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Abstract

In Sri Lankan advanced level mathematics curriculum, teachers are required only to provide the intuitive idea of the concept of limit. The purpose of this study is to explore the strategies used by mathematics teachers to achieve this. Twelve in-service secondary mathematics teachers working in government and private schools participated in the study. Data was collected through lesson observations and field notes. Video recorded lessons were transcribed and qualitatively analyzed. Teaching-With-Analogy model was used as a guiding framework to evaluate how teachers integrate real-life examples in classrooms. The analysis revealed few appealing approaches as well as various flaws and misconceptions. In fact, seven teachers took deliberate efforts to link the concept to real-life and among them, four teachers gave an intuition on how x approaches a specific value, through real-life scenarios. However, many were not successful in mapping features of the real-life example to the features of the concept. Few teachers superficially and negligently applied irrelevant metaphors, analogies and colloquial terms. There were gaps in teacher knowledge when incorporating real-life connections to the limit concept. We emphasize the importance of teachers being mindful when using real-life examples in fostering intuition since inappropriate choices of examples could contribute to misconceptions.

Introduction

Mathematics contains numerous abstract concepts that are difficult to comprehend which makes mathematics a daunting subject for many students. Connecting mathematical concepts to real-life is an effective strategy to eliminate such complexities and rigor in mathematics. “Teacher, when are we actually going to use these concepts in real-life” is a famous question that most mathematics teachers encounter during their instructions. Students are usually motivated to learn mathematics if they know when and where to use it in everyday life. There are various kinds of real-life connections that a teacher could utilize in teaching mathematics such as simple analogies, metaphors, classical word problems, real-world data analysis, modelling real life phenomena, etc. (Baki et al., 2009; Gainsburg, 2008; Karakoç & Alacacı, 2015). Such approaches enable students to understand an abstract concept in an intuitive way without referring to any rigorous definitions or formulas.

Most curriculum standards set by countries in the across the world, for instance National Council of Teachers of

Mathematics (NCTM, 2000), emphasize the ability of solving problems related to real-life phenomena. Previous research claim that integrating real-life connections create a utility value, increase student engagement, addresses issues related to confidence (Stoehr et al., 2015), supports brain to see meaningful patterns by connecting different knowledge, subjects and distinct topics (Baki et al., 2009). Arthur et al. (2018) found an increased student interest towards mathematics when mathematics is connected to other subject areas. These finding implicitly claim the role of relatedness in mathematics to real world and the way it motivates learner.

From a teacher's perspective, real-life connections facilitate teachers to deliver an abstract concept in a concrete manner. A teacher could induce open-mindedness, intuitive insights, excitement for learning and perseverance (Harlan, 1896) through everyday experiences. Creating a space for students to share their own life experiences, for instance through a story, enables teacher to discover more about students and their conceptions (Stoehr et al., 2015). Glynn's (2008) work on the way teachers use simple analogies in science classrooms, found that analogy, if used properly, could increase understanding science concepts that are hard-to-visualize. However, in numerous prior studies, mathematics teachers have demonstrated limited knowledge on real-life connections and have used inappropriate and irrelevant examples that lead to misconceptions (Baki et al., 2009; Gainsburg, 2009; Karakoç & Alacacı, 2015; Kula & Guzel, 2014).

Gaps are eminent in school curricula and teacher education programs in emphasizing the importance of real-life connections (Baki et al., 2009). There is a growing body of research focused on evaluating teacher knowledge on real-life examples through questionnaires and interviews. This study contributes new knowledge to the field of mathematics education by exploring the way mathematics teachers integrate real-life connections in actual classroom environments in teaching the concept of limit.

The Concept of Limit

Calculus is one of the most important branches in mathematics which provides a necessary foundation to learn advanced mathematical concepts. However, Calculus has "inherently difficult concepts" (Tall, 1992, p. 14), for instant *limit of a function*. Learning the concept of limit offers the necessary foundation to access almost all other topics in calculus. Continuity, convergence, derivatives and integrals are concepts that are derived using the concept of limit. Therefore, learning calculus can be viewed as the *study of limits.* Regardless of its vast importance, learners usually face cognitive challenges in comprehending the abstract content of this concept.

According to previous research, students exhibit incomplete and unstructured conceptual understanding of the concept of limit (eg. Cornu, 1991; Davis & Vinner, 1986; Juter, 2005c; Roh & Lee, 2011; Tall, 1992; Tall & Vinner, 1981). During the last few decades, growing amount of research have been conducted to explore these difficulties. For instance, students often find difficult to understand the rigorous $\varepsilon - \delta$ definition of a limit of a function (Tall, 1992) and the $\varepsilon - N$ definition of convergence (Roh & Lee, 2011). Literature also report various student misconceptions that are confounded with the notion of limit (eg.; Fernández-Plaza, 2011; Moru, 2009; Sebsibe & Feza, 2020; Williams, 1991) which could arise due to limited content knowledge of the teacher, linguistic ambiguities and over generalization.

The Concept of Limit in the School Curriculum

There are variations to how the concept of limit is included in school curriculum across the world. In countries like England and Sweden, the *informal* definition of limit is taught in the secondary school by way of showing how the gradient of a chord gradually becomes closer to the gradient of the tangent (Viirman et al., 2022). In such countries, the *formal* definition (usually referred to $\varepsilon - \delta$ definition) is introduced only at tertiary level, particularly in courses for mathematics majors (Tall, 1992; Viirman et al., 2022). There are countries like France that introduce the formal definition (without symbolic quantifiers) directly at high school. In Sri Lankan Advanced Level combined mathematics syllabus, "Calculus" is a fundamental topic which is taught under four main parts; Part 1: Limits (18 periods), Part 2: Derivatives (30 periods), Part 3: Applications of Derivatives (15 periods), Part 4: Integration (28 periods).

Students are expected to learn these four major topics in calculus during grade 12 and 13. *Introduction to the Limit of a function* is the very first lesson under "Limits". This lesson is generally taught during second term of grade 12 and the allocated teaching time is two periods (80 minutes). According to learning objectives set by National Institute of Education (NIE, 2017) in Sri Lanka, students are expected to get *an intuitive idea* of $\lim_{x \to a} f(x) = l$. This includes, knowing the meaning of the limit and to distinguish the cases where the limit of a function does not exist. Hence, the usual practice of the teachers in Sri Lanka is to teach the *informal* definition of the limit through tabular, graphical or symbolic representations. Direct methods of evaluating the limits of a function using "Limit Laws" are introduced subsequently.

Role of Intuition

The word intuition has derived from the Latin word *intueri* which means "to look inside" (Tirosh & Tsamir, 2014) and in layman terms, "gut feeling" or "instinct". In education literature, intuition has been defined as the *immediate apprehension* (Waks, 2006), or "the intellectual technique of arriving at plausible but tentative formulations without going through the analytic steps by which such formulations would be found to be valid or invalid conclusions" (Bruner, 1977, p. 13). The role of intuition in mathematics has been long addressed by Fischbein (1987, 1997, 1999) and defined intuition as a cognition that appear subjectively to be self-evident, immediate, certain, global and coercive. It is believed that *Intuitive thinker* would leap to the final answer without careful consideration of well-defined steps and logic (Bruner, 1977) which is opposed to *analytical thinking.* Intuition has been acknowledged as an efficient technique for teaching and learning (Bruner, 1977) as it facilitates delivering an abstract concept in a more concrete way.

Intuition of the Concept of Limit

Different teaching and learning approaches have been proposed in literature to develop students' intuition of various topics in calculus (Roh & Lee, 2011; Roh & Lee, 2017). Real-life examples, metaphors, analogies, table of values, visualization, hands on activities or even a simple explanation could give an intuition to the concept. A study reported designing a task for an introductory real analysis course to bridge the gap between mathematical rigor and student intuition (Roh & Lee, 2017) and integrated an ε–strip activity in the lesson. This study claims that ε – strip activity supports development of students' intuition and ultimately facilitates understanding $\varepsilon - N$ definition in proving theorems in convergence. Therefore, visualization is a key idea in development of theory of limits and continuity and students who rely on textbooks and lectures have ill-defined conceptual knowledge when compared with those who rely on intuition (Siddiqui, 2021). Intuitive idea to the limit can be given through a simple graph and a table of values. For instance, figure 1 shows an example extracted from a calculus text book which provides a tabular and graphical representation to give an intuition to $\lim_{x\to 0}\frac{\sin x}{x}$.

x	$\frac{\sin x}{x}$
±1.0	0.84147098
±0.5	0.95885108
±0.4	0.97354586
±0.3	0.98506736
±0.2	0.99334665
±0.1	0.99833417
±0.05	0.99958339
±0.01	0.99998333
±0.005	0.99999583
±0.001	0.99999983

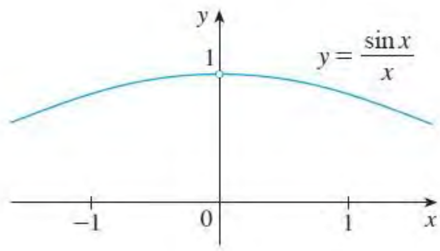

Figure 1. Different Representations to give Intuition to Limits (Stewart, 2011, p. 53)

These approaches could give an "aha" feeling to student without understanding the informal definition of limit. It provides an immediate comprehension that, when x *approaches* to 0 (from the left hand side and from the right hand side) the value of the function $\frac{\sin x}{x}$ *approaches* to 1. Hence, students intuitively guess that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. A simple real-life example, if chosen properly, could also facilitate intuition to simplify rigor in the concept of limit. Integrating one rich example may be sufficient to comprehend an abstract concept than spending hours of explanations. Analogies and metaphors can also be used to bring real-life sense to the classroom and often facilitates understanding a complex phenomenon. This is usually done by comparing and borrowing the language from a familiar system in describing a complex system (Genter & Genter, 1983). For instance, a *water-flow* analogy is being used as mental models of electricity to give an intuitions to the flow of electricity. *"An electrical system can be compared to a water system. Water flows through the pipes of a water system. Electricity can be considered as "flowing" through the wires of an electrical system. Wire is the pipe that electricity "flows" through….*" (Genter & Genter, 1983, p. 108). However it is not possible to match each and every item in analogy to the concept which we are trying to describe. Instead, it will convey the basic idea it entails.

Teacher Knowledge on Real-life Connections

Knowledge demands of a teacher in teaching mathematics is substantial. Through the lens of Mathematical

Knowledge for Teaching (MKT) framework (Ball et al., 2008), the knowledge of integrating real-life examples is a kind of Specialized Content Knowledge (SCK) of the teacher. However, one can also categorize the systematic way of elaborating a real-life connection as pedagogical content knowledge (Guess-Newsome, 1999 cited in Glynn 2008). Even though mathematics education community have emphasized the need on real-life connections in teaching mathematics, there is a little evidence on its' practice in the actual classroom environment (Ball et al., 2008; Gainsburg, 2008; Stoehr et al., 2015). Studies have found that some teachers lack knowledge in incorporating real-life scenarios to the concept of limit and develop misconceptions through inappropriate use of everyday experiences (Gainsburg, 2008; Karakoç & Alacacı, 2015; Sebsibe & Feza 2020; Tall, 1992).

Integrating real-life connections is mainly backed by attitude and perception of the teacher. Teachers believe that heavy content load in the high school curriculum and lack of time refrain them from using real-life connections whereas connecting with unrealistic examples which are not related to student experiences is assumed to be ineffective (Karakoç & Alacacı, 2015). They also believe that it is not suitable to offer examples related to real world in each and every topic in the high school curriculum. According to a study conducted by Gainsburg (2008) based on 62 secondary mathematics teachers, when selecting examples, teachers value context which brings interest to students rather than connecting those to future careers. Further, some teachers believe that real-life connections should be made after mastering a particular concept by students. Lack of training teachers receive, lack of resources and lack of ideas have also been reported as obstacles in applying real-life connections during instructions (Gainsburg, 2008).

Analogies or other sorts of real-life connections are known as double-edged swords (Glynn, 2008), as those can foster learning, but at the same time could induce misconceptions making concepts further difficult to comprehend. In a study conducted to evaluate misconceptions emerging from teachers' instruction, it was found that pre-service (student) teachers (Kula & Guzel, 2013) explained the concept of limit by referring to words used in everyday language causing students to assume that limit is a *maximum value* or a *boundary* which cannot be exceeded. Karakoç and Alacacı (2015) explored the feasibility of using real world connections based on the opinions of sixteen high school mathematics teachers and eight academics in mathematics education. Some of them had mention highway *speed limit, temperature, height* as connections of limits to real life. According to literature, highway speed limit is a popular example in order to introduce the mathematical word *limit*. However it can be taken as an example for the upper bound concept but not for limit.

The above mentioned literature supports the fact that real-life examples should be carefully scrutinized prior to presenting in the class. Teacher may question themselves: why I am using this analogy, how it can help learning, what are the connections which best explains the concept, etc. This preparation will guide teachers to choose the best possible example to support teaching and learning. In the discussion on how an analogy is integrated in classroom instruction, Glynn (2008) mentioned an example "A cell is like a factory" to introduce a *simple analogy*. This study pointed out that these assertions are not likely to scaffold learning and does not support concept development. Glynn's recommendation is to use an *elaborated analogy* in which features in the analogue are systematically mapped with the features of the concept that is intended to learn.

The aim of this study is to explore the way teachers integrate real-life connections to give an intuition to the concept of limit. In order to achieve the aim of the study, we have formulated following objectives.

- To investigate how teachers relate real-life connections to give an intuitive idea to the limit concept.
- To evaluate the appropriateness of the real-life connections used by teachers.

Method

Participants

Data for this study comes from of a two year ongoing research aiming at investigating MKT of Sri Lankan in-service secondary mathematics teachers in teaching limits and derivatives. Twelve in-service mathematics teachers working in ten government and private schools located in three districts (Gampaha, Kalutara & Colombo) in Sri Lanka participated in this study.

Instruments

This study followed a qualitative design. Data was collected through lesson observations and field notes. After obtaining necessary approvals, the schools visits and lesson observations were initiated in year 2021. Teachers were regularly contacted by the research team to find out the time during which the lesson is physically taught in the school. Each lesson on "Introduction to the limit of a function" lasted for two periods (80 minutes). Lessons were video recorded and the observer did not interfere with the teacher and students when the lesson was in progress. Video recorded lessons were transcribed and qualitatively analyzed. To maintain the confidentiality and the privacy of the data, pseudonyms were assigned to all 12 participant teachers as A, B,…L.

Chunks of excerpts which discussed the real-life connections were located by searching the common key words or *linguistic markers* used by teachers. Most teachers initiated their discussions by saying "I will give you an example for limit", "let's assume like this", "I'll tell you another example", "for an example", "where do you use the word limit in real-life?", "what do you mean by limit", "limit is something like this", "I'll give an example to explain how we approaches to *a*". Some teachers directly started their examples and concluded saying "that is the idea behind the limit", "this is what we meant by limit", "That is the idea that is being expressed here as well". These chunks of words covering the entire example and the background discussions were highlighted and analyzed for their relevancy and appropriateness for the context. However, all transcripts were read repeatedly to identify other forms of real-life examples or analogies which couldn't be located through above *linguistic markers*.

To achieve the first objective, we analyzed the examples through theoretical lens of Teaching-with-Analogy (TWA) model adopted from Glynn (1989, 2004, 2008). This model was initially developed through a task analysis of science text books and exemplary teacher instruction in teaching science. The use of TWA model in experiments and classroom settings have shown favorable outcomes in terms of student learning and interest towards science lessons (Glynn, 2008; Harrison & Treagust, 1993). The steps (or operations) of TWA model are as follows.

1. Introduce the target concept – Introduce the concept which is unfamiliar to students (e.g. the concept of limit).
2. Introduce the real-life connection (RC) – This is the familiar concept for students. In this particular study, it could be an analogy, past experience, real word example, etc. that can be used to give an intuition to the target concept (e.g. real life scenario on finding instantaneous velocity).
3. Identify key features of target and RC – Both RC and target have their own features. (e.g. x, $f(x)$, time, distance, how time approaches to zero, etc.)
4. Map similarities (Mapping) – A systematic comparison of features of the RC with features of the target
5. Indicate where RC breaks down
6. Draw conclusions

In practice, the change of the order of steps of TWA is accepted, however, it is recommended to perform all the steps in the model (Glynn, 1994b).

Results

Analysis of transcripts revealed that all teachers in the sample employed one or more methods of representations: graphical, tabulated, numerical and symbolic forms to introduce the concept of limit. However, in this report, we limit our analysis and discussions to the instances where teachers integrated real-life connections on their way towards conceptualizing the limit of a function. Mainly, it was discovered that ten out of twelve teachers in the sample presented different sorts of real-life connections during their instructions. Table 1 shows a summary of real-life connections used by teachers.

Table 1. Summary of real-life connections given by teachers

	Teacher	Description of the real-life connection	Discussion time
1	B	Finding the position of a ball in a cricket match at a given time	Beginning of the lesson
2	E	Boundary wall of the school as the limit of the school	Beginning of the lesson
3	L	Climbing a mountain from two sides, finding the temperature of a campfire, speed limit, capacity limit, unlimited data, Nolimit	Beginning of the lesson
4	J	Breaking a rod continuously, dissolve blue liquid in a water container	During the lesson
5	D	Breaking a rod into small parts continuously	During the lesson
6	G	Example on how salt dissolves in water as water is being added	During the lesson
7	K	Example on how salt dissolves in water as water is being added	During the lesson
8	A	Behavior of children when principal approaches to the class	During the lesson
9	F	Knowing the friend of a stranger	During the lesson
10	I	Person vising to a place through a Google map	During the lesson
11	C	No example was given	-
12	H	No example was given	-

According to Table 1, seven teachers: B, E, L, J, D, G and K took deliberate efforts (through *elaborated analogies*) to bring the real world sense into the classroom. The time spent on discussing those examples ranged from 1- 3 minutes. Only three teachers B, E and L initiated the discussion on limit by relating to real-life examples whereas others approached to the lesson using tabulated, graphical or symbolic representations. The pattern of examples used by J, D, G and K was similar whereas A, F and I used *simple analogies* when explaining certain sections of the concept of limit. Only two teachers, C and H, did not refer to any kind of real-life example during their instruction and only used tabulated, graphical and symbolic representations to give an intuition to the concept of limit. For ease of reference, we have grouped all ten examples into five main cases (cases 1 to 5) based on their similarities and will be discussed accordingly. We will first evaluate the examples given by first three teachers mentioned in table 1, since all of their examples were presented at the very beginning of the lesson. Their example discussions occurred prior to graphical, tabular and numerical explanations. At a glance, it was apparent that all these teachers were trying to prepare a context to begin the lesson through integrating familiar scenarios.

Case 1

As per the TWA model, teacher B first introduced the *target* (i.e. understanding the meaning of the concept of limit) by saying "Ok, Let's discuss what does this concept of limit mean". For this purpose, teacher brought an incident related to two popular sports: football and cricket by which he tried to grab immediate attention of students. Teacher walked around the class questioning the standard duration a football match. An active participation of students was observed and different answers were emerged from them. Thereafter, teacher explicitly recalled students' personal experiences related to watching a football match on the television and reminded how the time is displayed on the TV screen. He drew a diagram on the whiteboard (see Figure 2) to elaborate his example further.

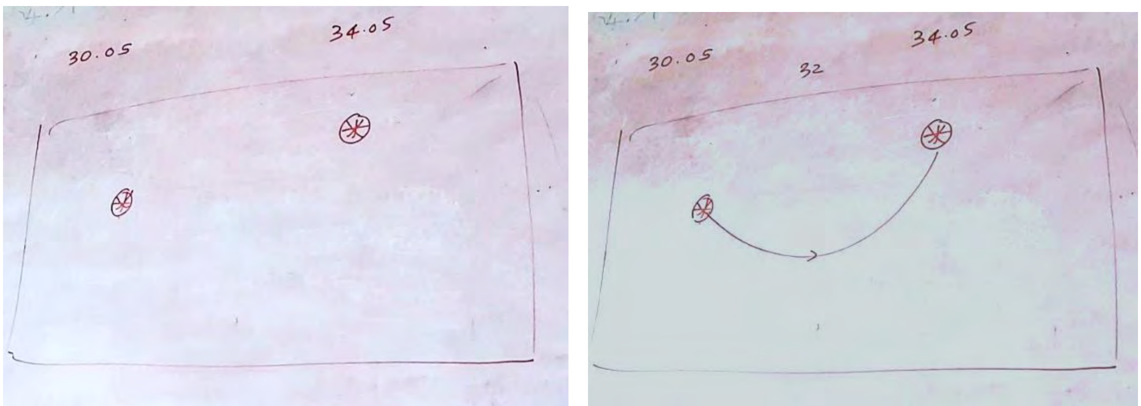

Figure 2. Teacher's Sketch to explain the Position of a Football at a Given Time

Thereafter the conversation (translated from Sinhala to English) continued as follows. The first column contains the excerpts of the transcripts of the teacher and the second column indicates which *step* (in the TWA model) is covered by the discussion.

Table 2. Analysis of Real-life Example given by Teacher B – Part 1

Teacher excerpt	Step
Okay, let's assume like this. This is the whole ground [teacher draws a diagram on the blackboard - refer figure 2.]*...here we think 30.05 ... The fifth second of the 30th minute, the ball is in a place like this…ok …next we have 34.05... The ball is in a place like this...next I'm going to ask from someone... tell me* [by looking at students] *where the ball can be in the 32nd minute?*	2

Teacher B questioned about the position of the football at 32nd minute. One student responded as "*in-between*". Teacher then confirmed the student answer as correct, but further explained that it may not be exactly in the middle of 30.05 and 34.05 minutes, but somewhere in between the two points. The conversation continued as follows.

Table 3. Analysis of Real-life Example given by Teacher B – Part 2

Teacher excerpt	Step
Now we assume, a person like Ronaldo spin the ball like this..it moves like a curve .. then we cannot say this is a linear route, route can be anything...okay ..it can move and in the 32nd minute, it can be anywhere in between these two pints…isn't that so?	3

Teacher B added some features to his analogy, for instance: the locus of the ball as a curve, but not a straight line and implicitly activated students' *prior knowledge* on straight lines and parabolas. He then diverted the students' focus to a cricket match and stimulated students' prior experience on how rewinding the recorded video back and forth enable the third umpire to take a decision on a slip catch or a run out. He explained it as follows.

Table 4. Analysis of Real-life Example given by Teacher B – Part 3

Teacher excerpt	Step
Now you'll see this again in cricket matches…after directing it to the third umpire.. let's say it is a slip catch or a run out ..What is shown in a run out...First watch the view…how the full video is…then it is said to rewind, it is said to rewind back and forth...sometimes in this case you cannot guess whether it is an out or not ? We cannot find whether it is on the line or outside the line since it is a moment in between ... because of the digital system there…Let's say ... we have to go to the 32nd minute right?	2, 3
let's now assume that we are watching a recorded one, what do we do?….bring it slowly from the 30th minute to the 32 side to see where it is in the 32nd minute [teacher explains the content using diagram in figure 2] *what do we do now… getting closer to 32 .. we can come from this side* [Refers to left hand side of the diagram in figure 2.*] to get closer to. from left to right….or from this side…* [Refers to right hand side of the diagram in figure 2] *…and then we can find the exact location of the ball in the 32nd minute.*	2, 3
But there's a small thing .. In limits, we don't check exactly 32, but doing this in close proximity. 32 gradually getting closer to the minute...	5

Teacher B explained how the officials keep rewinding the recorded video back and forth carefully to find out the actual position of the ball before giving the decision of whether the player is out or not. He also emphasized that they may not be able to stop the video exactly at 32^{nd} minute, but could come very close to 32^{nd} minute.

Then he tried to explain the forward rewinding from a point which is less than the 32^{nd} minute and backward rewinding from a point which is greater than 32^{nd} minute. It was apparent that, teacher B was trying to elaborate the real-life example and its' corresponding features (steps 2 & 3 of TWA model), but did not explicitly map similarities between the example and the target during any point of the lesson. However, his closing statement, "*but there's a small thing .. In limits, we don't check exactly 32, but doing this in close proximity..32 gradually getting closer to the minute...* ", covered step 5 in TWA model. That statement indicated where real-life connection breaks down and emphasized the main structural difference between the example and concept of limit.

Through this example, teacher B tried to elaborate how one could make the initial guess further accurate by bringing the time sufficiently close to 32^{nd} minute. Also, by rewinding back and forth from 34^{th} minute and 30^{th} minute respectively, teacher was trying to give a real-life sense of two sided limit, which could otherwise appear as meaningless to most students. This example was likely to facilitate an immediate cognition about the limiting processes associated with the concept of limit.

If he is to match this example with formal definition of limit implicitly, he could have mapped it by saying "it is possible to reduce the *distance* between the ball's actual and the current position *as much as we please* (which can be mapped to $|f(x) - L|$) by reducing the *distance* between current time and 32^{nd} minute (which can be mapped to $|x - a|$)". Teacher could have referred back to this example as soon as he started explaining the informal definition of limit which he didn't. If so, this example could have covered many aspects in TWR model and could be taken as a rich example to give immediate apprehension to the overall concept of limit.

Case 2

As an approach to the lesson, two teachers E and L referred to the Sinhala language equivalent of the word English *limit*, which is "සීමාව" and thereby tried to activate students' *prior experience* on limits. The excerpt obtained from the transcript of teacher E is given Table 5.

In this example, teacher's primary goal was to give a meaning to the mathematical word "limit" through its Sinhala language equivalent. For this purpose, she used *simple analogies*: "It is like a boarder", "nearest area" which are familiar contexts to students. She also referred to a "school boarder" (an analogy) and elaborated some features of it (step 03). However, she was unable to map the features of the analogy with the features of the concept of limit (step 04), but simply explained the analogy. Hence, her approach did not exemplify the most important operations in the TWA model (step 4 & 5) and was not sufficient to serve the actual purpose of using a real-life example.

Table 5. Analysis of Real-life Example given by Teacher E

Teacher excerpt	Step
As soon as we say "සීමාව" [Sinhala term for limit], *something comes to our mind ... what is a limit?*	1
It is just like a border, not exactly a point. Is it clear? Where it reaches…a border...that area... the nearest area…..so now we are talking about the limit of a function under the limit lesson….right? You all have some understanding of what this limit is…that means you know what the limit is in Sinhala.	2
So if we consider an area. what happens if there is a wall...now there is a wall around our school [draws a diagram on whiteboard-see Figure 3] *Is that clear? We call the edge of this* [see Figure 3] *wall as the nearest area to the school .. the nearest place .. the closest to the wall .. close to this side too* [shows the outer boundary of wall in Figure 3] ... *Is that clear? So think of this edge [*shows the outer boundary of wall in figure 3*] also as the limit of our school ... is this clear? Okay? The outer boundary doesn't belong to the school. Outer section doesn't belong to us. But we know that it is the boundary of the wall. Did you realize that it is the limit of the school? So now in mathematics we're talking about the limit of a function.*	2, 3

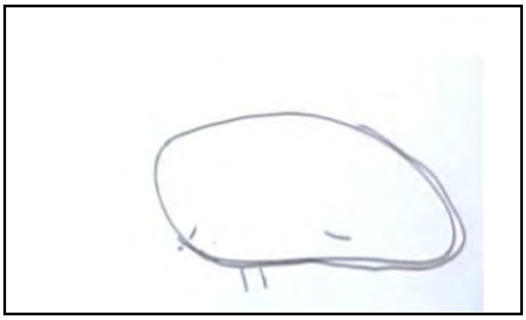

Figure 3. Teacher E's Drawing to Explain the Boundary of the School

There are several dangers bundled with incomplete analogies or examples. By leaving room for students to do interpretation in their own way, it may induce misconceptions and could negatively affect the concept image. It is likely that the example has not been scrutinized by the teacher for its appropriateness and relevancy to the topic. Since the mathematical meaning of the word "limit" is not equivalent to its Sinhala colloquial meaning "සීමාව", such example cannot be considered as appropriate to give an intuition to the mathematical meaning of the word "limit".

In a similar way, teacher L too started off the discussion referring to the everyday use of the English word "limit" as well as its Sinhala equivalent "සීමාව" as an approach to the lesson. The discussion of teacher L was initiated as in Table 6. In the example, teacher accomplished first three steps in the TWR model by introducing the target, introducing the analogies and few features of it. By saying that "It is a very famous word", teacher L tried to remind students about all the other English and Sinhala words associated with the word *limit*. Teacher recalled about "unlimited data" which is a common word related to a mobile package and gives the meaning of no upper limit (or maximum limit) for the data that a person could spend when browsing internet. Also the example on "speed limit" describes the maximum rate of speed that one is allowed to drive on a highway, "capacity limit*"* recalls on the maximum possible weight that a vehicle could transport. Thereafter, students spontaneously added a word familiar to them. The word "Nolimit", which is a name of a popular clothing store in Sri Lanka, was an example given by a student as it contains the word "limit". She finally concluded (step 6 in the TWA model) by saying "What we are going to talk about today under the limit of a function at a point in mathematics is same as the limit that you all talk about everywhere." Such conclusion, even though it signifies the connection of mathematics to real life, cannot to be accepted as mathematically precise.

Table 6. Analysis of Real-life Example given by Teacher L

Teacher excerpt	Step
We need limits to talk about calculus. Therefore, to begin, today we'll be doing the lesson on limit lesson, Now this word [limit] even though you don't know the word limit, but [refers to the Sinhala equivalent word of limit] is used.	1
It is a very famous word. What is the most important word used when you use your internet…? Unlimited data…that means there is data is available without a limit. Then where else will the word limit be used?	2, 3
Student : Nolimit [students laugh] *Nolimit [teacher laughs]….next capacity limit, the capacity limit for those heavy vehicles and big trucks, then there is a speed limit when you are going on the highway.*	2, 3
What we are going to talk about today under the limit of a function at a point in mathematics is same as the limit that you all talk about everywhere.	6

Similar pattern was noticed in the examples given by teachers E and L. Table 7 summaries the words used by two teachers in introducing the word limit. The word limit, when it is interpreted using either Sinhala or English colloquial language, gives a meaning as a boarder or a boundary that cannot be exceeded. However, meaning of a mathematical term cannot be always explained/replaced by colloquial language. Such examples tend to develop misconceptions that limit is the maximum value that cannot be exceeded or a boundary that should not be passed (boundary of the school). When students get an intuition that limit as the maximum value (e.g. capacity limit), it could hinder understanding the two sided limits. As a result, a conceptual conflict may occur and could affect student's conceptual understanding of the concept of limit.

Table 7. Examples Used by Teachers E and L to Explain the Word "Limit" in Mathematics

Words/ Phrases used to explain limit	Description of the word
Boarder	Boundary of a wall
Capacity limit	Capacity limit of heavy vehicles
Speed limit	Speed limit of heavy trucks
Unlimited	Unlimited data package
Nolimit	Reference to a particular clothing store

However, to rectify such misunderstanding, teachers E and L could have emphasized the difference between the colloquial meaning and mathematical meaning of the word *limit*. Teachers require adequate subject matter content knowledge to differentiate and rectify how all composite terms associated with the word *limit* (e.g. unlimited data) do not explain the mathematical meaning of the concept of limit. Teachers need to inform the danger in substituting or borrowing words from *colloquial register* to explain terms in *mathematical register*.

Case 3

For ease of reference, the next sequence of examples given by teacher L were grouped as Case 3. These examples were unique and different from examples given by rest of the participants. After the general discussion on the usage of the word *limit* in everyday life (discussed in Case 2), teacher L extended her discussion prompt to convey the general purpose of learning limit of a function. Presented below (see Table 8) are all three examples pretended by teacher L.

Table 8. Analysis of the Real-life Examples given by Teacher L – Part 1

Teacher excerpt	Step
Now why are we learning this today?	1
[Example 1]	
If I say for example ... if we want to know the exact temperature in the middle of a campfire. If we want to know the temperature in the middle of a fire we cannot jump into the middle of the fire. So what to do? We can use some technique and to find the temperature of nearby point and we can get a little idea of the temperature in the middle of a fire.	2, 3
[Example 2]	
Similarly, let's say….we want to send someone to the top of a very tall building but what if it affect his blood pressure ? He's likely to die ... So we can send him as much as we can without sending him to the upper limit and get an idea about that top point…and could make a decision whether we can send him to that work or not	2, 3
Then that is the practical experience we have on limits.	6

In the two scenarios as shown in Table 8, Teacher L indicated how one can get a general idea about a particular point or a place which we cannot physically reach. According to her first explanation, to find the temperature of the middle of a campfire, one can find the temperature of *proximity points* and get an idea (or a prediction) for the temperature in the middle of the fire. Her second example implies, if it is risky to send a person to a high altitude point or the rooftop of a very tall building, one can send him to a *vicinity point (or point close by)* and thereby could get a general idea about the tallest position of the building. Teacher L was trying explain how vicinity points are helpful in explaining the behavior of a particular point, without actually reaching to the point itself. Above examples signify the general role of vicinity points and how such points supports in knowing an unknown point. However. such examples could give an implied feeling to students that if person cannot reach beyond a particular point, that maximum reachable point is the *limit* or *limiting* point for that person. Thus, there is chance to create a misconception that limit is only an *approximation*. Also this example did not attempt to explain *how much* closer one should reach in order to make the correct guess on a particular point.

Figure 4. Teacher L's Drawing to Explain Example of a Person Climbing a Mountain

In her third example, she started telling a small story and drew a diagram on the whiteboard (see Figure 4). It was about a person who was climbing a mountain from bottom to top, happened to see an abyss (or a hole) at the middle of the road. The person was panicked and stopped climbing as he thought he would fall into the abyss if he would reach there. Another person who came along the same road, but walked from top to bottom, also saw the same abyss. The second person too stopped walking. Then teacher explained, if both of them continue to walk and if they come very closer to this abyss, then they would realize this abyss is just a small hole which they could easily jump over. Finally teacher elaborated that one should not decide on what is happening at a particular point without reaching very closer to that point. Subsequent to this discussion, teacher L mapped the scenario discussed to the function as follows.

Table 9. Analysis of Real-life Example (3) given by Teacher L

Teacher excerpt	Step
In a function like that we have to go close to a point and see if there are things that we can talk about at that point..about the behavior...Go closer to the point, not too far away ... to reach that, we give the word 'what' is the limit ... reach out, get very close ... reach out and see what the behavior of the function is......	4
that's the primary idea of limit…	6
then can we go here and set foot there..can't set foot there..If we set foot there we will fall.	3
so the story of a function at this point is not important to us..we want to talk about this function when we get close to this point from two sides. . This is called limit. Is that clear?.	4, 6

Unlike in the previous example (see Table 8), this scenario is likely to convey the conceptual importance *getting very closer* to a particular point in order to understand the behavior of that point (target). It supports the fact that why we need to select very closer points to "*a*" in explaining the informal definition of limit. For instance, to find $\lim_{x \to 2} x^2$ using tabular method, points are picked (for *x*) in such a way that those are arbitrarily very close to 2 such as 2.01, 2.001, 2.0001, 2.0001 from right hand side or 1.9, 1.99, 1.999, 1.999 from left hand side. Students may doubt or even question the reason for not taking points that lie far away from 2 such as 10, 8, 9, 4 or -5, -4, 0 when evaluating the limit of a function. The above example is capable of responding to such *why* questions during instructions. It is likely to give an immediate comprehension on why we should not talk about a particular point by staying too far away from that point.

It was seen that teacher adequately connected the example to the target through mapping features in the real-life connection with the target concept. By saying *"the story of a function at this point is not important to us. We want to talk about this function when we get close to this point from two sides",* this was likely to conclude that the limit of a function at a particular point does not depend on whether function is defined or not defined at that point. Even if the function is not defined at a particular point, still a limit could exist at that point. However, the analysis of her full transcript revealed that she did not map the similarities between the examples with the limit concept at any point of the lesson.

Case 4

Four Teachers in the sample: G, J, K and D presented examples from everyday life and a similar patern was identified in all four examples. All of them tried to illustrate the process of how *x* approaches to a particular value *"a"* through a real-life connections.

Table 10. Analysis of Real-life Example given by Teacher G

Teacher excerpt	Step
Now we will discuss little about x reaching towards a [simple letter a] like this.	1
Now if you put some salt [repeats]into a water container what would happen to the salt? If I add more water, what would happen? What would happen to the salt taste…it decreases? Isn't it? Can it be reduced to zero? ... Is it possible? You can't make it zero, can you? Though we added lot of water, the melted salt will be there ... isn't that so?	2, 3
That is something like this. Right? This is getting closer too.	4
Now x will not fall to a ... but it will get closer	6

Teacher G started the discussion by saying "*now we will discuss little about x reaching towards "a" like this",* which was her target (step 01). Her intention was to give an intuition to how x approaches to a particular value *a* without actually being equal to *a* $(x \rightarrow a\ ;\ x \neq a)$, by matching it with students' personal experience. Teacher G elaborated a real-life scenario on how the salt level in a water container gradually dissolves when the water is added continuously (steps 02 & 03). In this case, teacher wanted to explain the fact that as we add more water to the container several times, the density level of salt gradually decreases and approaches to zero, but would never be equal to zero (which implies $salt\ level \rightarrow 0\ ;\ salt\ level \neq 0$). She concluded the example by saying "*x will not fall to a ... but it will get closer ...*". Teacher was able discuss the example by completing steps 04 & 06 in TWA model by mapping similarities and giving a conclusion.

Teachers K and J also used same kind of examples to give an intuition to the same process of how x gets closer and closer a without being equal to *a*. In both cases, teachers tried to convey the students that no matter how much water is added to the container, the concentration of blue liquid (or salt) approaches to zero but the concentration will never become zero. At the end of the example, teacher J claimed "*that is the idea behind this limit*". However, appropriateness of such conclusive statement is questionable since the said example only conveyed the idea of how *x* approaches to *a* without *x* being equal to *a*. There was no explicit reference to the behavior of the function values, thus not sufficient to give an intuition to the overall concept of limit.

Table 11. Analysis of Real-life Example Given by Teacher K

Teacher excerpt	Step
Lets say the value of x approaches to a real number a. When x approaches to a, but x does not equal to a, and if fx approaches to L, where L is a real number, then we can say the limit of fx when x approaches to a as L….Okay..then next we will see..I told you… I will give you an example to explain the concept "x approaching to a"...	1
let's say there is a salt solution...let's think two moles per liter ... Then you adds water into this jar little by little and measure the concentration... then what happens..? Concentration is getting... the concentration two is getting lesser and lesser…but does it becomes zero at some point?...it never becomes zero…it only reaches zero but does not become zero..	2, 3
That is the idea that is being expressed here as well…right?	6

Table 12. Analysis of Real-life Example given by Teacher J

Teacher excerpt	Step
Limit has a meaning like this…	1
Contd…[explained her first example] *Then if we take another example...a practical one say that you add some liquid blue to the water…take some water and add liquid blue…then the water turns into blue colour .. then you add more water…and add more and more water to it… what will happen? .. the concentration of blue decreases .. add more .. as you add more water, the concentration of blue decreases ... even though the concentration of blue decreases… does the blue finish? will the blue disappear from there ?. No, it means that no matter how much water is added to it, the concentration decreases but the blue does not disappear.*	2, 3
That is the idea behind the limit.	6

Two teachers J and D explained a scenario on breaking down a stick into parts as a way of giving an intuitive idea to the limit. The extract of one such example presented by teacher D is shown below. D started the lesson with a traditional deductive approach by introducing the concept of limit straight away with geometric and symbolic representations. During her explanations, she discussed the following example to give an intuition to the concept of limit in Table 13.

After the above explanation, she converted her real-life example into mathematical expression as [This was written on a piece of paper as she was using smart board during the lesson] $\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \dots \dots \frac{1}{2^n}$. This example elaborated the concept of limit through a scenario related to convergence of sequences. According to teacher D's explanation, when n approaches to infinity, the length of the rod (which is $\frac{1}{2^n}$) approaches to zero, but never becomes equal to zero. Hence teacher concluded that length of the rod as n approaches to infinity is not zero, but its' limit is equal to zero and wrote it as $\lim_{n\to\infty} \frac{1}{2^n} = 0$. Since the students have not yet learned about the convergence of sequences under the limit lesson, teacher might have used this example to give an intuition to how x gets closer and closer a without being equal to a by mapping it to a rod. Like in the salt dissolving example, the length of the rod approaches to zero, but not equal to zero. However, she finally concluded that the limit of the length of the rod is zero, thereby mapped the features with the features of the concept of limit.

Table 13. Analysis of Real-life Example given by Teacher D

Teacher excerpt	Step
Now…suppose I have a rod. This is a rod. I'm splitting it in two. [repeats]*… when it is divided into two, this one part becomes one half of the entire rod...Clear? Then I divide that half into two again … then this one part becomes a quarter … and then I divide this one quarter again into two parts … then he becomes one eighth. Now children, you would see some pattern here… I can write him as one divide by two to the power two … If repeat this n times what would happens to that one part? It becomes one over two to the power n [pause]*	2, 3
Contd…Now I'm keep on breaking this nth one …the n is going towards infinity...then what's going to happen? This part becomes an insignificantly small... then we say that its' value is zero...Is it actually zero? No…It just approaches to zero…We are denote it like this..when n reaches infinity, one over two to the power n is equal to zero ..	2, 3
So what has been done here... when n approaches to infinity n, the value of one over two to the power n is approaching zero..so did it really become zero? No…its' limit becomes zero	4
So did you get a rough idea of what the limit is? It doesn't mean n is infinity. Infinity is not defined...not a real number…	6

Case 5

Three teachers A, F and I used *simple analogies* to explain few intangible concepts in the limits lesson. Teacher F explained the idea of evaluating the limit of a function by way of *surrounding points* using a real-life incident relevant to students. Teacher F discussed the meaning of evaluating the limit at x equal 1 for the function $f(x) = \frac{x^2-1}{x-1}; x \neq 1$ shown in Table 14.

After discussing the fact that the behavior of the rational function $f(x) = \frac{x^2-1}{x-1}$ is unknown (or not defined) at x equal 1, teacher explained why she was interested in knowing the functional values of surrounding points at x equal 1, which was her target (step 01). She used the statement "*even in class, if we do not know anything about* *a particular child, we will look at the friends who associate with that child and get an idea*" to introduce her analogy and some features (step 02 & 03). Teacher mapped the *friends* to *surrounding points* (step 04) and wanted to convey the idea of looking at surrounding friends to know about unknown friend. She has taken this as an analogy to knowing the functional values of *surrounding points* of x equal 1 to get an idea about the function value at x equal 1. In the next section of the discussion, teacher elaborated on surrounding points located to the left and to the right of x equal 1. Similar to the example given by teacher L in Case 3, teacher F too tried to give an intuitive feeling on the primary role of surrounding when calculating the limit of a function. Also she took an effort to teach the complex intangible concept through a simple analogy which could break the abstract nature of the lesson and connects it to students' personal experience. In this case, F was able to maintain the proper flow of TWA model and accomplished many parts in the model.

Table 14. Analysis of Real-life Example Given by Teacher F

Teacher excerpts	Step
Tell me what do you mean by evaluating limit? I don't know what's happening exactly at x equal 1 for the function...I don't know what is going on [stressed the word] *exactly at x equal 1 for this function. I don't know. I don't have a value. I don't know anything about that… but I am interested in* [stressed the word] *finding out the behavior of the function around* ***x*** *equal 1**..** around**.*** [paused]	1
Even in class, if we do not know anything about a particular child, we will look at the friends who associate with that child and get an idea. Ok?	2, 3
So by looking around x equal 1, the behavior of the function… we are going to determine.. Ok we are going to come to a conclusion what will happen to function when x approaches 1. What will happen to this function fx. Ok. So what do you mean by evaluating limits?	4
Student : check surroundings of that function	6

Teacher A also used a different analogy during the lesson. She explained the limit as "*the behavior of function when x coming towards a".* In order to explain the changes in the behavior of the function, she used an example related to school context. She refers to the children in a primary class who change their behavior when the principal arrives. Then she added "*will your behavior change even more when principal get closer*? [teacher laughs] *It is similar to the behavior of this function."* Even though this example did not adequately give an intuition to the particular concept being taught, teacher was trying to integrate her own random example (or an analogy) spontaneously to explain how the behavior of a function changes.

After explaining left/right hand limits and elaborating that the approaching value x never equal to a, but getting close to a, teacher I used a statement, "You know what? when we are using a google map, then the map says, you are close to a point, but you are not actually going there". However, the purpose of using such analogy related to Google map to explain how x approaches to a particular value was doubtful and did not cover many steps in the TWA model. Table 15 illustrates the overall evaluation of real-life connections through TWA model. It is revealed that many teachers were able to present their examples covering steps 1, 2 and 3, but were not successful in implementing step 4 & 5. Majority of the teachers just concluded (step 6) the example without properly mapping and connecting it to the target.

Table 15. The Overall Evaluation of Real-life Connections through TWA Model

	A	B	C	D	E	F	G	H	I	J	K	L
Step 1: Introduce the target	√	√	-	√	√	√	√	-	√	√	√	√
Step 2: Introduce RC	√	√	-	√	√	√	√	-	√	√	√	√
Step 3: Identify features of RC		√	-	√	√	√	√	-		√	√	√
Step 4: Map similarities			-	√		√	√	-			√	√
Step 5: Indicate Breakdowns		√	-					-				
Step 6: Conclusions	√		-	√	√	√	√	-			√	√

Discussion

According to Sri Lankan GCE advanced level curriculum, teachers are expected to provide only an intuitive idea about the concept of limit, whereas teaching the rigorous formal definition is not included in the syllabus. Based on the analysis of transcripts, it was revealed that all the teachers in the sample used either two or more representations: tabular, graphical, numerical and symbolic, to teach the concept of limit in an informal way. This is the most common approach mentioned in the literature (Tall, 1992) or in the calculus textbooks (Stewart, 2011) in giving an intuition of the concept of limit. This study is limited to reporting the special instances where teachers integrate real-life examples during their effort in conceptualizing limit of functions. Even though some analogies and metaphors were not significant to report, we highlighted such conversations as it provides insights to future educators to be mindful and thoughtful about their language usage during instructions. To begin the discussion, we would like to recall the first objective that this study intended to achieve.

How do Teachers Relate Real-Life Connections to Give an Intuitive Idea to the Limit Concept?

Main purposes of using real-life connections were identified as: to explain how x approaches to a specific value a without actually taking that value $(x \to a\ ;\ x \neq a)$, to explain the need to reach from both two sides to get an idea about a particular point, to explain the importance of vicinity points, to explain the convergence of a sequence and to explain the usage of limit in real-life. The participants offered different kind of real-life connections such as simple analogies, personal experiences, and real world examples to explain the mentioned concepts. Seven out of twelve teachers took deliberate efforts to integrate a real world sense to the classroom at different times. There were differences in the type of examples used by teachers. Some of them referred to extended real world scenarios (or *elaborated analogies*) while others used *simple analogies* to explain the concept of limit.

Three teachers brought few examples at the beginning of the lesson to give a meaningful approach whereas other four teachers discussed their examples during their conceptual explanation of one sided or two sided limits. Majority of the teachers used their examples to give an intuition to how x approaches to a specific value a without actually taking that value. Three participants in the sample tried to give an intuitive idea to the same concept using the incident of dissolving salt (or liquid blue) in the water and how the concentration of salt (or liquid blue) approaches to zero when water is being added continuously. However, some teachers emphasized this dynamic process as the meaning of limit. These results are consistent with Sulastri et al. (2021), a study which claimed that prospective student-teachers explained limit as x approaches to zero or as x is close to zero, but does not touch zero. Sulastri's research further claimed that prospective teachers have understood the concept of limit which is completely different from formal definition and students tend to develop conception that limit is as a process, but not as a value or the result of a process. Most of these results are consistent with what we revealed through this study.

Most teachers showed lack of knowledge and focus in mapping the real-life connection with the target concept. As mentioned in literature, even though the change in the order of the steps in TWA model is recommended, eliminating steps is not recommended. Leaving room for student to map the real-life connection to the target in their own style could create misunderstanding (Glynn et al., 1994b). These results are similar to the study conducted by Bitterlich (2020) which revealed that, when the connection between real-life example and the mathematical concept is not explicitly mentioned, it could result misunderstandings and misconceptions. Such approach has been named as a "decorative mask" since it does not serve the purpose of explaining the concept. Therefore, an example without proper mapping may not scaffold learning a concept. Teachers need to be mindful when integrating real-life examples to explain a particular abstract concept. Scholars have warned the danger in translating abstract concept into concrete. According to Tall & Schwarzenberger (1978, p. 44), "*This translation process contains two opposing dangers. On the one hand, taking a subtle high level concept and talking it down can mean the loss of precision and an actual increase in conceptual difficulty. On the other, the informal language of the translation may contain unintended shades of colloquial meaning."* Therefore extra care must be taken in explaining mathematical concepts using every day experiences.

What is the level of appropriateness of the examples given by teachers?

It was revealed that teachers borrow words from the everyday language to give meanings to technical terms in mathematics. Some teachers gave the first impression to the *limit* by comparing it to a "boarder", "speed limit", "unlimited data", "capacity limit" and "boundary". Referring the limit to a "border" is a well-documented misconception among both students and teachers. Some examples gave an implicit idea that limit cannot be reached, limit is an approximation and limit is the maximum point. These findings are consistent with outcomes of many international research (Karakoç & Alacacı, 2015; Kula & Guzel, 2013; Nagle, 2013). The example of breaking a rod into halves continuously (but the length of the rod will never become zero) may give an implied meaning that limit involves reaching to a point infinitely, thus limit cannot be actually reached. Laing (2016) pointed out that even some text book authors use real-life examples or analogies which could induce such misconceptions.

Misconceptions arising through teacher instructions largely affects student concept image. Since students hear the notion of limit for the first time in their life at grade 12, narrowing the scope of the limit (to a *boarder* or a *maximum point*) could affect concept image which is at the very early stage of development. As a consequence, it may create difficulties when learning the formal definition of limit at university courses. According to Williams (1991), some misconceptions that are engrained in student concept image are irreversible. Literature acknowledge the “good” and “bad” of the intuition (Bruner, 1997). Intuitive beliefs need to be consistent with the definition of the concept of limit which would otherwise create misconceptions. These daily experiences and real-life scenarios, when combined with new knowledge, tend to become part of student’s concept image (Tall & Vinner, 1981). According to Tall and Vinner’s description of concept image and concept definition, all these experiences, mental images, visual representations associated with a the concept can be translated by students into formal definitions. Therefore, the possible danger in intuitive conceptualization is that students could create their own concept images that are not consistent with the concept of limit resulting a cognitive conflict. Language that teachers use, each drawing they make, each example they present have a high power in invoking and developing student’s concept image. Also, metaphors and analogies must be carefully selected by the educators due to several reasons. Mainly, inappropriate metaphors could mislead and affect student’s conceptual understanding of the topic. Also student’s individual interpretation of a metaphor could be complete in opposite to what teacher is trying to elaborate. Therefore, teachers need to scrutinize the examples prior to presenting those in the classroom.

According to Laing (2016), students entering into calculus classrooms usually have an idea on limit based on their everyday experiences. For instance, a student may say that “my limit of running is two miles, and three hours is my limit to keep continuous working” (Laing, 2016, p. 37). If students do not change the understanding of limit based on everyday experiences to mathematical meaning of limit, they will not be able to upgrade their intuitive understanding to abstract level in future studies. Such prior experience on limits could create learning obstacles and hinders learning new concepts. When mapping such instances to our study, the real-life examples given by teachers were unjustifiable as such examples could further stabilize the already established students’ misconceptions.

This analysis exemplifies the gaps in teacher knowledge on integrating real-life connections to the notion of limit. Scholars in Mathematics Education (Ball et al., 2008; Shulman, 1986) claim the importance of having the prior knowledge on student’s misconceptions related the topic of interest. In Ball’s (2008) six domain MKT framework which defines the teacher content knowledge required to carry out the *work of teaching mathematics*, knowledge of student’s misconceptions has been given a greater value and categorized under the subdomain of Knowledge of Content and Students (KCS). Hence this study informs the urgency of developing teacher knowledge both in terms of subject matter knowledge and pedagogical content knowledge.

Conclusion

This qualitative study explored the approaches initiated by of twelve in-service secondary mathematics teachers in Sri Lanka in integrating real-life connections to give an intuition of the concept of limit. Many teachers presented different sorts of real-life connections such as analogies, personal experiences and real world scenarios to make limit comprehensible to students. Majority of the examples were used to give an intuition to the process of how x approaches to a without actually being equal to a and to explain the importance of knowing about the vicinity points. However, many examples were presented without systematically mapping onto the target concept, leaving room for misconceptions. This study further revealed the danger in borrowing words randomly from colloquial language to explain terms in the mathematical register. Examples picked from web sources needs to be scrutinized before presenting in the class. This study informs on developing teachers' Subject Matter Knowledge (SMK) on integrating real-life connections and Pedagogical Content Knowledge (PCK) on student misconceptions in order to standardize instructional practices in schools.

Recommendations

This study revealed a lack of sufficient set of real-life examples to give an intuition to the concept of limit. Therefore, it would be a better approach for teachers to have a pre-determined common set of real-life examples related to mathematics. These examples need to be carefully selected, critically analyzed and scrutinized before actually presented in the classroom.

Curriculum needs to be reformed in order to promote intuitive grasping. In Sri Lankan content, teacher guides lacks real-life examples to be used in instruction. Therefore, including real-life connections in text-books and teacher guides, emphasizing its importance, is essential in practicing this task. This approach is a solution for teachers, especially in the advanced level section, who tend to believe that real-life integration is an extra burden for them. Also it motivates teachers to use such validated examples without any hesitation.

Lack of knowledge on students' common errors and misconceptions also calls for initiatives to develop teacher pedagogical content knowledge aspects. Education authorities could organize workshops and awareness programs for secondary mathematics teachers to discuss how, when and why real-life examples should be presented in the classroom to give an intuition to a certain topic of interest. It is recommended to include a special module to Mathematics Education programs on bringing intuition through real-life context. Moreover, pre-service teachers can be trained to access current literature on mathematical concepts and to gain necessary prior-knowledge on student misconceptions. Teacher training sessions could inform teachers on the importance of following a standard frame work for instance, Glynn's Teaching-With-Analogies model, in presenting real-life examples.

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