formulas more than ideas, using textbooks and curriculum documents.

Lockwood et al. (2017) stated that more research is needed to investigate how teachers can learn to support students in acquiring knowledge based on the potential of discrete mathematics practice in their classroom teaching. Studies that provide an overview of the actual classroom are essential for students to obtain valuable insight and motivation for a deeper understanding of how the knowledge needed to teach permutations and combinations is learned. Similarly, research has not been widely carried out for mathematics teachers who participate in developing professional programs in Indonesia. Therefore, this research aims to explain how PCK in-service teachers in fills three cases vignettes, Co-Re and Pap-eRs when teaching permutations and combinations. This investigation's results will significantly impact how content knowledge and PCK courses are developed and improved in teachers' education programs.

Learning Permutations and Combinations

Similar to other countries in the world, in Indonesia, permutations are taught in the twelfth grade of Senior or Vocational High School with the expected competence that students can solve contextual problems (Melusova & Vidermanova, 2015; Permendikbud No. 37, 2018). According to NCTM (2000), the importance of developing an understanding of permutation as a computational technique. This concept plays an essential role as a foundation for building combinatoric analysis (Abrahamson & Cendak, 2006), and the context of the problem enriches students' mathematical thinking through the process of scheduling procedures (Kuo, 2009).

As a fundamental concept in combinatoric operation, the combinations provide the base for studying probability and statistics (Smith, 2007). Permutations and combinations are important parts of statistics introductory courses in some universities (Garfield & Ahlgren, 1988). Solving combination problems can develop the mathematical reasoning required for advanced mathematics (Lockwood et al., 2015) and activate their solving strategies (Lamanna et al., 2022b).

Teaching and learning permutations and combinations are rich with the relevant real-world problem that lend themselves to pedagogically desirable explorations. These topics are generally taught by requiring students to memorize formulas and categories of problems solved using the appropriate formula (Chotikarn et al., 2021). Students who learn these topics tend to succeed in scoring highly on end-of-chapter exams in textbooks, but they often lack the conceptual understanding required to deal with problems in the real world. Quinn and Wiest (1998) designed a lesson to teach permutations and combinations using a constructivist approach which focuses on the student's judgments regarding the issues of order and repetition. They suggested that students actively construct formulas rather than memorize the procedure to solving permutation and combination problems. Busadee and Laosinchai (2013) encouraged high school mathematics teachers to develop authentic problems in teaching permutations and combinations because its emphasis was on real-life problems capable of enhancing high school students' conceptual understanding of their achievements. Lamanna et al. (2022a) assessed the effect of instruction on students' strategies in solving permutation and combination problems. The result showed that those in the group without instruction showed a greater prevalence of differences in the main strategies used for enumeration and dividing a problem into parts. In contrast, students in the instructional group commonly used formulas and product rules.

Difficulties and Misconceptions of Children about Permutations and Combinations

Some students may experience misunderstandings in solving combinatoric problems because of their limited ability to make appropriate connections with the initial situation in the problem to be solved, despite having the potential and deep curiosity to learn the concept. Eizenberg and Zaslavsky (2004) stated that many students find combinatorial issues challenging with the main difficulty related to not recognizing the structure of the problem due to the inability to make connections between counting problems (English, 2005). Students need to develop their combinatorial thinking skills using formulas/expressions, counting processes, and sets of outcomes to solve associated combinatorial tasks (Lockwood, 2011, 2013). Several literatures have reported students' difficulties in solving combinatoric problems related to the concepts of permutations and combinations and the various misconceptions (Batanero et al., 1997; Fischbein & Gazit, 1988; Lockwood, 2011; Martin, 2001; Sukoriyanto et al., 2016b). Fischbein and Gazit (1988) stated that students need the ability to carry out a variety of combinatorial problems easily, particularly those related to combination, because they thought permutation problems are more challenging. This is explained using the fact that children give up using their intuitive empirical methods once the complex formula for the number of combinations is presented. According to preliminary studies, some students solve permutation problems using the combination concept (Sukoriyanto et al., 2016b). Another difficulty in solving combinatorial problems on the operating dimensions are through arrangements, permutations and combinations and the dimensions of the properties of the elements combined, namely lift, letters, people, and objects (Batanero et al., 1997). Preliminary studies found that the type, nature, and instructions of the problem are associated with the nature of the elements, and the type of error. Martin (2001) reported that one of the reasons students find counting difficult is due to "numerous formulas, and each problem seems to be different." Lockwood (2011) stated that students often find it difficult to understand when and why to use permutation and combination formulas to solve problems. One way to do this is through understanding students thinking and making anticipation to avoid their mistakes through adequate teacher knowledge of content and knowledge of pedagogy. Therefore, this study aims to determine the ability of in-service mathematics teachers to detect student mistakes related to permutations and combinations, as well as the underlying reasons associated with these mistakes, and to design a teaching process for its rectification.

Misconceptions caused by experience and basic knowledge of combinatorics involving the basic concepts of permutations and combinations in secondary school can adversely affect the learning process of mathematics at the tertiary level (Sukoriyanto et al., 2016b). According to Korkmaz and Şahin, (2020), the characteristics of mathematics are understood as a flowing and sequential science. It is challenging to teach a concept to children when they cannot understand the foundational concepts accurately (Turkdogan et al., 2009). Difficulties can continue to occur when the teacher has not been able to overcome student errors, which can be resolved through the adequate development of combinatorial thinking (Batanero et al., 1997). Middle school teachers must introduce many discrete math ideas into the curriculum to help improve students' problem-solving skills and prepare them for college (Salavatinejad et al., 2021; Siegel, 1986). Students can learn meaningfully and successfully if their learning challenges and the reasons for those difficulties are well understood (Yetkin, 2003). Therefore, teachers need to find ways to help students improve their combinatorial thinking by understanding the nature of their difficulties when solving combinatorial tasks (Batanero et al., 1997). Due to this, it is critical to assess the mathematics teacher's ability to identify students' errors in permutations and combinations and develop instructional strategies to correct them. This investigation's results will significantly impact how content knowledge and PCK courses are developed and improved in teacher education programs.

Pedagogical Content Knowledge

Teachers with adequate PCK can easily assess student errors and misunderstandings, provide appropriate explanations to teaching problems, and create learning tailored to their needs and cognitive level (Korkmaz & Şahin, 2020; Yunianto et al., 2021). The PCK concept, established in the 1980s by Shulman, has had a major influence on the intensively researched teacher training program. According to Shulman, a competent teacher should also have strong content and pedagogical knowledge to provide opportunities for various teacher professional development programs to make better improvements (Shulman, 1986).

According to Shulman's definition of PCK, teaching requires a combination of content and pedagogy in the teacher's area, as well as an "understanding of how" a particular topic, issue, or problem is organized, represented, and adapted to the various interests and learning styles of students (Shulman, 1987). Shulman stated that there are two essential elements of PCK, namely the capacity to recognize ideas that are simpler for pupils to comprehend, their errors and misconceptions, and understanding the characteristics of their learning. Another component is the teacher's knowledge of learning strategies, which is the use of analogies, illustrations, examples, explanations and demonstrations in learning. Ball et al. (2008) described mathematics teacher PCK in three domains, which include knowledge of content and students, knowledge of content and teaching and knowledge of content and curriculum. These three components are also considered in the Mathematics Teacher Special Knowledge model proposed by Carrillo-Yañez et al. (2018). Furthermore, PCK is conceptualized into three sub-domains, namely knowledge of the features of mathematics learning, which refers to the teacher's need to understand and be aware of how students think about mathematical activities and tasks. Knowledge of mathematics teaching is the teacher's knowledge in choosing various sources of representation and learning resources to assist students in overcoming their misunderstandings. Knowledge of mathematics learning standards includes knowing curriculum specifications, as well as the relationship between learning topics studied at each level. The essence of PCK described in each teacher knowledge model focuses on assessing how teachers comment on student errors and how they provide suggestions as an effective way of teaching to eliminate student errors. This research aims to assess teacher knowledge in understanding students' errors and misconceptions in solving permutation and combination problems. It also provides teaching alternatives that can eliminate these errors.

Based on the description above, this research aims to analyze the following question: "What is the in-service mathematics teachers pedagogical content knowledge on teaching permutations and combinations?" Furthermore, the two interrelated questions include:

1. What are the mathematical errors properly recognized by in-service mathematics teachers?
2. What is the instructional-strategies knowledge recognized by the in-service mathematics teachers on teaching permutations and combinations?

METHODS

This is a qualitative descriptive research, which allows researchers to explore individuals' social characteristics, behaviors and meanings (Lapan et al., 2012). Teacher responses in the form of vignette, CoRe and PaP-eRs were analyzed using the Qualitative Content Analysis method (Lune & Berg, 2017). According to Kuckartz (2019), the objective of the qualitative content analysis method is to create a category system (coding frames) and work with categories (codes). Therefore, many texts were converted into codes and summarized into categories.

Participant

A total of 33 teachers, consist of 20 Junior High School and 13 senior or vocational high school teachers, are participating in the teacher professional development program. In this study, we choose 13 teachers because of permutation and combination teaching in twelve grades. We coded them as IST1, IST2, … and IST13. Participants are teachers who have taught permutations and combinations, as shown in Table 1.

Table 1. Information of research subject

Aspect	Category	Total
Gender	Female	12
	Male	1
Ages	31 – 40 years old	13
Teaching experience	5-10 years	3
	10-20 years	10
Formal education	Mathematics education	12
	Mathematics	1
Region	West Java Province	3
	South Sulawesi Province	2
	Banten Province	1
	Bengkulu Province	1
	Central Java Province,	1
	East Java Province	1
	Central Kalimantan Province,	1
	Riau Island Province	1
	The Province of South Sumatra,	1
	North Sumatra Province	1

Teacher professional development program is required for teachers with more than five years of teaching experience who are permanent in public or private schools. This program is completed online within four months and allows teachers to study independently and with supervisors using learning management system (LMS) and other sources. Additionally, the program is also carried out for two months with a focus on the content and pedagogical knowledge, while the next two months in the practical field are learning activities in their respective schools (Kemdikbud, 2020). During the deepening course, teachers learn various basic mathematics topics, one of which is combinatorics and statistics.

Instrument

The instruments used in this study were CoRe, PaP-eRs and vignettes. A CoRe is a table representing teachers' understanding of the content for a particular topic. This is carried out by asking teachers to consider the central or "Big Ideas" of the permutation and combination. These include the essential tenants of the content that students are to learn, such as the "Big Ideas," which form the column headings. The rows consist of eight prompts that aim to reveal the teachers' reasoning behind pedagogical choices/activities and knowledge of their students, such as alternative conceptions, difficulties, points of confusion, and ways of assessing student understanding. PaP-eRs are linked to the CoRe attempts to draw out aspects of teachers' PCK. They are a detailed description and reflection of a teacher's reasoning and thinking about a particular lesson based on a particular part of the content from the CoRe. A PaP-eRs is commonly presented as a narrative account of the lesson from the teacher's perspective in line with questions, such as what they did and why it was conducted. It does not represent the complexity that makes up teachers' PCK, but a collection certainly further explores the differing elements (Loughran et al., 2004). Vignette is an example of a student error case in solving permutation and combination problems.

The first vignette explains the permutation of k objects from r different elements, with the focus of the misconception being the confusing concept. The second vignette is on the cyclic permutation subtopic with the circular misconception of certain conditions. The third vignette is the subtopic of the combination of k objects from n using some of the same objects with a misconception of concept combination. Examples of errors, actions or situations with each vignette comprising two open questions. The first and second researchers developed the vignette into three stages, namely the pre-design, design and post-design stages (Rungtusanatham et al., 2011). At the pre-design stage, the students' common errors obtained from their written tests after studying the permutations and combinations topic by paying attention to the findings of several types of students' errors in solving combinatoric problems by Batanero et al. (1997). Additionally, information about common mistakes that students often make was collected from LMS reviews. Data on student errors in permutations and combinations topic were gathered throughout the design phase and transformed into scenarios and situations as vignettes. Furthermore, a written vignette form was submitted to validate the post-design phase, and the revised results changed the design of the open-ended questions. This open-ended question aims to access teacher knowledge and is designed to tap into the cognitive aspects of teacher content knowledge (Tchoshanov, 2011). The open-ended questions used in this study are designed to provide data about teacher pedagogy and content knowledge. The illustration of the three vignettes is shown in Table 2.

Table 2. Permutation and combination vignette illustration

Vignette illustration	Open-ended question
Case 1: It is known that the word PELUANG consists of the letters P, E, L, U, A, N and G. How many possible arrangements can be obtained from 4 different letters? Then two students named Afkal and Bella made the following solutions Afkal's answer: “Known: The word PELUANG consists of the letters P, E, L, U, A, N, G, n=7, r=4	(a) Write your comments on Afkal’s and Bella's answers! (b) How would you design the proper instruction base on this case?

$$C_r^n = \frac{n!}{(n-r)!r!} = \frac{7!}{(7-4)!4!} = \frac{7!}{3!\cdot 4!} = \frac{7\cdot 6\cdot 5\cdot 4!}{3\cdot 2\cdot 1\cdot 4!} = \frac{210}{6} = 35 \text{ arrays”}$$

Bella's answer:
“PELUANG consists of 7 letters
3 vowels (E, U, A), 4 consonants (P, L, N, G)”

$$P_r^n = \frac{n!}{r!(n-r)!} = \frac{7!}{4!(7-4)!} = \frac{7!}{4!\times 3!} = \frac{7\times 6\times 5\times 4}{4\times(3\times 2\times 1)} = \frac{840}{24} = 35 \text{ arrays”}$$

Case 2:

A teacher gives the following questions to students:
A father, mother and three children are seated in a circle, with the youngest child sitting next to the mother. Determine the number of ways they can sit in a circle and solve the problem by discussing it with other classmates. The following is an excerpt from a conversation that took place in the discussion between Carly and Desna.

Carly: To make it easier, of the total of 5 people, 2 people, namely the mother and the youngest child, are considered 1. Therefore, the result is obtained by multiplying 5 elements by 2.
Desna: So, you mean the answer is 5! multiplied by 2!
Carly: Yes, the important thing is that we multiply the known elements of the problem.
Desna: I don't agree with you because yesterday, the teacher gave an example of a circular sitting arrangement as a cyclic permutation with the formula (n-1) factorial. Therefore, the result can be (5-1)! multiplied by 2!
Carly: So, you mean the circular seating arrangement is $4! \times 2! = 48$ ways huh?
Desna: Yes, but there is something wrong. I also need clarification.
Carly: How about we ask the teacher?
Desna: Yes, I agree.

(a) Write your comments on the conversation between Carly and Desna!
(b) How would you design the proper instruction base on this case?

Case 3:

Given questions:
How many ways can a group of 3 boys and 2 girls be arranged out of a total of 4 boys and 4 girls!
One of the students named Erni answered as follows:
“Known:
1 group: 3 boys and 2 girls out of 4 boys and 4 girls."

$$\begin{array}{ll} \text{Boys: } 3 \times 4 = & 12 \\ \underline{\text{Girls: } 2 \times 4 = } & \underline{\;\;8 \quad +} \\ & \textit{20 ways} \end{array}$$

a) Write down your comments on Erni's answer!
(b) How would you design the proper instruction base on this case

Data Analysis

The data analysis in the study was sourced from the teacher's response to the vignette, CoRe and PaP-eRs, which was analyzed using a directed content analysis approach and coding categorized deductively (Hsieh & Shannon, 2005). Independent coding reliability and complete data encoding associated with discrepancies were resolved through discussion for mutual agreement. Therefore, the need to use an interrater reliability method other than individual coding, followed by discussion, is important.

These three instrument documents were used to investigate teachers' content knowledge and pedagogy on topic permutations and combinations based on student errors. Each teacher was asked to provide written comments for the three vignettes of student misconceptions and provide suggestions if similar cases were found in their class. Furthermore, interviews were not conducted in this research to confirm the answers of the thirteen teachers. Each aspect of the CoRe and PaP-eRs is the result of adapting the aspects developed by Loughran et al. (2012), which analyzes the information provided for teachers' to attain more knowledge of teaching permutations and combinations. The analysis of the first open question in three cases vignetted aims to classify the nature of the teacher's PCK in identifying student errors. This process was obtained using the error type proposed by Batanero et al. (1997) to classify students' answers into several sub-categories and then paying attention to assign a comment category to errors.

Furthermore, analysis of CoRe and PaP-eRs and the second open-ended question from the three cases vignette was carried out on all teacher responses coded under the themes referred to as sub-categories and analyzed to obtain sub-categories. The PCK analysis of in-service mathematics teachers is guided by the framework developed by Chick and Beswick (2018), which states that the PCK framework is divided into three categories. The first is "clearly PCK," where pedagogy and content are closely related with eleven sub-categories comprising of the following topics: "knowledge of resources and curriculum," "knowledge of students' conceptions and misconceptions," "knowledge of representations in mathematical concepts," and "knowledge of teaching strategies for mathematics" (pp. 479–480). Six sub-categories are included in the second category, known as "content knowledge in a pedagogical context." It addresses the type of content, provides a deep comprehension of basic mathematics, breaks down knowledge into its core parts, creates awareness of mathematical structure and linkages, procedural knowledge, and problem-solving strategies (pp. 480-481). In the framework, "pedagogical knowledge in a content context" is the third category. It comprises four sub-categories that deal with "knowledge of learning goals, strategies for focusing students and knowledge of classroom techniques in a specific content area" (pp. 481-482). This framework provides an opportunity for the strengths and weaknesses of the PCK of high school mathematics teachers in Indonesia to be analyzed. Therefore, the findings were adapted to the categories and sub-categories in the framework.

RESULTS AND DISCUSSION

This section is divided into two parts, the first outlines the pedagogy content knowledge used by the teacher to address student mistakes in the permutations topic. The second outlines the teacher's pedagogical content knowledge for handling student mistakes on combinations topic.

Pedagogical Content Knowledge of In-service Mathematics Teachers in Permutations

After collecting data on the vignettes, CoRe and PaP-eRs from thirteen teachers, an analysis process was conducted. The first open-ended questions in vignettes 1 and 2 require the teacher to comment on the students' work in solving permutation problems. The content analysis results gave rise to eleven sub-categories, with the teachers' responses grouped in the first and second vignettes. Each teacher gives more than one response to student errors, which leads to more than one sub-category. Furthermore, this sub-category was classified into two, namely the teacher's response to understanding mathematics concepts and problem-solving step. The presentation of the two categories with examples of teacher responses for each sub-category is shown in Table 3.

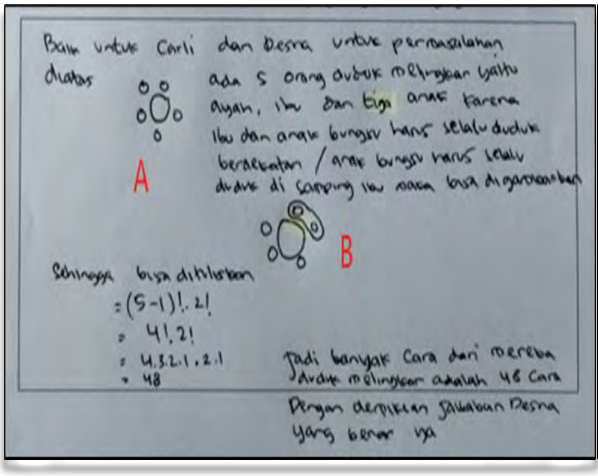
Table 3. Examples of teacher responses in identifying student work

Category	Sub-categories	Sample Responses selected for the first and second cases on vignettes
Teacher's response to understanding Mathematics concepts	Misinterpretation of the problem statement (n=17)	Both Afkal and Bella have not been able to distinguish the problem that involves repetition.
	Error of order (n=4)	Students have been unable to distinguish the problems that pay attention to order and those that do not.
	Understanding the problem (n=10)	Afkal understands that the number of arrangements of four different letters is the selection of 4 elements from the 7 available ones using the combination concept.
	Problem relates to instructional (n=3)	Students make mistakes because the teacher's explanation when conveying concepts is too fast, and they lack reading literacy. Hence, they do not understand the problem.
	Students' understanding of the correct concept (n=5)	Students should use the concept of permutations from the number of choices of r elements from the available n-type by paying attention to the order.
Teacher's response to student problem-solving steps	Error of formula (n=4)	There are still misconceptions about using permutation and combination formulas
	Giving an incorrect solution (n=4)	They are wrong in doing calculations where the n is not 5 but 4 so that $P_{cycle} = (n-1)! = 3! = 6$
	Comment on formula errors and write the correct formula (n=8)	Bella's permutation formula was wrong, it should be $P_r^n = \frac{n!}{(n-r)!}$
	Error in formula writing (n=3)	Bella's answer is not correct because she wrote wrongly writing the permutation formula as a combination formula $P_r^n = \frac{n!}{(n-r)!}$
	Comment on calculation errors with the formula (n=12)	Afkal and Bella both use combination formulas instead of permutation.
	Stating the student's answer Wrong (n=5)	The answers of both students are wrong

Table 3 explains that the first category is the teacher's response to understanding mathematical concepts involving five sub-categories. According to Batanero et al. (1997), some students' mistakes when solving combinatoric problems involving the concept of permutations are the errors of order and formula. Based on the thirteen teachers' responses, seventeen similar responses were included in misinterpreting the problem statement sub-category. Four teacher responses stated the student's error as an order, while ten others were based on students' understanding of solving problems. An example is "Bella understands the problem involving permutation rules because the rules are from the number of choices of r elements from the available n elements by paying attention to the order" (IST11). The teacher's statement in the sub-category of understanding the problem was classified into various categories, with responses stating the relationship between students' misunderstandings and the teaching process. It is further classified into the sub-category of problems related to instruction, such as "The first case of the permutations concept is students remembering the problem, which is presented by the teacher in solving daily problems" (IST9). This response is in line with the research carried out by Lamanna et al. (2022a) that the effect of instruction can activate various strategies to solve permutation and combination problems. Other comments that indicate errors related to students' understanding of mathematical concepts are student errors of formulas, such as the misconceptions in using permutation and combination formulas" (IST10). Subanji (2015) stated that disturbances such as thinking occur when students know construction from two related concepts. Sukoriyanto et al. (2016a) further added that global inference occurs when viewing permutation material as combination and permutation problems.

In the second category, namely "teachers' responses regarding problem-solving steps," a classification based on the teacher's procedural knowledge in commenting on errors and the correctness of the problem-solving steps were taken by students in both cases. It comprises four sub-categories, namely, the teacher's response containing the incorrect solution, the correct formula, errors in writing the formula, and comments stating the student's answer was wrong. Most teachers commented with the correct answer, even though some failed to fully understand the students' mistakes in both cases of permutations, hence they failed to provide concrete reasons for the mistakes. This finding is in line with the research by Fischbein and Gazit (1988) and contradictive with the finding of Lamanna et al. (2022a), which stated that permutation problems were found easier than combination, especially in case selection.

In the second vignette, one of the teachers gave an inappropriate response to the student's error case solving the cyclic permutation problems. The teacher tried to provide comments by visualizing the problem by describing the circular sitting position of the five family members and the sitting position of the mother and child side at a round table, as shown in Figure 1. The calculations by Desna were correct, which means the teacher cannot comment on students' experience regarding the concept of cyclic permutation. This finding indicates that the teacher still needs to understand the case of student errors related to permutations. Although both cases are frequently encountered in permutation learning, the lack of knowledge of teacher content will likely lead to the incorrect identification of student errors. Hubeňáková and Semanišinová (2020) stated that without broad additional experience, the teacher could not make the formulation of students' misconceptions possible. It means they were missing a part of the student's understanding of this content.



Translation: "For Carly and Desna, the problem can be described as having 5 people sitting in a circle, namely the father, mother, and three children, represented by A as a large circle surrounded by five small circles. Because the mother and youngest child must always sit side by side, it is depicted as in B where a big circle is surrounded by three smaller circles that are mutually exclusive and two circles that are joined together. So that it is obtained = (5-1)!.2! =4!.2! =4.3.2.1.2.1=48. So, the number of ways they sit in a circle is 48 ways. Hence, Desna's answer is correct."

Figure 1. IST1's written response in commenting on the second vignette

The second open question from the first and second vignettes aims to determine the Teacher's PCK in proposing learning strategies to eliminate students' mistakes when teaching permutations topic. The teacher's responses in filling out vignettes, CoRe, and PaP-eRs were classified into seventy-three types, and all can provide more than one suggestion for teaching improvement. The PCK category for school mathematics teachers from the framework Chick and Beswick (2018) was used to carry out this research. The suggestions for improving teaching were classified into three main categories, namely (1) clearly PCK with 7 sub-categories, (2) content knowledge in a pedagogical context with 3 sub-categories and (3) pedagogical knowledge in a content context with 4 sub-categories. One of the teacher's responses to each teaching improvement suggestion is shown in Table 4.

Table 4. Examples of teacher PCK responses in teaching permutations

Category	Sub-categories	Teaching improvement suggestions	Sample Responses Selected from Teachers
Clearly PCK	Teaching strategies	Provides a variety of practical questions	I will give various practice questions
		Uses the role-play method	I will explain by conducting a direct practice of the sitting position in accordance with the given problem.
		Gives awards	I will reward students' answers when they can solve the problem.
		Use discovery strategies	My inability to directly say the student's answer is wrong, therefore, it is ideal that I ask them to analyze the problem first, understand the problem and guide students to find solutions to the problems given.
	Representation of concepts	Illustrate the problem in the form of a picture representation	I will explain to the two students by describing the sitting position in a circle.
	Explanation	Explain the difference between permutations and combinations	I will explain the difference between permutation and combination problems.
		Provide concept reinforcement	I will strengthen my understanding of cyclical permutation and adapt to the problem.
		Reinforce prerequisite topic	Recalling counting rules, additional rules, multiplication rules and factorial definitions presents some permutation problems.
		Explain concepts in order	Explain permutation of n elements from n different objects, permutation of the same object and cyclic permutation
	Knowledge of examples	Give examples of contextual problems	- I will give contextual questions such as determining the committee's composition consisting of a chairman, secretary, and treasurer. - Count the number of ways cars and motorbikes line up in different sequences and determine locker codes, suitcase codes, and vehicle number plates.
	Knowledge of resource	Using digital learning resources	Make teaching resources in the form of electronic books and use several Learning videos related to permutation topics from YouTube.
		Using learning resources available at school	Using the Ministry of Education textbook for class XII in 2018 and the module of Mathematics High School for grade 12 for students to understand the topic being taught
		Using learning designed by the teacher	Using teaching resources and worksheets that are designed by themselves for students to understand the concept of permutations better
	Curriculum knowledge	Mastering the order of the topics	I will explain the concepts in the permutations taught in grade 12 under the subject of probability with a subtopic of counting rules, which includes addition and multiplication rules.
	Propose of content knowledge	Understanding the prerequisite topic	Before studying permutations topic, students must understand the rules for filling slot and the definition of factorial.
		Give the right concept	It ensures that students are not wrong in applying the concept of permutations to problems involving combinations and vice versa.
Content knowledge in a	Deconstructing content to key components of mathematical	Identify subproblems	I will ask students to think about what is known from the problem. For instance, what are the number of letters in the word "PELUANG."? How many

pedagogical context			different letters will be arranged? What rules would be used for the different typesetting problems? Next, students are asked to solve the problem again.
	Procedural knowledge	Give the right answer	The correct answer is the word probability that all the letters are different is chosen 4 means a permutation of $P_4^7 = \frac{7!}{(7-4)!} = 840.$
		Explain the steps to get the right answer	I explained this cyclical permutation by asking students to remember the formula, followed by analyzing the known elements with the youngest child and mother side by side so that $n = 4. For\ (n-1)! = (4-1)! = 3 \cdot 2 \cdot 1 = 6$. There are 2 ways to arrange mother and youngest, namely mother-youngest and youngest-mother. Therefore, the total number of ways becomes 6 x 2 = 12 ways.
	Methods of solution	Explain the best solution to the problem	There are two possible sitting positions. (i) if the mother is on the left and the youngest is on the right equal$(4-1)! = 3! = 6$, (ii) if the mother is sitting on the right, the youngest is on the left, equal $(4-1)! = 3! = 6$. Hence, the different arrangement that occurs is 6+6 =12 ways.
	Assessment approach	Assess the cognitive aspect	I chose to do a written test using essay questions and multiple choice.
		Assess the affective aspect	I choose to assess through observing student activity and performance.
		Assess the cognitive, affective, and psychomotor aspect	The assessment that I chose is affective, cognitive, and psychomotor
		Conduct group and individual assessments	Conduct assessments through group discussions and individual assignments
Pedagogical knowledge in a content context	Goals of learning	Explain the general objectives of learning according to the curriculum	Learning objectives are students that can understand the concept of permutations and solve related contextual problems
		Explain the specific purpose before studying the topic of permutations	Students must understand the concept of counting rules and the definition of factorial before understanding the concept of permutations
	Getting and maintaining student classroom techniques	Using a verification strategy (looking back)	Ask students to pay attention to the completion of their answers and mistakes.
		Using the formula	Ask students to solve problems using formulas.
	Classroom techniques	Making student groups discussions and guiding them	I will divide students into several groups and then guide them in discussion groups
		Facilitate students with various activities	Facilitating students with teaching resources and worksheets, dividing them into groups, discussing, and presenting the discussion results.

This research used the framework by Chick and Beswick (2018) to analyze and capable of making suggestions for improving teachers learning in mathematics. A total of 73 teachers generated codes for teaching permutations. Furthermore, 11 teachers suggested forty-nine strategies for improving teaching, which is classified into seven sub-categories under the "clearly PCK" category. Some suggestions for improving teaching to increase student motivation include using various practice questions, role play, giving rewards, discovery strategies, illustrating problems using image representations and providing examples of contextual problems. Knowledge of the content and order of presenting the topic proposed by the teacher in the categories of explanation include knowledge of learning resources, curriculum, and content knowledge. In the process of fixing the mistakes that students made when performing permutation and combination problems, it was discovered that in-service teachers favored the teacher-centered instruction, such as direct instruction or meaningful learning approach, more commonly. Meanwhile, senior high school students are better adapted to learning through case study group investigations and digital visualization, which in-service teachers rarely prefer. According to Zhang (2015), teachers need more teaching strategies and methodology knowledge.

In the "content knowledge in pedagogical context" category, twelve suggestions for improving teaching were given by six teachers, who were classified into three sub-categories. It is identified by examining the sub-problems, providing the right answers, explaining the stages, and demonstrating the best solutions. This shows that the teacher's procedural knowledge is quite good in guiding students to correct their mistakes.

The third is the "pedagogical knowledge in a content context," which comprises as many as twelve suggestions for improving teaching given by nine teachers under four sub-categories, including cognitive, affective, a combination of both with the psychomotor, group, and individual assessment. Eight teachers gave the same response for learning objectives in solving permutation problems and general learning objectives adapted to the curriculum. Three teachers provided three learning strategies, namely verification strategies and formulas under the sub-category Getting and maintaining student classroom techniques.

Pedagogical Content Knowledge of In-service Mathematics Teacher in Combinations

The first open question in the third case of the vignette requires the teacher to make a response to combination problems. Five sub-categories emerged from the analysis of teachers' responses to vignette under two categories, namely "comments on understanding mathematical concepts" and "comments based on problem solving steps." Table 5 shows the categories and examples of teacher responses for each sub-category.

Table 5. Examples of teacher responses in identifying student work

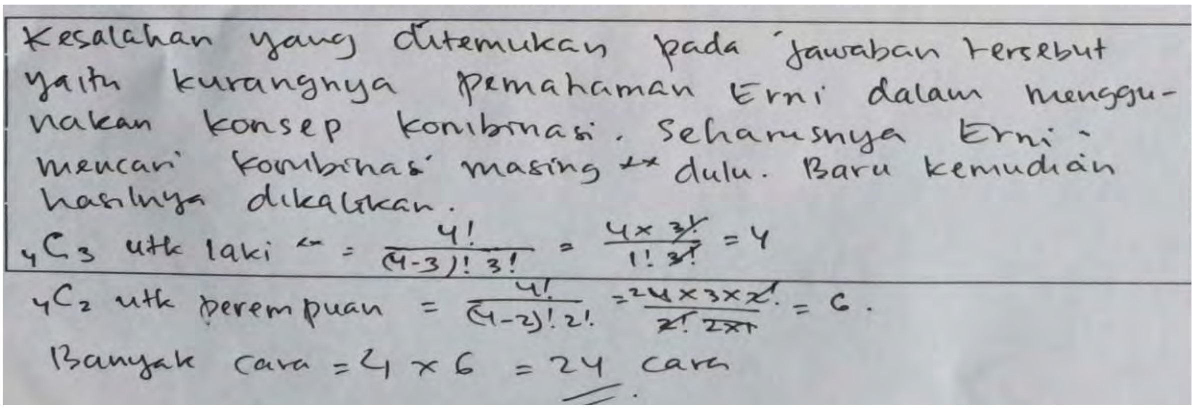
Category	Sub-categories	Sample Responses selected for the third case on vignettes
Comments on understanding mathematical concepts	Error of order (n=13)	Erni should have used the concept of combinations.
	Students thinking (n=7)	Erni understands that many ways of arrangement can be obtained by multiplying the selection of n elements with the number of available ones.
	Misconception of concept (n=10)	Students are wrong because they do not understand the concept of permutations and combinations
Comments on problem-solving steps	Mistaken intuitive answer (n=4)	Students directly multiply the number of men they want to choose by the number of available men as well as for women then add up
	Correct solution (n=5)	Students should answer with C_3^4 for boys = 4 and C_2^4 for girls = 6. So, there are 4 x 6 = 24 ways

Table 5 shows that there are three sub-categories under the first category known as "Comments on understanding mathematical concepts." According to Batanero et al. (1997), one of the students' mistakes in solving the combination problems is the error of order. All teachers can comment on specific errors as the first sub-category, known as error of order. There are seven comments of teachers refer to the students' way of thinking in understanding the problems identified in the second sub-category. Ten teachers chose to provide responses related to conceptual errors, which were categorized in the sub-category of misconceptions of concept. Six responses pay attention to the connection of permutations and combinations concept such as the following example.

"Students did not understand whether the problem was a permutation or a combination" (IST1)

"Students do not understand the concept of permutations and combinations" (IST 7)

This finding suggests that teachers can quickly respond to student errors by understanding their mistakes caused by the ambiguity of the two topics. Combination problems are considered like permutation and vice versa. Additionally, some teacher responses not only paid attention to conceptual errors, but also those in the stages of solving combination problems, which are classified into teachers' comments in the problem-solving step. This category includes two sub-categories, namely mistaken intuitive and correct answer. In this third vignette, nine teachers provided comments, which are classified in both categories. For example, the Teacher (IST4) made a comment about a mathematical concept and presented the correct solution, which is classified into three sub-categories under two different categories as shown in Figure 2.


Kesalahan yang ditemukan pada jawaban tersebut yaitu kurangnya pemahaman Erni dalam menggunakan konsep kombinasi. Seharusnya Erni mencari kombinasi masing² dulu. Baru kemudian hasilnya dikalikan.

$_4C_3$ utk laki² = $\frac{4!}{(4-3)!\,3!} = \frac{4\times\cancel{3!}}{1!\,\cancel{3!}} = 4$

$_4C_2$ utk perempuan = $\frac{4!}{(4-2)!\,2!} = \frac{{}^{2}\cancel{4}\times3\times\cancel{2!}}{\cancel{2!}\,\cancel{2\times1}} = 6$.

Banyak cara = 4 × 6 = 24 cara

Translation: "Erni's lack of understanding in using the combination concept, Erni should have looked for the combination of each result first and then the results were multiplied, C_3^4 for men = $\frac{4!}{(4-3)!.3!} = \frac{4\times3!}{1!.3!} = 4$; and C_2^4 for women =$\frac{4!}{(4-2)!.2!} = \frac{4\times3\times2!}{2!.2\times1} = 6$. So, there are = 4 x 6 = 24 ways".

Figure 2. IST4's written response in commenting on the third vignette

Furthermore, the teacher's response analysis in filling out the third case of vignette, CoRe, and PaP-eRs were identified in twenty suggestions for teaching improvement. The result showed that of the thirteen teachers, 8 gave more than 1 suggestion for teaching improvement. Chick and Beswick (2018) classified the developed PCK framework for improving the teaching process into thirteen sub-categories under three main categories. Examples of teacher responses for each teaching improvement suggestion are shown in Table 6.

Table 6. Examples of teacher PCK responses in teaching combinations

Category	Subcategories	Teaching improvement suggestions	Sample responses selected from teachers
Clearly PCK	Teaching strategies	Using various examples	I will provide various examples of questions as an exercise
		Ask students to review answers	I will ask students to review their answers to find and correct errors
	Explanations	Explain the concept of permutations and combinations	I will explain again about the concept of permutations and combinations
		Explain concepts and the relationship between concepts	I will explain the concept and its relationship in the application of problems involving addition, multiplication, and combination rules

		Explain the prerequisite topic	I will explain the filling slot rules, and define the factorials and permutation rules
		Analyze the problem	I will ask students to analyze the problem by paying attention to the characteristics of the problem that involve the concept of combination
	Knowledge of examples	Provides a variety of contextual problems	Examples of problems involving combinations include: - Basketball team selection - Problems related to color mixing
	Knowledge of resource	Using self-designed teaching resources	Electronic books and learning videos uploaded on one of the Teacher's YouTube channels
		Using learning resources provided by the school	Using the Ministry of Education textbook for grade 12
	Curriculum knowledge	Explain the order of topics in the curriculum	I will remind students that the topic of combinations is taught after the permutations and explain the concept of combinations to solve problems.
	Propose of content knowledge	Associating goals with the real-life problem	The benefit of teaching students a combination topic is that they can tackle real-life problems.
		Understanding the concept	I will give the right concept to ensure students are not ambiguous in understanding the problem of permutations or combinations.
	Deconstructing content to key components of mathematical	Construct the problem in several sub-questions	I will provide guiding questions for students to solve problems independently, including what is known, what is being asked, determining whether the problem is concerned with order or not, and what rules should be used.
Content knowledge in a pedagogical context	Procedural Knowledge	Explain the steps to get the right answer	This is a combination problem. First, the number of arrangements is counted for the male group, followed by the process of calculating the arrangement for the female group after obtaining the results and then multiplying. (1). How to choose 3 boys from 4 boys is $C_3^4 = \frac{4!}{3!1!} = 4$ ways. (2) how to choose 2 girls from 4 girls is $C_2^4 = \frac{4!}{2!2!} =$ 6 ways. So, the number of ways to form a group = 4x6 = 24 ways
	Method of Solution	Demonstrating the best solution of the problem	It is a combination problem and must be solved using the formula $C_r^n = \frac{n!}{r!(n-r)!}$ Therefore, the number of ways to choose 3 men and 2 women from a total of 4 men and 4 women is $C_r^n = C_3^4 \times C_2^4 = 24$ ways
	Assessment approaches	Provide test and non-test assessments	I will give a test assessment through a written test and observation with a sheet so that students can understand topics on the learning combination.
		Conduct assessments throughout the teaching process	I will conduct an affective and cognitive aspect assessment at the end of the lesson, as well as a skills assessment during a group discussion
Pedagogical knowledge in a content context	Goals of Learning	Explain the learning objectives according to the curriculum	The learning objectives are that students can analyze combination problems; understand the concept, and are able to solve associated problems
	Getting and maintaining student classroom techniques	Using teaching by question	I will guide students with questions to analyze the problem and determine if it is a permutation or combination. What are the characteristics?
	Classroom techniques	Using class discussion and performance methods	I will ask students to work on worksheets in groups, make presentations and conduct classical discussions.

Suggestions for improving teaching proposed by teachers are adapted to the PCK framework by Chick and Beswick (2018). Furthermore, 48 teachers generated codes proposed suggestions for teaching improvement for the combinations topic. Thirteen teachers coded into six sub-categories under the clearly PCK category proposed thirty-two suggestions for improving teaching. Examples of practice questions include reviewing answers, explaining concepts, relationships between concepts, prerequisite topics, analyzing problems, and providing various contextual problems. Others include using various learning resources, both provided at school and designed by the teacher, explaining the sequence of topics, linking learning objectives and their application, and providing appropriate concept understanding. There are seven similar teacher comments in the explanations category, classified into one suggestion for teaching improvement. For example, teacher comments such as "I will explain the concept of permutations and combinations again" was found repeatedly.

Eight suggestions for teaching improvement suggested by five teachers were coded into three sub-categories under the "Content Knowledge in a Pedagogical Context." Two teachers suggested improving teaching by explaining the steps to get the right answer. One teacher provided a problem-constructing strategy in several sub-questions to explain student errors in the sub-category using the deconstructing content to key component's mathematical structure and connections. In addition, two teachers provided suggestions for improvement by demonstrating the best solution to the problem.

Under the pedagogical knowledge in a content context category, five teachers gave eight suggestions for improvement, which were coded into four sub-categories. This improvement suggestion was used to provide test and non-test assessments throughout the teaching process in the Assessment approach category. Additionally, for the goals of the learning category, the teacher provided suggestions for improvement in making learning goals according to the curriculum. The responses indicate the teaching-by-question strategy, included in the sub-category of getting and maintaining student classroom techniques. Meanwhile, five teachers suggested conducting class discussions and performances for the classroom techniques category. Therefore, in-service mathematics teachers that provide general responses, such as giving written tests, making learning objectives in accordance with the curriculum, and teaching by question and discussion concerning eliminating errors, are very high. The results obtained from this study are similar to literature studies in terms of providing teacher centered strategies to correct students' errors (Korkmaz & Şahin, 2020). Furthermore, in this study, teachers were seen to design their teaching process by giving correct solutions, class discussions, conducting tests, and others. The reason behind the usually superficial responses of in-service mathematics teachers regarding designing processes to eliminate student errors is assumed to be a lack of actual teaching experience. Therefore, insufficient in-service mathematics teacher PCK can be considered a common occurrence.

These research findings show that most teachers' responses indicate that they know students' mathematical misconceptions when learning permutations and combinations. This implies that the teacher is not able to correctly address student errors in solving these problems. Some of the teacher's responses also showed they could explain the causes of student errors and demonstrate good knowledge of the errors. Of the thirteen teachers who filled out the vignette, 9 were unable to identify the students' mistakes on the topic of permutations. They could not identify the missing descriptive statements in the student's written work or in the students' conversations. Only four teachers correctly showed the vignette when filling in students' errors in permutations and combinations topic. When examining the explanation given by the in-service mathematics teacher, it was found that there were misconceptions related to knowledge of the subject matter. The results of this study are in line with the preliminary studies, which show that students have misconceptions about permutation and combination materials. In the first and second vignettes, the in-service teacher responded to the misconceptions about how to obtain the arrangement of alphabets as a group selection and about cyclic permutations under certain conditions. When examining the explanation given by in-service mathematics teachers on the question related to filling in vignette, CoRe and PaP-eRs, it was observed that knowledge of permutations and combinations was limited to the arrangement of objects that paid attention to order and memorization of formulas. The results are in line with the research on misconceptions, such as the assumption that remembering formulas will only work in the short term but also leads to lack of understanding of the context of real-world problems, strategies for verifying answers, and dimensional operation (Batanero et al., 1997; Eizenberg & Zaslavsky, 2004; Fischbein & Gazit, 1988; Quinn & Wiest, 1998).

The findings on student errors align with the facts, which show that teachers have received a lot of criticism due to their limited knowledge of combinatoric topic. According to Hubeňáková and Semanišinová (2020), teachers with low content knowledge are unable to distinguish the nature of the error correctly because they have trouble determining the correct one. They need to look closely at the connections to help students overcome the documented difficulties they face when counting (Lockwood, 2011). Furthermore, they lacked conceptual knowledge to support teacher readiness to teach on different topics (Hill et al., 2008; Loong, 2014). A better understanding of teachers in teaching and learning can increase their knowledge of ways to educate students with various learning styles (Sandefur et al., 2022). Teacher success is primarily determined by better analysis of each student's responses in solving permutation and combination problems (Batanero et al., 1997). Additionally, Soto et al. (2022) stated that solving combinatoric problems can jointly build the teacher's teaching experience on students' understanding and, at the same time, promote students' thinking. However, the teacher has the knowledge to provide suggestions for correcting errors by explaining the basic concepts and procedures that need to be mastered by students, namely the multiplication and addition rules, and factorial definition. Suggestions for improving teachers' skills have paid attention to the prerequisite topics by involving students to analyze the problems, and the teacher can explain the sequence of content. Several learning activities suggested by teachers are oriented towards meaningful learning to help students eliminate errors and activate permutation learning. Most suggestions for improving teaching permutations topic are framed in the "clearly PCK" category. The teacher's suggestions for improvement have focused on activating students through activities providing various problem-solving questions, role-playing, giving awards, using image representations, and analyzing and reverifying the problem. Teachers in learning still carry out the sub-category of providing explanations of concepts by emphasizing the prerequisite concepts. Teachers know how to choose diverse learning resources to activate and facilitate students' learning during a pandemic. Therefore, technology in online classes provides more opportunities for students to visualize mathematical concepts (Cao et al., 2021).

Like the teaching strategies teachers offer in permutation learning, in combination, most of the suggestions are categorized as clearly PCK. Based on the results of the analysis of vignettes, CoRe and PaP-eRs, this research findings show that teachers have not been able to provide deep conceptual knowledge, especially in the cognitive demand of task category, profound understanding of fundamental content and structure and connection of content in the framework of Chick and Beswick (2018).

CONCLUSION

In conclusion, this research revealed that it is possible to evaluate teachers' PCK of permutations and combinations topic using vignettes, CoRe, and PaP-eRs. Most teachers have the knowledge to understand students' mathematical errors in determining the problem-solving steps for the concepts of permutations and combinations. In addition, some teachers have not been able to recognize student mistakes correctly. Although most of the suggestions for improving learning using teachers' knowledge, can reduce student errors, they are still more dominated by general explanations. Teachers training in developing lessons on permutation and combination problem-solving skills will better comprehend students' thinking. Teachers need to use constructivist pedagogy in lesson planning to help students build strong conceptual understanding and prevent errors. This study mainly examined the topic of permutations and combinations with differing findings dependent on the topic. Therefore, given the small number of research subjects, this study cannot be generalized.

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