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Preservice Elementary Teachers' Understanding of Fraction Multiplication and Division in Multiple Contexts

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Abstract

The present study examined preservice elementary teachers' performance on the problems of multiplication and division of fractions and compared their performances and analyzed the misconceptions. An instrument including 11 fraction multiplication and division tasks was given and the task involved three contexts: making own story problem, computations, representing operation using visual model. The findings reported that among the three contexts, making a diagram was the most challenging task for both operations, and their division performance varied depending on the division problem types. The author suggests that specific emphasis with rich story problem with different whole(s) in fraction, carefully designed context with different types of division concept, and building fractional number sense can help both PSTs and students reduce misconceptions and enhance deeper understanding of fraction operations.

Keywords:

Teacher Education in Elementary Mathematics Education, Teacher Education, Preservice Teachers' Learning Mathematics

Introduction

The pivotal role of real-world contexts and visual representations in teaching fractions are outlined in policy and recommendation documents (e.g., National Council of Teachers of Mathematics [NCTM], 2014; Common Core State Standards Initiatives [CCSSI] 2010). Despite of the importance of this concept, the struggle with teaching and learning fractions has been an ongoing issue for in-service teachers, pre-service teachers (PSTs), and elementary students (Newton, 2008; Park, 2013). Unfortunately, elementary teachers' knowledge of fractions is often weak (Ball 1990a; Ma, 1999; Park, 2013) and deficiencies in fraction knowledge continue to exist for both preservice and in-service teachers (Ma, 1999). In particular, the multiplication and division of fractions are known for being the least understood and the most mechanical topic in elementary school mathematics (Izsák, 2008; Tirosh, 2000).

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These studies document PSTs' lack of conceptual understanding of multiplication and division with fractions and address how conceptual understanding is critical for their future students. For instance, without a deep understanding of the concepts, their students cannot reproduce meaningful instruction of fraction division (Ma, 1999), and teachers' knowledge often predicts students' achievement gains (Newton, 2008). Iskenderoglu (2018) emphasized that if we want the students to have conceptual understanding of fractions, we need to prepare PSTs to have robust conceptual understanding first.

Then, how can we support preservice teachers? The present study aims to make some suggestions for teacher educators through exploring elementary preservice teachers' conceptual understanding of fraction multiplication and division. This content area was a focus because fraction multiplication and division is a critical component of elementary school mathematics that relates to students' learning algebra and rational numbers in the later grades (Ball, 1990a; Luo, Lo & Leu, 2011). In addition, research studies continuously report preservice teachers lack conceptual understanding in this area (Lou et al., 2011; Ma 1999; Son & Lee 2016; Tirosh 2000; Tirosh & Graeber, 1990). The goal of this study is to first assess PST's performances in multiplication and division of fractions in various contexts and identify and analyze most common difficulties on the tasks. The three contexts are creating story problems, solving real-world application problems, and visually representing the algorithmic solutions. The rationales for these contexts are: (1) Creating story problems plays a pivotal role in establishing links between real-life situations and operations with fractions (Abu-Gyamfi et al., 2019). (2) providing real-world context when engaging in fraction division problems is critical because it helps not only to build a strong foundation for understanding fractions (Kent, Empson, Lynne, 2015) but also gives meaning to the division of fractions to students. (3) making connections among stories and diagrams in the problem with fractions are important because symbols help students make sense of fraction operations (Cengiz and Rathouz, 2011). Furthermore, this study examines solution strategies and the misconceptions of fraction operation tasks to better understand what it means to divide or multiply with fractional numbers (Newton, 2008). Taken together, the present study aims 1) to examine PSTs' conceptual understanding of fraction multiplication and division in three different situations, b) to explore PSTs' specific difficulties or misconceptions in engaging with multiplication and division.

Theoretical Framework

PST's Understanding of Fraction Multiplication and Division.

Research on teacher education continuously reports on the lack of conceptual understanding among preservice teachers on fraction multiplication and division. First area of struggles is the connection between the correct application of fraction operations and fraction word problems (Graeber & Tirosh, 1989; Ma, 1999; Seaman & Szydlik, 2007; Tirosh 2000). For instance, Graeber and Tirosh (1989) administered a written test to 129 preservice elementary teachers, and the test included 26 multiplication and division fraction word problems. The PSTs were asked to write an appropriate expression for the given word problems. The findings indicated that 25% of the respondents incorrectly wrote a division expression as an appropriate method to the solution for the problems involving fraction multiplication context. A parallel finding was documented in Tirosh's later study (2000) that PSTs provided multiplication expressions for fraction division problems. Tirosh explained that the participants' mistakes in finding the appropriate operation for the problems resulted from the misconception that multiplication makes always bigger, and division always makes smaller. Seaman and Szydlik (2007) revealed another example of PSTs' misuse of operational symbols in the word problem of fraction multiplication. For the problem "Brooke has a $\frac{3}{4}$ pound bag of M&Ms. If she gives $\frac{1}{3}$ of the bag to Taylor, what fraction of a pound does Taylor receive? (p.173)" all the participants were uncertain how to approach this story problem and most of them tried to apply fraction subtraction initially because of the word "give away". In the meantime, Ma (1999) focused on the conceptual analysis of the fraction division word problems written by Chinese and U.S. in-service teachers. Her research uncovered that many of US teachers were not able to come up with correct word problems that would match the given fraction equation, $1\frac{1}{2} \div \frac{1}{2}$. Among 23 U.S. teachers, six failed to create a story and 16 made up stories with misconception. Only one teacher was successful in creating a fraction division problem. These studies generally indicated that when asked to write an equation based on the word problems involving multiplication or division of fractions, PSTs tend to come up with incorrect operation in their equations and the connection between the two is a challenging task.

In a similar vein, other research group investigated PSTs' conceptual knowledge of fraction multiplication and division in terms of its multiple representations (e.g., pictorial illustration, graphical representation using area model, linear models, part-whole model, etc.). (Adu-Gyamfi et al., (2019); Izsák, 2008; Lee & Lee, 2021; Lee et al., 2011; Luo et al., 2011; Son & Lee 2016;). One big idea shared in these studies was that many PSTs' struggle to visually represent the fractions when it comes with fraction multiplication or division. For instance, in the study of Lee & Lee (2021), PSTs were asked if a square-shaped cake on a geoboard was equally shared among 3 people and to justify if each part represented $\frac{1}{3}$ of the cake. They reported that majority of incorrect answers were associated with a misconception of area and a visual-dependent appraisal. In the study of Son and Lee (2016), 60 PSTs were asked to find $\frac{3}{4}$ of $\frac{3}{4}$ in the word problem context using graphical representations as if they were teaching to 5th graders. They found that 62 % of PSTs recognized the given word problem correctly as fraction multiplication but 11 % of PSTs failed to provide correct visual representations to explain the problem. The comparative study of Lou et al., (2011) reported also addressed that that many PSTs in both USA and Taiwan was not able to find the incorrect visual models for fraction multiplication equations such as $\frac{3}{4}$ of $\frac{4}{5}$.

Another main idea of the related research was about the pedagogical benefits of using visual representations in this operation. Son & Lee, 2016 underlined that various representations/models can be more helpful to make sense of an algorithm than representing fractions with a symbolic-only format, and it also can be used as a means of reasoning about fractional quantities (Izsák, 2008; Lee et al., 2011). More recently, Morano & Raccomini (2020) reported that PSTs strong conceptual knowledge of fraction operations were associated with the ability to accurately model fractions multiplication and division using visual representations and story problems. Their findings support the earlier findings of Adu-Gyamfi et al., (2019). Adu-Gyamfi and his colleague addressed that teachers' ability to draw visual representations is essential knowledge to teach fraction division conceptually because it allowed them to understand what kind of situations lead to fraction division and what kind of reasoning occurs in fraction division situations.

Mathematics Knowledge of Fraction Multiplication

The related research indicates that the important conceptual knowledge required in fraction multiplication is the concept of the whole (unit) (Lee et al, 2011; Mack 2001; Son & Lee, 2016). Son & Lee asserted that to fully understand fraction multiplication procedures understanding a fraction as part of a whole (two levels of unit, e.g., ¾ means "three of four parts") is not enough. These two levels of understanding should be expanded to three levels of unit understanding (e.g., ¾ means three-fourths of one whole). According to the study of Mack (2001), students were more successful with fraction multiplication problems when they demonstrated an understanding of the three levels of a unit such as various uses of portioning and units. In a similar vein, Lee at al., (2011) explained that to understand the meaning of fractions, multiplying fractions such as $\frac{2}{3}$ $\frac{2}{3}$x $\frac{2}{3}$, one should know the relationship between the fractional number and the referent unit. For instance, $\frac{1}{5}$ x $\frac{1}{5}$ can be interpreted as taking $\frac{1}{5}$ of a unit, and the referent unit is ⅔ of the whole and not $\frac{1}{5}$ of the one whole. Hence, the answer of $\frac{2}{15}$ should refer back to the same whole to which the $\frac{1}{5}$ referred to. The important role of the unit in the fraction is also emphasized in the study of Izsák (2008). He studied teachers' mathematical knowledge for teaching fraction multiplication, and this case surrounding teachers highlighted how important it is to have a full understanding of the three levels of units, not only to illustrate drawings correctly but respond to students' questions. In his study, the teacher explained that $\frac{2}{3}$ is bigger than $\frac{2}{3}$ by using the common denominator strategy (computation) but struggled to visually show why $\frac{3}{4}$ is bigger than $\frac{2}{3}$ using the area model of fractions. In addition, when representing $\frac{1}{5}$ of $\frac{1}{5}$ on the number line, the teacher had trouble responding to a student who was confused between one-fifth of a whole and one-fifth of a third. Through these findings, Izsák (2008) emphasized that teachers' understanding of the multi-level structure of the unit is necessary when using drawings to reason about fraction multiplication. These studies provide evidence that the knowledge of the unit is important to successfully perform fraction multiplication problems beyond its computational strategy.

Mathematics Knowledge of Fraction Division

In this study to explore PSTs' conceptual understanding of fraction division, the researcher applied the concept of whole number division because fraction division can be understood in connection with whole number division (Son & Crespo, 2009). The different conceptualizations of whole number division were addressed in many research studies (Ball 1990a; Ma 1999; Lo & Luo 2012; Van de Walle, 2010). Among them, division as a measurement and partitive concept was discussed before the data collection based on the course textbook (Van de Walle, 2010). The concept of whole number division, i.e., 10 (dividend) ÷ 2 (divisor) = quotient, can be interpreted in two ways. The first concept was finding the number of groups when the size of the group is known. For instance, if there are 10

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cookies and each individual will have 2 cookies, how many people will have 2 cookies? This concept of division was described as measurement division (Ball, 1990) and the answer can be found by thinking of how many groups of 2 fit into 10 with the answer being 5.

The same equation (10 ÷ 2 = 5) can be interpreted differently such as in a sharing context when things are equally shared among the groups, i.e., 10 cookies are equally shared between 2 people. This type of division concept is known as partitive division. Prior research suggested that a sharing context (partitive division) would be a helpful approach to teaching fractions because of its intuitive nature (Van de Walle, 2010) but, at the same time, it limits the conceptual understanding when the divisor is a fractional number such as $\frac{2}{3}$ of a person (Lo & Luo, 2012).

As discussed within the section on fraction multiplication, understanding unit/unitizing is highlighted as an important piece of knowledge to build a conceptual understanding of fraction division (Lee, 2017; Lo & Luo, 2012). They explained, in measurement division, $2 \div \frac{2}{3}$ can be interpreted as "how many $\frac{2}{3}$ kg bags of rocks can you make from a total of 2 kg of rocks?" Students need to be able to conceptualize $\frac{2}{3}$as a unit to measure 2 kg of rocks. However, this fraction equation is not easy to solve when considering partitive division problems because it is hard to imagine sharing 2 objects/things with $\frac{2}{3}$ people. It also requires students to think of division as the inverse of multiplication. In this scenario, the unit is not as clear as the measurement concept because the divisor is a fractional number. However, the situation would be different if a divisor is changed to a whole number such as $\frac{2}{3} \div 2$. This would be easier to solve with the partitive concept of division because students can think $\frac{2}{3}$ left of the cookie is shared by 2 friends solving for how much of the whole cookie each will have. As the author discussed the two meanings of whole number division with the participants, the author was interested in examining how their knowledge of whole number division would play out when they solve the fraction division problem.

In summary, this study had two major research questions. First, how preservice teachers would perform in solving fraction multiplication and division problems in three different contexts described above? Second, are there are any different patterns of difficulties (or misconceptions) in solving multiplication and division problems?

Methodology

Participants

The data for the present study came from 46 elementary preservice teachers (PSTs) who were enrolled in the elementary teacher preparation program at a mid-sized university in the western United States. The participants were in their senior year when they take their mathematics methods course. The researcher is a mathematics professor who teaches elementary mathematics methods courses at the university. The data were collected in the middle of the semester before the researcher covers fraction concepts to capture PST's prior knowledge regarding their conceptual understanding of fraction multiplication and division. Before fraction instruction, the instructor taught whole number multiplication and division, and two types of division concepts– partitive and measurement – were discussed and practiced. The concept of division was applied to the fraction division problems during the survey. The purpose of this was to explore if their drawings and posing word problems would differ by different concepts of division or the number relationship between the dividend or divisor. The researcher hypothesized that if the dividend is greater than the divisor, such as in the case of Q3 ($2 \div \frac{2}{3}$), it would be more appropriate to use a measurement approach to find out the solution because one can reach an answer by thinking how many groups of $\frac{2}{3}$ are contained in the dividend two. On the contrary, the partitive interpretation would be easier to solve the division task when the dividend is smaller than the divisor such as Q4 ($\frac{1}{3} \div 2$) by using a sharing context (e.g., $\frac{1}{3}$ pizza is shared by 2 friends).

The data collection method was a paper-and-pencil assessment with 10 problems that consisted of four fraction multiplication items and six fraction division items. To measure PSTs' conceptual understanding of fraction operations, the focus of tasks extended beyond a basic computational competency. The items were categorized into three parts: Part 1) four writing a story problem from a given fraction symbol, part 2) three solving fraction story problems, and part 3) three problems that require representing a visual diagram from fraction equations. The task items in part 1 were designed to measure PST's conceptual knowledge. These conceptual tasks required PSTs to make their own story problems to correspond with the multiplication and division of fraction equations. As Adu-Gyamfi (2019) argued, teachers need to be able to explain the meaning of fraction multiplication and division to their students since it is important to examine whether PSTs can identify when we use fraction operations in real-life situations

In part 2, PSTs' computational knowledge was a major focus. The items in this category asked the participants to solve the fraction word problems and show the process of how they solved the problems. Lastly, the items in part 3 asked PSTs to provide pictorial representations that would match the given fraction multiplication and division expressions. Drawing a diagram was chosen because many teachers struggle with drawing diagrams to match algorithm solutions and such difficulties imply teachers' limited knowledge of fraction multiplication and division (Izsak, 2008, Lee, 2017). Part 3 was designed to measure PST's conceptual understanding of fraction operations along with their computational knowledge, especially to investigate if the participants were able to justify the algorithm procedures with appropriate visual representations. See below for the 10 task items.

The fractional numbers in this survey were selected based on 5th grade common core state standards because PSTs should know this concept first to be able to teach their students (Iskenderoglu, 2018). For instance, number and operation of fraction standards (5.NF.B.3) requires solving word problems involving division of whole numbers leading to answers in the form of fractions (Q5: 5 ÷ 2), or multiply a fraction x whole number (Q1: $3 \times \frac{1}{3}$) or fraction by a fraction (Q3: $\frac{1}{3} \times \frac{3}{4}$, Q8: $\frac{3}{4} \times \frac{1}{3}$). In terms of division standards (5.NF.B.7), students need to able to apply whole number division concept to divide unit fractions by whole numbers (Q4: $\frac{1}{4} \div 2$) or whole numbers by unit fractions (Q9: $5 \div \frac{1}{4}$). Common core state standards also ask students to use visual model to represent the fraction problems. (Q2: $2 \div \frac{2}{3}$; Q10: $\frac{1}{4} \div 5$). The 10 tasks are illustrated below.

The Task

Part I: From a symbol to a story

Using your own words, create word problems with the following fractional numbers:

Q1. $3 \times \frac{1}{2}$	Q2. $\frac{1}{3} \times \frac{2}{3}$
Q.2 $\div \frac{2}{3}$	Q4. $\frac{1}{3} \div 2$

Part II: From a story to an answer

Solve the following problems and show your work.

Q5. March and Jada share 5 yards of ribbon equally. How much ribbon will each get?

Q6. It takes half a yard of ribbon to make a bow. How many bows can be made with 5 yards of ribbon?

Q7. Paula has 9 pounds of candy bags. If she uses ½ of what she has for Halloween, how many pounds will she have used?

Part III: From a symbol to a diagram

Multiply or divide the following fractions. Draw a model to explain your thinking.

Q8. $\frac{3}{4} \times \frac{1}{4}$	Q9. $5 \div \frac{1}{4}$	Q10. $\frac{1}{4} \div 5$

Analysis

The 10 tasks are designed to assess multiple aspects of PSTs' understanding of fraction multiplication and division. To analyze the data, the researcher used both a scoring and coding system. The stories in part I and diagrams in part III were first scored as accurate or inaccurate. If a PST demonstrated a story problem and diagram that would result in the correct concept, it was scored as accurate. For diagram analysis, whether the participant provided the correct diagram or not was the most important criterion. If the participant provided an incorrect diagram but reached a correct answer with an algorithm, this work was coded as an incorrect diagram. After that, the stories and diagrams were analyzed using error codes. The error codes included new codes developed by the researcher and the codes aligned with common errors previously identified by Morano & Riccomini (2020). Some examples are missing or incomplete responses, modeling the equation, misrepresenting (multiple) wholes, and representing the answers, etc. For the items in part II, solutions to the story problems were scored as correct or incorrect and their strategies were also analyzed. First, the researcher categorized if the solution was correct or incorrect and then identified what strategies were used to solve the problems. Since multiple strategies were used and there were cases that one of the strategies was incorrect, the researcher coded each different case of the strategy used. At times, there were correct solutions with incorrect algorithms or diagrams, and the researcher counted them as correct with an incorrect algorithm because the correct solution was considered as a higher category. As part of the review process, the problems were analyzed by the author and one Ph.D. student. Initial analysis was performance individually between two of us. Later, it was discussed in depth multiple times in order to come up with the consensus. Table 1 summarizes how 10 tasks were coded with the coding rationale.

Results

Research Question One

The first research question of the study was to explore PSTs' general performances when engaging multiplication and division of fractions in various contexts. Figure 1 below shows the overall picture of PSTs' percent of accurate performance.

Among the 3 contexts of fraction problems (Part I, II & III) PSTs were most successful with Part II, solving a story problem, in both multiplication and division. In the meantime, we can observe the difficulties to create a story problem and represent visual models with fraction in both operations. The average of correct answers for making a multiplication story problem was slightly lower (31%) than division (35%) but the gap

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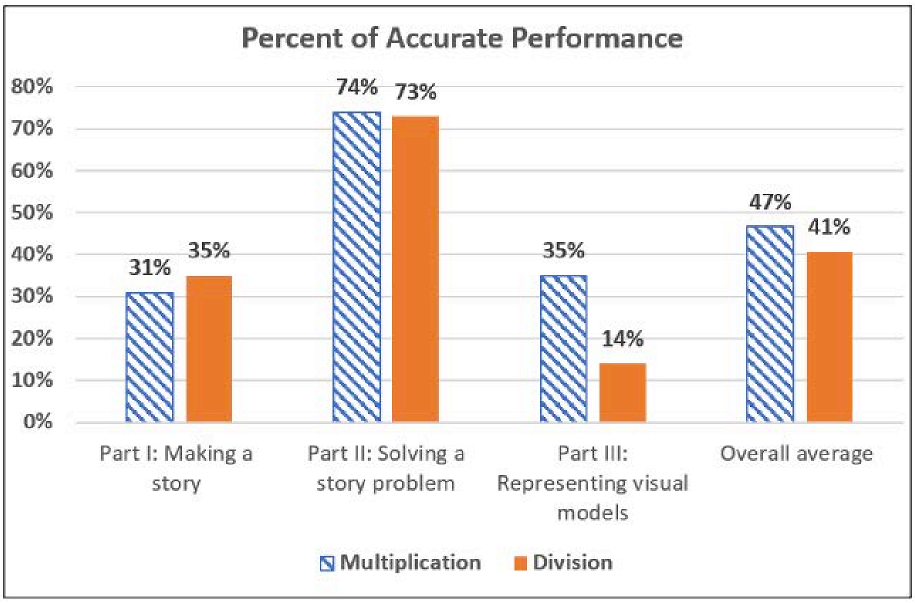
Table 1

Task Items and Analysis Chart

Task Items	Descriptions	Examples
Part I From a symbol to a story Q1: 3 x $\frac{1}{2}$	Accurate stories if the posed problems reflect: an understanding of multiplication concepts (e.g., repeated addition or part of the whole)	I have 3 friends. Each friend gets ½ a bag of candy. How many bags of candy did my friend get? (Part of the whole) I have 3 friends. Each friend gets ½ of a bag of candy. How many bags of candy did they have? (Repeated Addition)
Q2: $\frac{1}{3}$ x $\frac{3}{4}$	an understanding of redefining the whole (e.g., three levels of unit)	Sally has 1/3 of a yard of fabric and she uses ¾ of this fabric for a craft project. How much fabric does she use for her craft project? (Redefining of the whole)
Q3: 2 ÷ $\frac{2}{3}$	an understanding of measurement division (equal grouping)	If you have 2 pies and each pie is cut into 3 pieces, and everyone ate 2 pieces How many people had pie?
Q4: $\frac{1}{3}$ ÷ 2	an understanding of partitive division (equal sharing)	1/3 of a pizza was shared by 2 people. What fraction of the whole pizza was eaten by each person?
	In accurate stories, if there was a missing or incomplete problem	
	An inaccurate story if it models incorrect operations (e.g., used multiplication and subtraction concepts for division problem)	2 ÷ $\frac{2}{3}$: Jenny has 2 pounds of chocolate. If she uses 2/3 for the party tonight, how many pounds will she have left?
Part II From story to an answer Q5. March and Jada share … Q6. It takes half a yard. Q7. Paula has 9 pounds…	Scored as correct if the solution was accurate The strategy was analyzed and recorded. Examples of Incorrect algorithms: 5 ÷ $\frac{1}{2}$ = 2.5, 2 x $\frac{1}{2}$ = 2.5 5 x $\frac{1}{2}$ = 10	Correct answer with Correct Algorithm (CA) Correct Diagram (CD) Both CA & CD Incorrect Algorithm (IA) Incorrect Diagram (ID) CA with ID / IA with CD Incorrect answer Missing or Incomplete IA or ID
Part III From a symbol to a diagram Q8. $\frac{3}{4}$ of $\frac{1}{4}$ / Q9. 5 ÷ $\frac{1}{4}$ / Q10. $\frac{1}{4}$ ÷ 5	Accurate diagram if it visually shows the process of the problem with the correct answer. Inaccurate diagram if it lacks the process of the problem or the answer is missing	Accurate diagram (See Figures) Circle model or bar model Inaccurate diagram (See Figures) Missing or incomplete Modeling the equation Misrepresenting (multiple) wholes Representing the answers Unclear

Figure 1

Overall Percent of Accurate Performance



between the operations was much bigger in part III. Only 14 % of participants were able to solve fraction division problems by representing visual models.

Table 2 reports the frequency of correct answers by fraction problem types. Note that the correct answer total in Part II corresponds to the sum of 'correct answers with correct procedures total' and 'the correct answers with incorrect procedure total'. As stated previously, to deepen the study, PSTs' solution strategies were also analyzed and coded.

On the tasks in part I, PSTs' performance with $3 \times \frac{1}{2}$ (48%) was much better than the task of $\frac{1}{3} \times \frac{3}{4}$ (14%). It seems that $3 \times \frac{1}{2}$ was easier because PSTs applied the concept of whole number multiplication to create the problem, such as 3 groups of $\frac{1}{2}$ chocolate bar, and they used repeated addition to find out the solution. This strategy was not applicable anymore for the fraction x fraction problems. For division tasks, 23 PSTs (50%) were successfully created a story problem for $\frac{1}{3} \div 2$ using sharing context (e.g., $\frac{1}{3}$ bag of candy is shared by 2 people) but only 20 % of PSTs wrote the conceptually correct division problems for $2 \div \frac{2}{3}$

The problems in Part II, PSTs demonstrated comparable computational competency with fraction multiplication and division but, it is noteworthy that among the correct answers, the trend of solution strategies were different between the operations. For instance, in the problem of multiplication (9 x ½), there were 32 correct solution strategies and 75% (24 out of 32) presented correct algorithm and 22 % (7 out of 32) presented correct diagram and only one PST demonstrated both correct algorithm and the diagram. On contrast, for division solution strategies, 69 % (20 out of 29) and 70% (19 out of 27) of participants successfully demonstrated correct algorithm and the diagrams for the whole number and fraction division story problems. This result indicates that majority of participants were successfully solved both fraction multiplication and division story problems with correct algorithms, but they struggled much more with multiplication story problems when it comes to the connection with visual representation.

Furthermore, it was also noticed that some of the participants arrived at the correct answers with

Table 2

Participants' Overall Percentage of Correct Answers for Each Category.

Task Category (n = 46)	Multiplication		Division	
Part I: Creating a story problem	$3 \times \frac{1}{2}$	$\frac{3}{4} \times \frac{3}{4}$	$2 \div \frac{2}{3}$	$\frac{1}{3} \div 2$
Conceptually Correct Story Total (Mul. 31% vs. Div. 35%)	22 (48%)	6 (14%)	9 (20%)	23 (50%)
Part of whole/Redefining whole	2	6	n/a	n/a
Repeated Addition	20	n/a	n/a	n/a
Measurement (equal grouping)	n/a	n/a	9	0
Partitive (equal sharing)	n/a	n/a	0	23
Part II: Solving story problems		$9 \times \frac{1}{2}$	$5 \div 2$	$5 \div \frac{1}{2}$
Correct Answer Total (Mul. 74 % vs. Div. 73%)		34 (74%)	36 (78%)	31 (67%)
Both Correct Answer and Procedure Total		32	29	27
Correct Algorithm Only		24	5	2
Correct Diagram Only		7	4	6
Both Correct Algorithm and Diagram		1	20	19
Correct Answer with Incorrect Procedure Total		2	7	4
Incorrect Algorithm		2	5	2
Incorrect Algorithm but Correct Diagram		0	2	1
Correct Algorithm but Incorrect Diagram		0	0	1
Part III: Representing Visual Models	$\frac{1}{3} \times \frac{1}{3}$		$5 \div \frac{1}{4}$	$\frac{1}{4} \div 5$
Correct Diagram Total (Mul. 35% vs. Div. 13%)	16 (35%)		8 (17%)	4 (9%)
Circle Model	0		1	0
Bar Model	16		7	4

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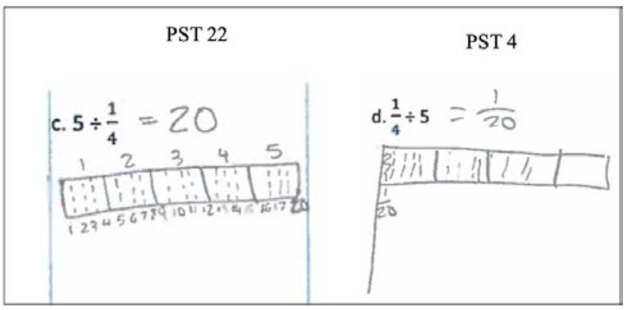
incorrect solution strategies in part II. For instance, two participants reached to a correct answer for multiplication problem, but the algorithm was incorrect. Similarly, 11 participants who wrote down the correct answers for division problems did not demonstrate either correct algorithm or correct diagram for their correct answers. In case of these participants who reached to a correct answer with incorrect procedures we may assume that PSTs found the answers easily from reading the written problems and they did not need to do any calculation to solve the problems. However, when they were asked to show the process for the solutions either using algorithm or diagram, they made mistakes or did not know how to draw diagrams to solve the problem. The analysis for incorrect strategies will be further discussed in the next section of the results.

In part III, the analysis revealed an interesting result in terms of division diagrams. PSTs performed better with a fraction divided by a whole number (e.g., Q10: $\frac{1}{4} \div 5$) for creating a story problem, but in terms of drawing visual representations, the percentage of accurate answers for Q10 was lower (9%) than the equation with a whole number divided by a fraction (e.g., Q9: $5 \div \frac{1}{4}$, 17%). For Q10, only nine percent of participants were able to solve the division equations using visual model drawings. Figure 2 shows the successful examples of Q9 (5 ÷ ¼) and Q 10 (¼ ÷ 5). PST 22 demonstrated that she/he tried to find out how many groups of ¼ were in 5, and PST 4 demonstrated that $\frac{1}{4}$ was divided by 5 parts and the shaded part represented $\frac{1}{20}$. In case of PSTs who demonstrated correct visual representations, we can assume that they have a good conceptual understanding, according to Lee (2017) and Newton (2008).

Figure 2

Successful Diagram Examples in Part III (Q9 and Q10)



Research Question 2

After looking at the correct answers with solutions strategies, this section reports the error patterns observed from participants and describes how they were similar or different across three different contexts. Since the error patterns vary depending on conceptual understanding of multiplication and division, this section addresses the error patterns in multiplication first and that of division in the following section.

Error Patterns in multiplication problems

Table 3

Participants' Overall Percentage of Incorrect Answers for Multiplication Problems.

Task Category (*n* = 46)	Multiplication	
Part I: Creating a story problem	$3 \times \frac{1}{2}$	$\frac{1}{3} \times \frac{3}{4}$
Conceptually Inaccurate Story Problems Total	24 (52%)	40 (86%)
Missing or Incomplete	21	31
Modeling Inaccurate Operation	3	5
Misrepresenting the referent unit	0	4
Part II: Solving story problems	$9 \times \frac{1}{2}$	
Incorrect Answer Total	12 (26%)	
Missing or Incomplete	9	
Incorrect Algorithm	3	
Incorrect Diagram	0	
Part III: Representing Visual Models	$\frac{1}{3} \times \frac{1}{3}$	
Incorrect Diagram Total	30 (65%)	
Missing or Incomplete	23	
Misrepresenting the referent unit	4	
Unclear/other	3	

In terms of the error code of multiplication of fraction problems, 'missing or incomplete' was the biggest error type in all three contexts. Besides the missing or incomplete work, two error patterns were noticeable in Part I and the first one was modeling inaccurate operation. For instance, some of the participants wrote fraction division problems for the multiplication equations and the written work of PST 34 and PST 21 are the example of this error code. PST 34 wrote for $3 \times \frac{1}{2}$ as '$\frac{1}{3}$ of a cookie needs to be split with 3 friends' and PST 21 wrote for $\frac{1}{3} \times \frac{3}{4}$ as '$\frac{1}{3}$ of a pizza is shared with $\frac{3}{4}$. Both written problems required a division operation not a multiplication.

Another error pattern observed in Part I was not understanding the three levels of unit concept in the multiplication problems (Lee et al., 2011). See the examples below.

> *⅓ x ¾: The phone display is only showing 1/3 of the screen. If he only sees ¾ of the ads on his phone, how much of the 1/3 of this phone can actually be seen? (PST 37)*

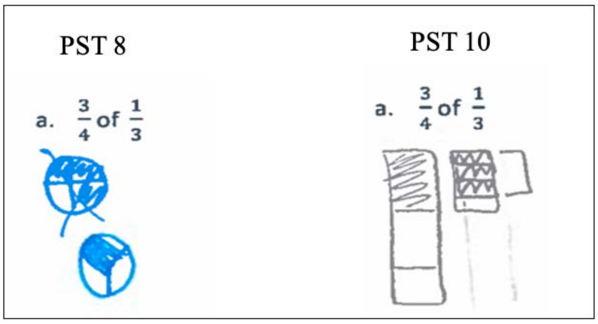
> *⅓ x ¾: ⅓ of the boys in the class, and ¾ of the girls are wearing red shirts, what fraction of the class is wearing a red shirt (PST 39)*

The response of PST 37 did not clearly refer to what 'his phone' means in the context (the whole screen or ⅓ of the screen), and PST 39 used two separate wholes (number of boys and number of girls). Both cases did not clearly represent the unit of each fractional number and what each number referred to in terms of whole and the referent unit of the fraction.

Tasks in part II explored participants' computational knowledge through solving the real-life story problems and this was the most successful task for the participants. There were only three errors and it seemed that PSTs simply made arithmetic errors. The analysis of error patterns in part III revealed that PSTs demonstrated similar error patterns in part I and part III, which was misrepresenting the referent unit. See Figure 3 below as the example. When representing visually $\frac{1}{3}$ x $\frac{3}{4}$, PST 8 drew $\frac{1}{3}$ of one whole and $\frac{3}{4}$ of another one whole and did not show how $\frac{3}{4}$ is the referent unit of $\frac{3}{4}$. The work of PST 10 was little more advanced. This participant represented $\frac{1}{3}$ of one whole first and used the size of $\frac{3}{4}$ to represent $\frac{3}{4}$ of $\frac{1}{3}$. However, the visual model of PST 10 did not clearly demonstrate why the answer was $\frac{1}{4}$ or $\frac{1}{4}$ using $\frac{1}{3}$ as referent unit. Both cases indicated lack of conceptual understanding of referent unit in multiplication (Son and Lee, 2016).

Figure 3

Examples of the Error Coded as 'Misrepresenting the Referent Unit'



Error Patterns in Division Problems

Like multiplication problems, Table 4 shows that in the division problems, 'missing or incomplete' category is the biggest error pattern and 'the use of incorrect algorithm' is the next common error pattern in Part I. However, unlike the multiplication context, it is worth to note that PSTs' error pattern of 'modeling incorrect algorithm' varied by the division problem types (either partitive or measurement) or the number relationships between the dividend and the divisor. For instance, in part I, of the provided incorrect story problems for Q3 (2 ÷ $\frac{2}{3}$), the most common error was switching the dividend and the divisor, and it resulted in an incorrect algorithm. For Q3, most PSTs attempted to write problems using a sharing context, and it resulted in the mismatch between the story and the given equation of 2 ÷ $\frac{2}{3}$. See the examples below:

> *2 ÷ ⅔: There was ⅔ left of a cake. Sam and John wanted to split what was left equally. How much of the cake will they get? (PST 24)*

> *2 ÷ ⅔: If I have two friends and ⅔ of a pie to share, how many pieces will each friend get? (PST 30)*

Table 4

Participants' Overall Percentage of Incorrect Answers for Division Problems.

Task Category (*n* = 46)	Division	
Part I: Creating a story problem	2 ÷ $\frac{2}{3}$	$\frac{1}{3}$ ÷ 2
Conceptually Inaccurate Story Problems Total	37 (80%)	23 (50%)
Missing or Incomplete	18	16
Modeling Incorrect Algorithm	16	6
Misrepresenting the Whole	3	1
Part II: Solving story problems	5 ÷ 2	5 ÷ $\frac{1}{2}$
Incorrect Answer Total	10 (22%)	15 (33%)
Missing or Incomplete	5	4
Incorrect Algorithm	5	11
Incorrect Diagram	0	0
Part III: Representing Visual Models	5 ÷ $\frac{1}{4}$	$\frac{1}{4}$ ÷ 5
Incorrect Diagram Total	38 (83%)	42 (91%)
Missing or Incomplete	34	32
Lack of conceptual understanding of division	1	5
Misrepresenting the whole	1	5
Unclear	2	0

For both examples, the written examples are represented as $\frac{2}{3}$ ÷ 2, not as 2 ÷ $\frac{2}{3}$.

Interestingly, the similar error pattern of switching the dividend and the divisor was rarely observed in Q4 ($\frac{2}{3}$ ÷ 2). It seems that these errors are associated with the two different conceptualization of whole number division (Ball 1990a; Van de Walle, 2010).

In the meantime, for the tasks in part 2, the incorrect algorithm arose from the lack of distinction between dividing by 2 and dividing by one-half. Q6 (5 ÷ 2) asked how much ribbon each will get if 5 yards of ribbon are shared by 2 people. More than half of the participants who wrote down incorrect solutions demonstrated incorrect algorithm. Some of the examples are 5 ÷ $\frac{1}{2}$ = 2$\frac{1}{2}$, 5 x $\frac{1}{2}$ = 10, and 5 ÷ $\frac{1}{4}$ = 2.5. It seemed that PSTs found the answers easily from the problem context but made mistakes when expressing the fraction division algorithm and did not recognize that their written equation did not produce the correct solution.

As already mentioned, division tasks in part III were identified as the most challenging tasks for PSTs and higher percentage of incorrect answers (83% & 91%) are the evidence of that difficulty. The common error types with division diagrams were 'lack of conceptual understanding of division' and (Figure 4)' and 'misrepresenting the whole (Figure 5)'. It seems that these error patterns are associated with the relationship between dividend and divisor because it was identified only once for Q9 ($5 \div \frac{1}{4}$) but five times more for Q10 ($\frac{1}{4} \div 5$). For instance, the work of PST 27 in Figure 4 illustrated that there are five separate wholes, and each whole was divided into fourths. It seems that this participant represented the given equation with a drawing but was not sure how to represent the meaning of division (e.g., how many fourths are in five) visually. Similar errors were identified more frequently for Q11($\frac{1}{4} \div 5$). Looking at the examples of PST 19 and PST 28, they drew one rectangle to represent one whole and partitioned the whole into fourths. Next, she/he divided $\frac{1}{4}$ into 5 parts successfully. Yet, the diagrams did not clearly show how the correct solution was $\frac{1}{20}$. In particular, in the work of PST 28, she/he shaded $\frac{1}{5}$ of $\frac{1}{4}$ to represent $\frac{1}{20}$ of the whole but was not able to find out what the shaded piece represented in the context of division (e.g., sharing ¼ of brownie among 5 people and finding a portion for each).

Figure 4

Examples of the Error Coded as 'Lack of Conceptual Understanding of Division'

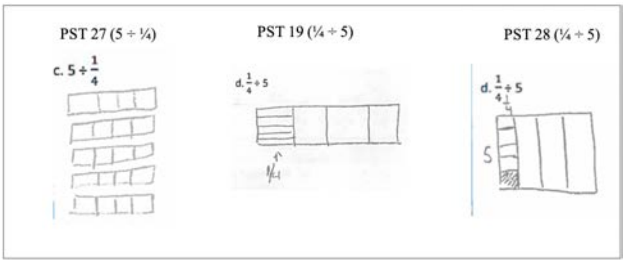
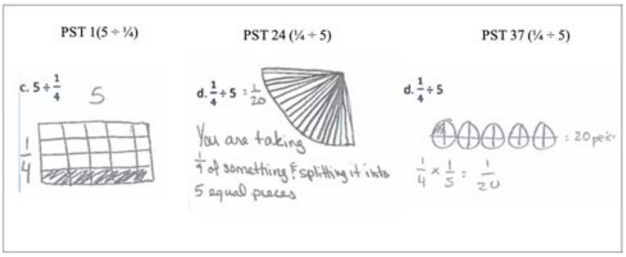


Figure 5

Examples of Error Type Coded as 'Misrepresenting Wholes'



The examples in Figure 5 reported the examples of misrepresenting the whole. In the first diagram of Figure 5 (PST 1: $5 \div \frac{1}{4}$), the one rectangle was represented as 5 wholes, and this whole was divided into four equal parts with $\frac{1}{4}$ represented by shading the part. The diagram was incorrect because the shaded part did not represent $\frac{1}{4}$ compared to the whole (1). Rather, it represented $\frac{1}{4}$ of five wholes. It seems that this participant misunderstood the five connected wholes as one whole by drawing one large rectangle instead of drawing five separate rectangles. When the fraction is defined based on the whole, what was shaded in this example was not $\frac{1}{4}$, but $1\frac{1}{4}$ (five groups of $\frac{1}{4}$ for each whole). When looking at the second picture (PST 24), the verbal explanation seems reasonable, but the pictorial illustration did not clearly represent what ¼ should look like in terms of one whole. It was not clear whether the presented circular sector represented the whole (1) or $\frac{1}{4}$. If PST 24 meant to represent the circular sector as one whole, the diagram could be right because it was divided into 20 equal parts despite the unclear representation of the dividend ($\frac{1}{4}$) in the picture. Yet, if one interprets the presented circular sector as $\frac{1}{4}$ (that shape typically is used to represent $\frac{1}{4}$ out of one whole circle), the diagram is inaccurate because $\frac{1}{4}$ should be divided by 5, not 20. Thus, this diagram was coded as incorrect due to its unclear representation of the whole (1) in the fraction.

Discussion and Implications

The purpose of this study was to provide a comprehensive description of elementary preservice teachers' conceptual understanding of fraction multiplication and division in three contexts and analyze the error patterns. Below the author summarizes the findings for each of the two research questions and suggests possible pedagogical implications for elementary mathematics teacher educators.

The first research question examined PSTs' overall performance on the fraction multiplication and division problems in three parts. Overall, the part II, solving word problems, was the most successful task in both multiplication and division of fractions with 73.5 % of correct answers. However, an in-depth analysis of solution strategies indicated that the majority of the PSTs applied algorithm to solve the problems and PSTs were more competent with fraction multiplication algorithm than the division algorithm. Of the correct solution multiplication strategies, 71% of PSTs solved the problem using algorithm only and two incorrect algorithmic mistakes were identified. On the contrary, with the fraction division word problems, just 10% of students solved the problems using algorithm only, and more frequent arithmetic errors were identified in division strategies. When comparing the use of diagram to solve the word problems, the percentage is much higher in division (more than 50%) than multiplication context (about 23%). One possible explanation is that an algorithm of fraction multiplication is simpler than fraction division because they just need to multiply numbers across fractions. Yet, this is not the case for division of fractions as the division algorithm requires the process called 'copy, dot, and flip'. Another possible explanation for the different use of strategies is that the context of the problems between multiplication and division. It seems that PSTs did not need to draw any pictures for the fraction multiplication problem

(Q7: $\frac{1}{2}$ of nine pounds of candy bags) because they could use mental math to find out the correct answer. However, for the division problems (sharing ribbons or measuring out the ribbon) they used drawings to represent the number or ribbons in the problems. The findings in part II are strong evidence that PSTs' fractional knowledge is procedure-based, and they are fragment (Ball 1990a; Park 2013). This finding emphasizes that teachers' knowledge needs to extend beyond computational competency to build deeper conceptual understanding of fraction concept (Adu-Gyamfi, 2019).

As past research has shown that division of fraction was most challenging especially with representing correct diagrams (e.g., Lee & Lee, 2019; Lee et al., 2011; Luo et al., 2011; Son & Lee 2016;), PST's lowest average rate of accurate diagrams in part III was not surprising. The PSTs' average rate of accurate answers of multiplication and division problems was the lowest with 20 % overall, but it was much lower with division problems. However, the findings from this study addresses one positive suggestion to support PSTs. It is true that PSTs were least successful with the drawings in part III, but as mentioned above, in part I, the average percent of accurate diagrams was much higher in the division problems than multiplication problems. The only difference was how the problem was provided. In part I, the fraction problem was provided using context and a story problem but in part III, only the equation was provided. The findings proved that providing context enabled PSTs to represent fractions with accurate diagrams in division problems. Thus, this study argues that providing context in division and multiplication problems is pivotal experience for PSTs to fully understand what situations lead to multiplication or division problems, and how to represent visual model correctly to teach fractions as sense making.

For the second research question, an in-depth analysis of data was conducted to identify the common errors observed across multiplication and division solution strategies.

According to the error analysis, the most common error of fraction multiplication and division was using incorrect application of fraction operations with word problems as described in the previous research (Ma, 1999; Tirosh, 2000). However, the findings of this study describe the difference of the errors between multiplication and division. The incorrect algorithm presented in multiplication problems were using different operation symbols (e.g., using division for multiplication problems). Yet, the error patterns in division problems were associated with the order of a dividend and a divisor (e.g., creating a story for either $2 \div \frac{2}{3}$ or $\frac{1}{5} \div 2$ or drawing a visual model for $\frac{1}{5} \div 4$ or $4 \div \frac{1}{3}$) because the different conceptualization of division may need to be applied (Adu-Gyamfi, 2019). PSTs' performance in this study was divergent depending on the division problem type. PSTs were more successful in creating story problems with the partitive division concept (e.g., $\frac{1}{3} \div 2$ or $\frac{1}{5} \div 4$) but when generating visual representation, their performance was much better with the measurement division concept (e.g., $2 \div \frac{1}{5}$ or $4 \div \frac{1}{5}$). PSTs are prone to develop the division problem by sharing context, but they have difficulties solving the problem conceptually using visual models. A discrepancy existed between creating the problem and meaningfully solving the problem. This finding suggests teacher education courses need to encourage PSTs to discuss the meanings of two different concepts of division in whole numbers, how this approach could be used in fraction divisions, and how this approach would differ depending on the relationships between the dividend and divisor in fraction problems. Further, this study recommends providing an opportunity to think about the pros and cons of applying the different concepts of whole number division into division with fractions. This conceptual challenge in learning division should be fully discussed prior to learning and teaching fractions. These in-depth discussions could support PSTs to understand what it means to divide and multiply fractions in our daily lives.

The finding of this study brings an attention to the PSTs' lack of fractional number sense. If PSTs understand the relationship between the dividend and divisor of the whole number, they should be able to identify Q10 ($4 \div \frac{1}{5}$) cannot have the same answer as Q9 ($\frac{1}{5} \div 4$) and to determine which answer should be bigger than 1 or smaller than 1. For instance, if one understands multiple conceptualizations of division concepts (e.g., finding a quotient, ratio, and measurement, etc.) she/he needs to be able to tell $\frac{1}{5} \div 4$ means either how many groups of 4 fit into $\frac{1}{5}$ or $\frac{1}{5}$ of pizza is shared by 4 children. In both cases, the answer should not be bigger than 1. The same logic can be applied to Q10 ($4 \div \frac{1}{5}$). As $\frac{1}{5}$ goes into 4 more than 1 time, the answer should be a lot bigger than 1. This type of practice could be helpful for students to connect a student's prior knowledge of whole number division with an earlier understanding of fraction division building fractional number sense at the same time. Fractional number sense can help PSTs and their future students to reduce misconceptions

Another common error identified in this study was the misrepresenting the whole in division and not understanding the referent unit in multiplication problems. Both are critical elements to conceptually understand fraction multiplication and division, thus, it is critical to practice identifying the whole(s) and the referent unit with carefully designed fraction problems. Concerning this, Mack (2001) posed a couple of suggestions. For instance, the problems start from an equal-sharing context, and move to finding a fraction of a whole number, taking a part of a part of a whole (e.g., ¼ of ⅘), and choosing fractional numbers carefully where the relationships between numerators and denominators are easily represented visually, etc. (e.g., ¾ of ⅔). These carefully chosen problems would be a great opportunity for students to practice partitioning units in various ways and conceptualizing the critical concept of fraction multiplication and division.

To this end, the results of this study summarizes some potential implications to support PST to be able to teach fraction multiplication and division meaningfully and conceptually. It includes:

- Helping PSTs to connect to context when solving and representing problems visually.
- Helping PSTs to explicitly connect the various meanings of fraction division and real-life examples.
- Providing ample discussion time about the conceptual meaning of division with whole numbers prior to multiplication and division of fractions with the emphasis of the relationships between a dividend and a divisor.
- Providing carefully selected fractional numbers considering the level of difficulties
- Helping PST to build fractional number sense to justify their answers.

There are still some questions that remain unanswered. Future studies will include interviews to explore PSTs' mathematical thinking in detail and will focus on pre- and post-test to find out which instructional strategies are helpful to enhance conceptual understanding. Further, a bigger sample size will make the study more robust. Also, it will be beneficial to broaden the scope and sequence of the task levels for each category such as including a conceptual understanding of whole number division. We hope future research will shed some light on promoting PSTs' conceptual understanding of the most challenging topic in elementary school mathematics.

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