Building Thinking Classrooms Online: From Practice to Theory and Back Again

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Abstract

In the COVID-19 era of adapting to pandemic lockdown protocol, teaching practices have become more negotiable and less tethered to the familiar and institutionally normative practices found in educational settings around the world. With a shift to online teaching, many practices are being adapted from face-to-face settings and being imported into online settings. However, this sort of adaptation is by no means trivial, and a direct transfer of practices may not necessarily be effective or plausible. While adaptation is undeniably necessary, a theory *for* teaching can offer guideposts around which adaptation may occur. Over many years of empirical investigation into how to enhance the synergy and capacity of students' thinking in face-to-face mathematics classrooms through systematically bypassing institutionally normative practices, the *Building Thinking Classrooms* framework offers a basis for one such theory. While this framework is used in many different contexts, one of these is in the education and professional development of mathematics teachers in tertiary and professional settings. However, with COVID-19 protocols in place, the tightly woven face-to-face practices of this framework had to evolve and be adapted. In this article, we discuss and exemplify how we drew from these face-to-face practices a set of principles, which served as guideposts for designing adaptations for engaging adult learners in mathematical tasks in a fully online setting. In our analysis, we consider not only the adaptations for online teaching we made, but the process of adaptation through a theory *for* teaching we used in designing effective and intentional learning settings for adults experiencing mathematics.

Key words: mathematics, online, teaching practice, teacher education, theory for teaching building thinking classrooms

Introduction

Adult learners return to the study of mathematics for a variety of reasons (e.g., to fulfill economic needs, for personal fulfillment, etc.) and the learning contexts in which they do so vary widely (e.g., parent education, financial literacy, workplace and vocational education, adult basic education, pre-service and in-service teacher education, etc.) (Safford-Ramus, Misra, & Maguire, 2016). Regardless of context, adult learners face various boundaries and barriers towards learning mathematics based on their past learning experiences and life situations (FitzSimons, 2019). Their personal responsibilities and life pressures make them aware of why they are learning something and how they can apply it in their lives (Knowles, Holton, & Swanson, 1998). In turn, they desire an active role in decisions and discourse in a learning environment (FitzSimons & Godden, 2000). Adults have also been "positioned by practices of curriculum (Popkewitz, 1997), pedagogies and psychologies about mathematical reasoning and learning (Popkewitz, 1988; Walkerdine, 1994), and textbooks (Dowling, 1998), [and] these practices are not neutral but reflect larger economic, cultural and political considerations" (FitzSimons & Godden, 2000, p. 15). The multiple and overlapping subjectivities adult learners carry are called up by a range of classroom practices and are further shaped by new classroom practices they encounter. As such, teaching practices used with adults who are learning mathematics in post-compulsory settings require careful attention about how they shape their experiences of doing mathematics, and in consequence, of thinking mathematically. It is thus important for practitioners to be reflective and cognizant of their own practices and how they acknowledge the adult learner's needs for engaging in the thinking process. Moreover, it is important for practitioners to consider how practices can be adapted when contexts change since every teaching context provides novel challenges related to engaging adults in mathematical thinking.

While there are many approaches to teaching adults mathematics, our interest in this paper is to examine our own teaching practices used with adults learning mathematics in tertiary pre-service teacher education and teacher professional development settings in which we adopted a Building Thinking Classrooms (BTC) (Liljedahl, 2016, 2020) model of instruction. In particular, we examine how we shifted our teaching practices from those that were appropriate in our face-to-face settings, to those we used in fully online settings with these populations in response to the limitations created by the COVID-19 pandemic. While we do not extend our discussion to the experiences of our adult learners in this paper, we choose instead to focus on our interpretation of adapting our teaching practices to meet the needs of our learners in a new context. By investigating our own practice, we are serving the global aims of improving the learning experience for adults learning mathematics in our context of tertiary and professional education. We are also revealing a viable approach to adaptation of teaching practice.

To this end, we first visit the roots of where our face-to-face practices emerge from by reviewing how the BTC model of instruction arose, what it is, and how we used it in our adult settings. We then reveal how we approached designing adaptations of the BTC model for the fully online environment and showcase how we implemented some of these adaptations

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The Emergence of Practice

The teaching of adults in tertiary and professional settings can look very much the same the world over. For the most part, it follows a model of demonstration and reproduction – what is often called an I do—we do—you do approach to teaching. To understand why this is, we first take a brief look at the origins of public education and consider where these normative practices arose from.

Looking back at when the first industrial revolution came to a close, countries around the world at this time realized that if they wanted to continue to grow their economies, they would need to educate their citizenry. Out of this realization was born the concept of public education (Katz, 1987) and with it the institution of school, which was constructed to create conformity and compliance. To achieve this, public education was built on a foundation of the three institutions that were, at the time, seen as successful (Egan, 2002).

1. The *church*, which already had a mandate to educate the masses and from which the early designs of classrooms were drawn.
2. The *factory*, from which we learned the principles of mass production.
3. The *prison*, where had learned how to manage and move large numbers of people.

Together, the influences of these three institutions shaped what the classroom looked like, and, in turn, what teaching looked like at the dawn of public education. It was at this time that we saw the emergence of a pedagogical model that we now call *I do—we do—you do*. This model capitalized on the efficiencies of the factory while maintaining the control of prisons, and it looked like church, with the teacher at the front and all the students facing forward.

And through the process of *cultural reproduction* (Bourdieu & Passeron, 1990), classrooms of today, and the teaching that takes place inside them, still look very much the same. These norms transcend the classroom (Cobb, Wood, & Yackel, 1991; Yackel & Cobb, 1996) and have woven themselves into the very fabric of the institution of school - forming what can only be referred to as *institutional norms* (Liu & Liljedahl, 2012). But these norms transcend K-12 (primary and secondary) education and have infused themselves into what it means to teach in general – at all levels from primary to tertiary and for all audiences from children to adults. This is not to say that education has not changed over the course of the last 150 years. Curricula have evolved, there have been efforts to create access and equity in education, and the role of technology has vastly altered what is possible in (and out of) the instructional setting. The desks have evolved from church pews to desks to tables, and we have gone from blackboards to greenboards to whiteboards to smartboards. But much of what happens in K-12, tertiary, and professional development settings today is not too dissimilar to what happened in these settings a century ago. That is, although there has been great evolution of *what* is taught in the last 150 years, the institutional norms that were laid down at the dawn of public education still dictate much of *how* teaching looks in tertiary and professional settings today. Learners are still sitting, and instructors are still standing. Instructors are still writing on boards and learners are still writing in notebooks. And instructors are still following the *I do—we do—you do* pedagogical routine.

In our efforts at designing effective and intentional learning spaces with adults learning mathematics in our tertiary and professional settings, we asked: How do we change this? How do we break the cycle of cultural reproduction to change the experiences of our adult learners? One of the ways we have achieved this is by drawing on the research of Liljedahl (2016, 2020) on how to build thinking classrooms. This research offers a set of teaching practices developed systematically out of challenging institutionally normative practices. Although it was enacted in the K-12 setting, we have found numerous points of connection with the world of adult education and have been able to transfer he ideas seamlessly into our adult education settings. Since our face-to-face teaching practice is based on this research, we first discuss its highlights.

Building Thinking Classrooms

In visits to 40 different K-12 mathematics classrooms in 40 different schools, Liljedahl (2016, 2020) found that in all cases, the lesson began with some form of teacher demonstration (I do), followed by student replication either individually or in groups (you do), which in turn was followed by some form of consolidation (we do). Although the details of how this looked, the amount of time apportioned for each activity, the degree to which students worked in groups, and the degree to which technology and manipulatives were incorporated varied, what did not change was a general adherence to this routine. Liljedahl (2016, 2020) further observed that in a typical lesson, there was very little opportunity, and even less need, for students to do much thinking. Closer examination of this observation (Liljedahl, 2020, Liljedahl & Allan, 2013) revealed that in a typical mathematics lesson only about 20% of the students did any real thinking and, even then, only for about 20% of the lesson. Instead, students relied on a slate of behaviors that included slacking, stalling, faking, and mimicking to slide through the lesson without thinking. Liljedahl (2016, 2020) attributed this to the aforementioned institutional norms that not only dictate many of the activities of teaching, but also the activities of learning.

Liljedahl (2016, 2020) posited that for this reality to change – in order to get more students thinking and thinking for longer – a radical departure from the institutional norms would be needed. And thus was born the *Building Thinking Classrooms* (BTC) project which, for over 15 years, sought to empirically emerge and test pedagogical practices that not only afford opportunities to think, but that necessitate thinking and increase thinking in the classroom. This work was organized around the 14 general categories of practice that all teachers adhere to in some shape or form.

1. What types of tasks we use.
2. How we form collaborative groups.
3. Where students work.
4. How we arrange the furniture.
5. How we answer questions.
6. When, where, and how we give tasks.
7. What homework looks like.

8. How we foster student autonomy.
9. How we use hints and extensions to further understanding.
10. How we consolidate a lesson.
11. How students take notes.
12. What we choose to evaluate.
13. How we use formative assessment.
14. How we grade.

Each of these general practices served as a variable in the research, which involved more than 400 K-12 teachers implementing thousands of two-week micro-experiments, each of which sought to measure the degree to which a specific practice impacted the amount of thinking observed. More details about methodologies involved and results can be found in Liljedahl (2016, 2020).

Emerging out of this research are 14 teaching practices, one for each general practice, that have been proven to produce more thinking in the classroom than the institutionally normative practices they sought to replace as well as more thinking than any of the other hundreds of practices experimented with (Liljedahl, 2020). These practices are described briefly below.

1. *The types of tasks we use*: Lessons should begin with good problem-solving tasks. At the beginning, highly engaging, non-curricular tasks are used, but after a period of time, they can be gradually replaced with curricular problem-solving tasks.
2. *How collaborative groups are formed*: At the beginning of every class, a visibly random method should be used to create groups of three to will work together that day.
3. *Where students work*: Groups should stand and work on vertical non-permanent surfaces (VNPS) such as whiteboards, blackboards, or windows, making work visible to the teacher and other groups.
4. *How we arrange the furniture*: The classroom should be de-fronted with desks placed in a random configuration around the room (but away from the walls) and the teacher addresses the class from a variety of locations within the room.
5. *How we answer questions*: Teachers should only answer the third of three types of questions that students ask: (1) proximity questions – which are questions asked merely because the teacher is close; (2) stop thinking questions – which are questions that aim to cease thinking e.g., “is this right” or “will this be on the test”; and (3) keep thinking questions – which are questions that get them back to work.
6. *When, where, and how we give tasks*: The teacher should give tasks verbally (as much as possible) at the beginning of the session from a non-central location in the room after gathering students around them. If there are data, diagrams, or long expressions in the task then these are written or projected on a wall, but the instructions pertaining to the activity of the task should be given verbally.
7. *What homework looks like*: Rather than assigning homework or practice questions, students should be assigned 4-6 questions for them to check their understanding. Students should have the freedom to work on these in self-selected groups or on their own, and on the vertical non-permanent surfaces or in their desks, and should be for self-evaluation and not marked or checked.

8. *How we foster student autonomy*: Students should interact with other groups extensively, both for the purposes of extending their work and getting help. As much as possible, the teacher should encourage this interaction by directing students towards other groups.
9. *How we use hints and extensions to further student understanding*: The teacher should maintain student engagement through a judicious and timely use of hints and extensions to maintain a balance between the challenge of the current task and the abilities of the students working on it.
10. *How we consolidate a lesson*: When every group has passed a minimum threshold, the teacher should pull the students together to debrief what they have been doing. This debrief should begin at a level that every student in the room can participate in.
11. *How students take notes*: Notes should consist of meaningful notes written by students to their future selves. Students should have autonomy of what goes in these notes and how they are formatted, and the notes should be based on work that has already taken place.
12. *What we choose to evaluate*: Summative assessment should honour the activities of a thinking classroom (evaluate what you value) through a focus on the processes of learning more so than the products. It should not in any way have a focus on ranking.
13. *How we use formative assessment*: Formative assessment should be focused primarily on informing students about where they are and where they are going in their learning. This requires, by necessity, several different activities from observation to *check your understanding* questions to unmarked quizzes where the teacher helps students to decode their demonstrated understandings.
14. *How we grade*: Reporting out of students' performance should be based on the analysis of the data, rather than the counting of points, collected for each student within a reporting cycle. These data need to be analysed on a differentiated basis and be focused on discerning the learning that a student has demonstrated.

Although this set of BTC practices emerged from research in the K-12 setting, they have been used effectively in adult tertiary and professional learning settings such as in adult basic education courses (Larsen, 2018a; 2018b), teacher education courses (Mellone, Pacelli, & Liljedahl, 2021), and in teacher professional development settings (Andrà, Rouleau, Liljedahl, & Di Martino, 2019; Liljedahl, Andrà, Di Martino, & Rouleau, 2015; Rouleau & Liljedahl, 2016; Rouleau, Ruiz, Reyes, & Liljedahl, 2019). While significant efforts have been made in adult mathematics education settings to support some of the BTC practices such as prioritizing collaboration (e.g., Gibney, 2014; Oughton, 2009), fostering autonomy (e.g., Larsen, 2015; Yen & Liu, 2009), and involving formative assessment (e.g., Looney, 2007), the BTC framework offers a complete set of practices that work in conjunction with each other to ensure a context that is rich in student thinking. In many cases, the implementation of this set of practices makes a radical departure from the prevailing norms of the context, particularly in larger institutional settings. In other cases, such as in more informal or progressive contexts, less so. Regardless, this set of practices marks a significant evolution of teaching from the institutionally normative practices that permeate and have permeated education for the last 150 years and offer us new ways to enact teaching in the face-to-face classroom

Disruptions of COVID-19: From Practice to Theory

But then in early 2020, COVID-19 hit. Suddenly, the practices that had taken 15 years to emerge out of face-to-face classrooms no longer seemed relevant to the online setting that COVID-19 necessitated. Digital modalities replaced every bit of classroom practice. Group work became more challenging to arrange, hands-on activities became next to impossible, and communication was constrained. Every aspect of practice was now challenged by new contextual obstacles. Teachers around the world were sharing this struggle and some found it easier than others to tailor their face-to-face practices into the constraints set by online learning tools.

We, the authors, were regularly implementing BTC practices in adult tertiary and professional learning settings prior to the COVID-19 pandemic. When the pandemic forced us to move into a fully online learning setting, we were faced with adapting practices developed in face-to-face environments to fit into an online environment unfamiliar to us or our students. Rather than adapting BTC practices to the online setting in a random fashion, we decided to conduct a thematic analysis of the BTC practices themselves to identify the core guiding principles behind each practice. Such an analysis would help identify aspects of the BTC practices that were important to retain as we adapted to teaching in a novel context (i.e., the fully online learning setting) while maintaining the integrity of their aims of prioritizing learner thinking over *I do—we do—you do*.

In essence, we aimed to extract from the empirically deduced BTC framework of practice a theory *for* teaching. We did so deliberately and with full knowledge of the controversy this can invoke. We understand that in mathematics education the idea of a theory is reserved for constructs that are empirically, philosophically, and theoretically deduced explanations for natural phenomena, such as learning, and not intentional actions, such as teaching.

While theory provides us with lenses for analysing learning (Lerman, 2001), the big theories do not seem to offer clear insights to teaching and ways in which teaching addresses the promotion of mathematics learning. (Jaworski, 2006, p. 188).

And, because of this, a theory *of* teaching is not possible.

Theories help us to analyse, or explain, but they do not provide recipes for action; rarely do they provide direct guidance for practice. We can analyse or explain mathematics learning from theoretical perspectives, but it is naive to assume or postulate theoretically derivative models or methods through which learning is supposed to happen. Research shows that the sociocultural settings in which learning and teaching take place are too complex for such behavioural association (Jaworski, 2006, p. 188).

However, we were not trying to create a theory *of* teaching. Rather, we were trying to extract from the BTC framework a theory *for* teaching–a theory that does not *explain* teaching, but that *guides* teaching. There is a distinction between theory *of* and theory *for*. In what follows, we show how we developed this theory *for* teaching as a set of *principles of practice* and then articulate how we used these results to adapt BTC practices for the online setting with adult learners, such as those made necessary by the COVID-19 pandemic.

Method

Towards developing a set of principles of practice from a series of practices, we chose to apply a thematic analysis. Thematic analysis (Braun & Clarke, 2012) is a systematic way of “identifying, organizing, and offering insight into patterns of meaning (themes) across a data set” (pg. 57). In our case, our data set was composed of results from the micro-experiments from which the BTC practices were developed. Our process for thematic analysis was therefore to consider each of the BTC practices and code them for their underlying principles.

When codes seemed too similar, they were confounded into a single code, and when codes were too broad, they were broken into more specific principles. Practices could have more than one code, but the codes themselves were compared and contrasted with each other until they were deemed unique. For example, the BTC practice of using vertical non-permanent surfaces could be coded as both guided by an effort to *decentralize control* as well as to *mobilize knowledge*, and these denote different principles of practice.

Both authors engaged in this process individually and then compared their results, adapting codes as necessary. This was done iteratively until the smallest possible set of principles of practice emerged that spanned all of the BTC practices.

Results

Through this process, a set of six *principles of practice* emerged that together underpin all the BTC practices. These six principles are listed in Table 1 below, along with the BTC practices they emerged from, and are briefly outlined in what follows.

Table 1. Principles of Practice and the BTC Practices they emerged from.

Principle of Practice	BTC Practices
Decentralize control	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14
Give them something to think about	1, 5, 6, 7, 9, 10
Do first	5, 6, 9, 10, 11, 12, 13
Diversity is a strength	2, 6, 9, 10
Mobilize knowledge	2, 3, 4, 5, 7, 8, 9, 10, 11
Assessment as communication	7, 8, 12, 13, 14

Decentralizing control is about the teacher relinquishing control over what students do and how students think. It is therefore also about how authority (in the sense of authorship) is distributed in the learning environment. Contrary to a traditional classroom, in which authority and control are centralized to the teacher, the BTC framework positions the teacher away from the center, granting more power to students to share their ideas and to have capacity in the space. Decentralizing control (and by consequence, distributing authority) in the learning space permeates and underpins almost all the BTC practices. This includes how the environment is set up (e.g., a de-fronted room), how the teacher acts in the space (e.g., how questions are answered), and how students act in the space (e.g., student autonomy). Although the teacher is a central organizer, their aims are (ironically) to step away from the center of organization, and to give value and voice to members of the learning space.

Giving them something to think about is about prioritizing meaningful content that is rich in ideational connections and that offers multiple access points for thinking. It is the guiding principle for what kinds of tasks are selected for a BTC environment as well as how the tasks are given, how questions are answered, how hints and extensions are used, how learning is consolidated, and what homework looks like. In every step where content is introduced or discussed, students ultimately need to have something to think about. If we show students how to do something before they have had a chance to attempt it themselves, we end up with students who mimic procedures without thinking about why they are doing them or how they could adapt them. Therefore, in all the micro- and macro-moves in a BTC environment, our guiding principle is to give students something to think about and to keep them thinking. That is the essence of what the BTC practices were designed to achieve.

Doing first is the foundation of prioritizing learner thinking over *I do—we do—you do*. It guides how we give tasks, how we use hints and extensions, how we answer questions, how students take notes, and how we consolidate the ideas after a period of activity. It also applies to how we evaluate what we value through the co-construction of rubrics after a period of doing, and in self-assessment. In a BTC environment, we engage in action first and then use approaches to consolidate ideas from the environment where activity occurred. This is in stark contrast to traditional teaching methods in which talking about how to do something happens first and is followed by doing what was 'instructed'. BTC practices are designed to reverse this order so that what learners do is rich in thinking, and therefore, not conducive to mimicking. This principle is closely related to *giving them something to think about*, but is more specific to the order in which things occur in a BTC environment.

Diversity is a strength is a perspective that contrasts traditional aims of conformity. Instead, diversity is seen as important for valuing and amplifying learner thinking. This principle influences how both macro- and micro-moves occur in a BTC environment and has to do with both diversity among agents and within their ideas. At the macro-move level, decisions about using visibly random groupings and their size are influenced by this view. Knowing that diversity is necessary in a group for proper group functioning means that when we choose group size, we make sure enough of the diversity can live in that group of agents. At the micro-move level, how we answer questions in groups and how we use hints and extensions are both highly influenced by the view of maximizing the diversity so that new thinking can prevail. It also pertains to consolidating learning, where we emphasize the diversity among ideas in the room. In a broader sense, diversity is seen as necessary for enriching a thinking space and acts as a mindset for a teacher when making decisions.

Mobilizing knowledge is an essential component of how a BTC environment functions. When thinking is at the forefront, any knowledge that is inert needs to be mobilized. This can happen at the individual level, at the small group level, or at the large group level. At the individual level, how we answer questions and how we use hints and extensions aids in pulling ideas out of individuals and possibly, in small groups. That is, it helps ideas become reified into a visible artefact. At the larger group level, how we foster autonomy helps with mobilizing knowledge between groups. The choice of using vertical non-permanent surfaces and visibly random groups in a defronted room aids in mobilizing knowledge since inert ideas are reified on vertical non-permanent surfaces and can then be moved around the room through micro-facilitation strategies. The facilitator can bring groups together or pass on ideas between groups through minor moves such as 'look over there'. Further mobilizing of knowledge occurs during consolidation, where the facilitator draws together the ideas from around the room and weaves them into a meaningful narrative. Mobilizing knowledge continues into the notetaking process where students mobilize the consolidated knowledge in their own personalized ways, and where they find ways to try it out on 'check your understanding' questions. Mobilizing knowledge is therefore the backbone of the entire knowledge cycle in a BTC environment from individual learners to the grand collective of learners and back to individual learners.

Assessment as communication is a mindset that influences how assessment occurs in a BTC environment and what counts as assessment. It reframes the traditional approach of using assessment to control behaviour and to rank learner abilities. Instead, it presupposes a feedback cycle that occurs in how formative assessment, summative assessment, and reporting out are conducted as well as how homework is framed. At the more granular level, assessment as communication also informs how hints and extensions are used to prompt students to think in different ways, thus assessing and communicating at the same time. Rather than making assessment a distinct event, it is occurring continuously as a form of communication in the learning process.

Teaching Online: From Theory to a New Practice

When the COVID-19 pandemic hit and all teaching was forced to move online, adaptations to practice needed to be made. This was inevitably challenging since many of the necessary aspects of a BTC environment, such as 'using vertical non-permanent surfaces' and 'defronting the room' were no longer viable in the online environment. Instead, digital applications dictated the possibilities for what adaptations to practice could look like. In our case, this involved the use of the Zoom conferencing platform to host synchronous online sessions, with the use of hyperlinks to other tools as needed. While there were some other similar platforms available at the time, this platform was chosen based on its feasibility in terms of cost, availability, and ease of use for all participants. Therefore, all the adaptations to practice we address in this paper are based on the possibilities and limitations of the Zoom conferencing platform. Platforms like Zoom typically support institutionally normative practices such as that of having a central authority deliver content to a set of listeners. While some collaborative tools are available (e.g., breakout rooms), limitations on what participants can see or hear still exist and are affected by the various technologies they have access to in their remote sites. These various constraints create new dilemmas around teaching design choices for which guideposts for adapting practice become necessary. To this end, the principles developed in this study became instrumental for creating an effective BTC environment online.

To illustrate how these principles of practice were used to tailor adaptations to the BTC practices for a fully online context, we (the authors) use examples from our own work as educators implementing these adaptations. During the 2020-2021 academic year when teaching moved into a fully online context, we were both working with both pre-service and in-service mathematics teachers in tertiary and professional development settings in various locations in Canada. This included semester-based pre-service teacher education courses at our home institutions as well as single session or series-based workshops for in-service teachers across Canada as part of their professional development. Prior to the pandemic, we had each been using the BTC practices with these groups of adults regularly to engage them in mathematical tasks and reflect on the teaching and learning of mathematics. These groups of adults ranged widely in age and teaching experience, as well as in their mathematical background. Group sizes ranged from as small as 16 in the first author's pre-service secondary mathematics methods course, to more than 100 in the second author's in-service teacher workshop sessions. And session meeting times typically ranged from 2.5 to 6 hours. Notwithstanding, our teaching now could only be conducted via online meetings, which we did via synchronous Zoom meetings, maintaining similar meeting frequencies but now cutting down meetings to about half as long as in pre-pandemic conditions.

In what follows, we discuss how three of the six principles manifested within the online environment: *decentralized control*, *diversity is a strength*, and *mobilizing knowledge*. While the examples we use are drawn from the pre-service and in-service mathematics teachers in our tertiary and professional development settings, our focus remains on our own development of practice. Our purpose in revealing these examples is to support how our teaching decisions were impacted by the principles arising from our analysis of the face-to-face BTC practices; that is, to illustrate how the principles can be harnessed as a theory *for* teaching.

Decentralized control

While almost all the BTC practices decentralize control in some way, the use of vertical non-permanent surfaces is perhaps most notable. In a face-to-face thinking classroom, students use vertical non-permanent surfaces (e.g., whiteboards) to notate their ideas while working in visibly randomized groups. Using such a medium was found to improve students' time to task, time to first mathematical notation, amount of discussion, eagerness to start, participation, persistence, and non-linearity of work (Liljedahl, 2019, 2020). Having students write on vertical non-permanent surfaces also grants them power to erase their work if they so choose. Instead of prioritizing ideas originating from the teacher, learner's ideas are given value in such a space, and can later serve as launch points for consolidation of learning.

When moving into the fully online environment via synchronous Zoom meetings, decentralizing control became more challenging since online meeting platforms such as Zoom are built for speakers presenting to 'the masses', and its tools are centered on showcasing the ideas of one central authority rather than providing opportunities for distributed authority among the agents. That is, it is very well designed for a presentation, but more limited when it comes to supporting collaboration. While making visibly randomized groups became easier with the use of the randomization option when creating breakout rooms in Zoom, decentralization of thinking through vertical non-permanent surfaces became more challenging.

To this end, we employed the use of an external tool that participants could follow a hyperlink to that allowed them some of the features vertical non-permanent surfaces would offer. Namely, we chose to use Google's Jamboard, a web application that allows a number of users to access and edit the document synchronously, with a variety of mediums for notations, and a slide deck organization option for easy access. While Zoom breakout rooms each have their own whiteboard options with some similar tools, these were at the time very limited (e.g., the eraser only would erase the whole screen not parts of it) and they were only visible to the participants in that group. As such, the Zoom whiteboard option offered limited capacity to act as vertical non-permanent surfaces do in a face-to-face setting. Moreover, our principle of decentralizing control shed light on the necessity of having a non-permanent work space for participants that could promote the notation of ideas during a group work task with opportunities for distributing control to participants. In Jamboard, they could create new pages, add new images or drawings, and could refer to various pieces of information that were preloaded into the slides (such as task information). This was not all as easy within Zoom itself.

For example, the Jamboard in Figure 1 below was used as a writing space for groups working on a mathematical task (in this case, a numbers and operations puzzle) in a session with in-service mathematics teachers.

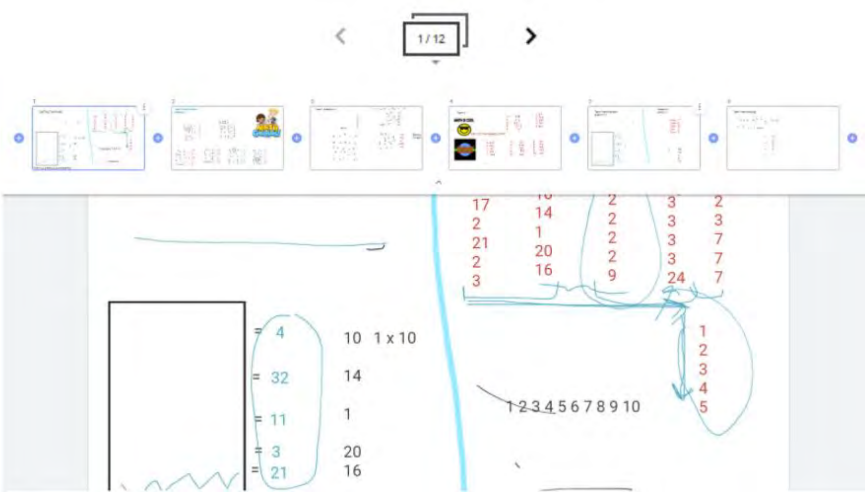

Figure 1. Example of multiple pages of different groups' working spaces on a Jamboard.

As may be seen, each group had their own workspace. As they worked, they could communicate about their ideas verbally via their Zoom breakout room's audio line. And they could add pages, refer to information on other pages, and easily erase or markup as needed. This was also all visible to the teacher, but the teacher was decentralized. Our awareness of the importance of decentralized control in this case supported our choice to use the Jamboard tool.

Diversity is a strength

However, merely implementing the use of a Jamboard did not necessitate a synergy of ideas among participants. When control is distributed, the ideational landscape hinges on having enough diversity in each group of agents (Larsen & Liljedahl, 2017). This means that we had to be careful with how we formed groups beyond that of making sure they were visibly randomized. The principle of seeing diversity as a strength guided us to reconsider group size.

In face-to-face thinking classrooms, research showed the ideal visibly random group size was three (with some twos): having only two students in a group risked the possibility of not having enough diversity among them to develop ideas for the task, while four students in a group risked the possibility of having so much diversity that the stability of the group would be compromised (Liljedahl 2016, 2020). With frequent and visibly randomized groupings of three, students were willing to offer an idea, irrespective of whether they believed their idea would lead to a solution, and students were entering into their collaborative groups willing to think rather than just follow. Students also had opportunities to show various capacities within each grouping, and therefore, could take on various roles each time they were in a new group. The diversity in each grouping easily came to surface since face-to-face settings allow for communication not only via verbal or digital markup tools, but also in gestural ways.

When we shifted to online teaching, the use of frequent random groupings seemed to create silence more so than thinking. Cameras or microphones would be turned off and the technological barriers would be used as a legitimate way to not participate. We believe this had to do with the communication tools available in the online setting. The incredible anonymity in the online setting grants power to students to choose to engage or not. Whereas in a face-to-face thinking classroom setting, students moved quickly into their groupings and promptly began communicating in at least gestural ways, students in an online setting often chose to disengage, cutting out the informal communication that would normally occur haphazardly. And if students end up being placed in a group with others who act this way, even those who would engage more passively are now left out of any engagement at all since they cannot see into other groups' work easily and cannot see what those hidden behind their cameras are trying. This left a smaller number of students in each grouping who were potentially willing to take the active move to communicate with the others they were grouped with.

The 'diversity is a strength' principle helped us identify that the issue with the transfer of this practice to an online environment had to do with the lack of diversity being mobilized in the online setting, leading to it being a diversity depleted space. Therefore, we needed to actively compensate for this tendency of depletion. We found we could compensate for this lack of diversity in two ways. First, we needed to increase the size of the groupings to five or six students rather than three. We found that doing this in the online environment created a similar synergy as when grouping in threes in a face-to-face environment. Doing this increased the number of those willing to engage in each group. As participants became more comfortable with the environment and were more likely to share, the grouping size could be gradually reduced. Second, we understood that sometimes participants needed to work on a task on their own prior to entering the online collaborative environment. This amplified the diversity of ideas in the group enough that when they were placed in the online grouping, they had things to discuss and were not entering empty handed. When we did this in a face-to-face thinking classroom, it created too much diversity for the groups to function cohesively. But in online environments, providing this individual thinking time prior to collaboration seemed to boost the diversity to a minimum functional level.

Mobilizing knowledge

However, the diversity of ideas not only needed to exist in a group, but also needed to be mobilized within and across groups in order for further ideational development to occur. In a face-to-face thinking classroom, knowledge moves between students and groups both through passive and active means (Liljedahl, 2020). Passive knowledge mobility occurs when a participant or a group casually looks to another group's workspace to glean a hint or an extension. Active knowledge mobility occurs when either the teacher pairs groups or groups pair themselves to have more active discussions about specific ideas.

With this in mind, Jamboard was chosen as a participant workspace for the online environment due to its capabilities not only of serving as a digital proxy for a vertical whiteboard, but for its possibilities for knowledge mobility. With its many frames, one Jamboard can accommodate several groups with the possibility of groups being able to easily navigate between frames to see what others are doing (see Figure 2 below).

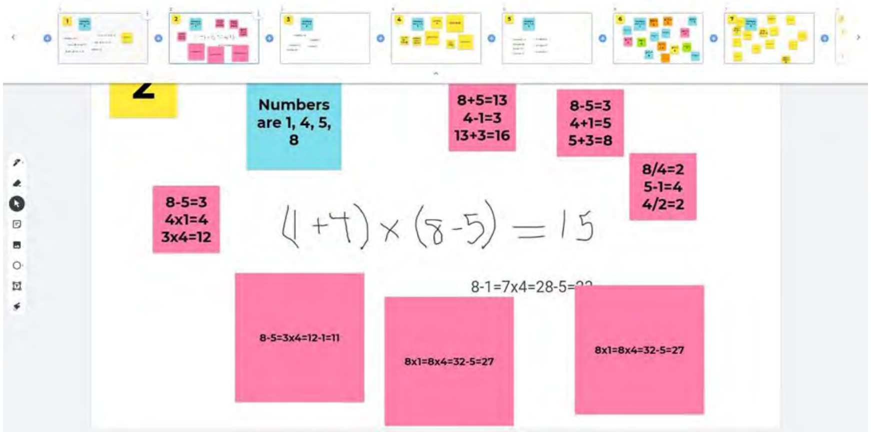

Figure 2. Example of a Jamboard with many group pages.

Jamboards allowed us to afford passive knowledge mobility, but only partially. We found early on in our work in online environments that despite the fact that groups had passive access to other groups' work, they did not make use of this in the same way as in in-person thinking classrooms. This is because navigating between frames in a Jamboard takes a certain active effort. Rather than being able to casually move their gaze from their own vertical whiteboard to another group's board and back again, they needed to click into a frame in order to see it. This, it turns out, was not the same as flipping between frames on a Jamboard. For one, it was more active than the casual glance over the shoulder. In addition, it made it very difficult to look back and forth between frames. We needed to find a way to recreate the casual over the shoulder look that is such a powerful medium for knowledge mobility in the face-to-face thinking classroom, and thus was born the *knowledgefeed*.

A knowledgefeed is a collaborative GoogleDoc that a teacher sets up for students to keep open on their desktop while working in certain settings and is populated with the kinds of things that students would see on VNPSs in a face-to-face classroom. This includes everything from the task at hand, to hints, extensions, and (pictures of) student work (see Figure 3 for an example).

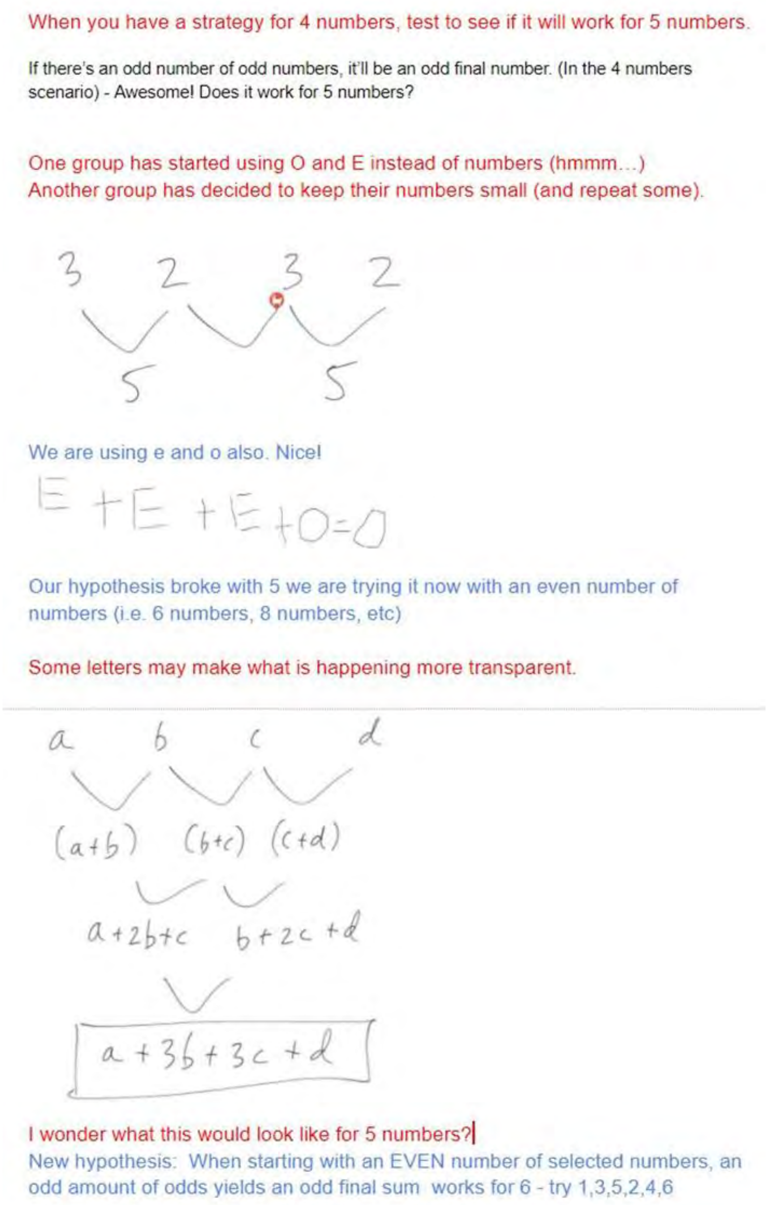

Figure 3. Example of a knowledgefeed for a problem-solving task in a professional development setting.

When we first started to use a knowledgefeed in our online settings, it was something that only we, as the instructors, posted to. After a while, however, it became apparent that participants' voices and images were missing from this feed. Yes, we were adding screen captures from groups' Jamboards, but we were choosing what to add as a way to focus the thinking of the groups. But this is not an accurate analogue of the thinking classroom. In an effort to create a proxy for knowledge mobility we were re-centralizing the control. So, we opened the knowledgefeed to allow students and groups to populate the space with their images, ideas, and questions of their own (see the blue text in figure 3). As they populated it, they also serendipitously encountered other ideas, which they could then bring back into their groups to continue working on their task (e.g., pursuing extensions).

Although allowing participants to contribute made the feed somewhat chaotic, it more accurately approximated the non-linear way in which knowledge moved around in a face-to-face thinking classroom. In this way, the knowledgefeed was a solution created in response to the need for mobilizing knowledge in a way that continued to decentralize control and amplify diversity. It illustrates another choice we made based on the principles identified from our thematic analysis.

Conclusions

Teaching practice requires many micro and macro-level decisions and can require swift adaptations when contexts change. While there is no such thing as a theory *of* teaching to guide necessary adaptations, theories *for* teaching are viable in this regard. In this paper, we explored how a set of principles of practice could be drawn from the BTC teaching practices that had already been established, reified, and validated in their effectiveness in face-to-face settings. These principles of practice served as a theory *for* teaching and helped guide our understanding of which aspects of the face-to-face BTC practices we needed to retain and which we needed to adapt when moving to a fully online teaching setting. By means of a thematic analysis, we moved from practice to theory, which then allowed us to move back to practice in a new setting (i.e., a fully online context). The implications of this are far-reaching.

Of primary importance, we now have a set of six principles of practice that have emerged from the BTC framework for enhancing learning. These principles can now be used to adapt BTC practices to other settings (e.g., online synchronous vs. asynchronous, hybrid synchronous and asynchronous, etc.). By taking into consideration these principles, we can justify our micro- and macro-decisions whenever we face new constraints. While the BTC practices give direction on what to do specifically, the principles give perspective on what is important. In our case, this is to decentralize control, give students something to think about, to do first, to remember that diversity is a strength, to mobilize knowledge, and to treat assessment as communication.

On a broader scale, in going through this process, we have developed an approach for tailoring a theory *for* teaching out of one's own current practice. By using a thematic analysis on the practices used in one context to emerge principles that guide these practices, any educator may then transfer their teaching practices to novel contexts with a more in-depth understanding of their own teaching decisions. That is, the adaptations to teaching practice can be justified and targeted rather than decided on at random. In our case, to determine how best to facilitate mathematics problem solving in adult tertiary and professional development online settings, we could have proceeded in many ways. For instance, we could have succumbed to institutional norms of following an *I do - we do - you do* model because of software constraints. Or, we could have aimed to mimic the BTC face-to-face practice as exactly as possible (e.g., keeping the same grouping sizes) while guessing at how to replicate practices that were no longer possible (e.g., defronting the room). However, emerging principles of practice before importing them into a new setting helped us, as educators, to be able to justify our decisions, to adapt intentionally, and to be more confident with implementation of the adaptations.

Given the variety of settings found in adult mathematics education, traversing teaching contexts can occur. This approach of reifying principles of practice may be a way to guide and justify adaptations. And given that the principles of practice we developed in this paper emerged from practices designed to challenge institutional norms, these principles may serve to break patterns of cultural reproduction in settings that remind adult learners of their subjectivities towards mathematics as rooted in their past experiences with learning mathematics.

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