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Research Article

The difficulties in geometry: A quantitative analysis based on results of mathematics competitions in Italy

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ABSTRACT

This paper focuses on the difficulties encountered by Italian students in performing geometry tasks. A quantitative analysis, aimed at understanding the extent of the phenomenon, is carried out using the results of district competitions from the year 2018 to 2020, comparing the scores obtained in geometry questions with those in other areas of Olympic mathematics. In addition, the answers given by the students to a questionnaire administered at the end of the 2020 district competition are analyzed in order to better understand possible motivations behind the phenomenon in question. The results obtained need further confirmation through future research on the topic but represent clear trends worthy of further investigation.

Keywords: geometry difficulties, geometry performance, quantitative analysis, high school students, mathematical ability, mathematical competitions

INTRODUCTION

Many studies have been carried out by researchers, teachers, psychologists, and sociologists in order to understand the reasons behind the difficulties that a large percentage of students, regardless of age, gender or country of origin, encounter in studying mathematics (Jordan et al., 2013; Mazza & Gambini, 2022; Mazzocco, 2007; Mutlu, 2019). The aim is to understand which mental processes are activated, which aspects represent a barrier to learning and what good practices and strategies can be used to overcome these difficulties (Craig, 2010; Hill & Wicklein, 1999).

Research studies on this issue generally focus on a specific task or, at least, on one of the different areas of mathematics, such as geometry (Gal & Linchevski, 2010; Kuzniak & Rauscher, 2011; Sulistiowati et al., 2019). In fact, each field of mathematics brings with it specific problems related to teaching/learning.

The focus of this article is on geometry and, in particular, on the difficulties students have in solving geometric problems and proofs. The research was carried out on a sample comprising all those who took part in the Roman district mathematics competitions from 2018 to 2020. It must first be said that the study of mathematical competitions is a relatively recent field and therefore the literature is proportionately scarce (Marushina, 2021; Nieto-Said & Sánchez-Lamoneda, 2022). As Kaiser states in her foreword to the ICME-13 monograph, “Competitions for young mathematicians: Perspective from five continents” (Soifer, 2017):

> "Despite this high importance of mathematical competitions, either as mathematical Olympiads or as mathematical tournaments of towns or other kind of mathematical competitions, there exists hardly any scientific research about mathematical competitions" (p. vii).

This research represents exploration of an aspect that has been abundantly studied in the literature, i. e., the difficulties possessed by students in geometry, but in this case with regard to a rather small and, as just mentioned, scarcely-studied sample represented by students participating in mathematics competitions. Geometry, in particular, is a substantial area in competitions; with a relatively small amount of knowledge, a large number of exercises can be developed, and of varying levels of difficulty (Nieto-Said & Sánchez-Lamoneda, 2022).

The approach adopted is purely quantitative, with the aim of obtaining general indications and broad trends about the extent of the phenomenon under study, while at the same time trying to outline possible interpretations without claiming to give any exhaustive explanation, nor to investigate the socio-cultural aspects underlying the phenomenon itself. Since this is a rather narrow time interval, there should be future research and in-depth studies on the subject. In addition to further quantitative analyses aimed at fully understanding the extent of the phenomenon, conducting qualitative analysis may allow us to understand the motivations behind the phenomenon. However, the research reported here presents clear trends from which to develop future research.

Students' Difficulties in Geometry: A Look at the Status Quo

The teaching of geometry in schools has many aims, such as developing students' intuitive spatial, graphic and linguistic abilities, stimulating the need for demonstration, getting them used to reasoning and giving a significant example of an axiomatic-deductive system. For centuries, geometry has inspired epistemological and philosophical studies, such as the theories of Plato and Kant. Although it is often seen as merely the study of points, lines and shapes, geometry has considerable applications in other fields such as music and architecture. In the Italian scholastic tradition, geometry and its centrality within the mathematics curriculum have experienced fluctuating phases (Sbaragli & Mammarella, 2010), although in the aftermath of national unity, a certain importance was given to the study of geometric properties with the inclusion of Euclid's elements in the high school mathematics curriculum.

Nowadays, however, geometry occupies an important place in education in many countries, including Italy, where the wide space devoted to analytical aspects of geometry has not led to an excessive reduction in the study (through a synthetic approach) of plane and solid figures. Although the classical (Euclidean) approach taken by many teachers and textbooks is to start with two-dimensional geometry and then move on to three-dimensional geometry, there is still an open debate within the research community about the need to reverse this treatment, at least at primary school level (Sbaragli & Mammarella, 2010). This is due to the belief that three-dimensional geometry represents a reality closer to the child's experience than a study that begins with concepts such as point, line and plane, which "do not exist, cannot exist in reality" (cit. Fischbein, 1993, p. 141) and as such are less intuitive (Arrigo & Sbaragli, 2004; Cottino & Sbaragli, 2004). In the world of teaching, there are also elements of inhomogeneity, both from country to country and within the same country or, sometimes, even within the same school where different teachers have different sensitivities and attentions towards geometry, which is sometimes relegated to the study of a few and limited theoretical notions without a real exercise in demonstration. This lack of sensitivity to the teaching of geometry is clearly revealed in light of the difficulties students display when called upon to perform demonstrations or tasks requiring special visual-spatial skills. Research in mathematics education has focused a great deal on the subject of demonstration, in an effort to understand the cognitive mechanisms and logical processes that are activated (Anderson, 1983; Herbst & Brach, 2006; Knuth, 2002; Selden & Selden, 2003). Some studies analyze the different stages of cognitive development at which students can produce demonstrations in a meaningful way. These include the theory of the development of geometric thinking by Dutch teachers Pierre van Hiele and Dina van Hiele-Geldof (Fuys et al., 1984; van Hiele, 1986). Noting the difficulties their students had in geometry, they developed a theory that introduces different levels of thinking that students go through as they move from simply recognizing a figure to being able to describe a formal proof (Clements & Battista, 1992; Crowley, 1987; Mason, 2009, Mason & Moore, 1997; Sbaragli & Mammarella, 2010; Usiskin, 1982). Five

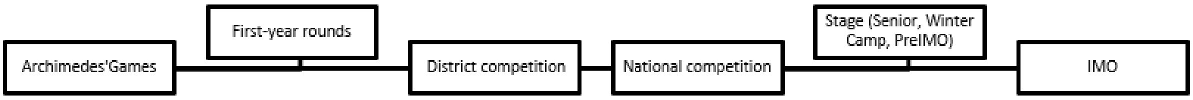

Figure 1. The different stages of mathematics competitions in Italy (Source: Authors)

levels are indicated by van Hiele (1986), which are sequential and hierarchical: level 1 (visualization), in which one recognizes figures only by their appearance without perceiving their properties; level 2 (analysis), in which one recognizes the properties of figures without, however, perceiving the relationship between them; level 3 (abstraction), in which one perceives the relationships between properties and figures without, however, yet understanding the role and significance of formal deduction; level 4 (deduction), in which one is able to construct demonstrations, understand the role of definitions and axioms and know the meaning of necessary and sufficient conditions; and, finally, level 5 (rigor), in which one understands the formal aspects of a demonstration. According to van Hiele (1986), writing a demonstration requires a high level of thinking, but many students are not able to do this because they need to do more work with specific tasks at a lower level. With specific reference to students who perform well in mathematics, if they sometimes seem to "skip" levels, it is probably only because they have developed logical reasoning skills in contexts other than geometry (Krutetskii, 1976; Mainali, 2019; Mason, 2009).

THE CONTEXT

Mathematics Competitions in Italy

Since this paper focuses on students taking part in Italian mathematical competitions, it is considered useful to provide an overview of the different stages of student competitions in Italy (**Figure 1**); see also Mazza & Gambini, 2022, for more details. The most widespread is represented by the Archimedes games, an individual competition organized by the UMI (Italian Mathematical Union) and held, generally in November, within each participating school (as many as 1,440 schools throughout Italy in the 2017/18 school year). The primary objective of this event is to disseminate a different way of looking at mathematics by setting problems that are different and, in some respects, more stimulating than those proposed during curricular hours. The number of participants has been around 200,000 in recent years, although the recent pandemic situation has led to a drastic reduction in extracurricular activities in schools and, consequently, a drop in participation in mathematics competitions. The Archimedes games and subsequent competitions deal with essentially four areas: algebra, geometry, number theory, and combinatorics. Since the youngest students (grade 9) could risk being left out of the next competition, a 'repechage' competition is reserved for them, in which around 2,000 students from all over Italy participate, in early February.

The next step is the district competition at the end of February, which is attended by the highest scorers in the Archimedes games and the grade 9 students' competition in each district[1]. About 10,000 students take part in this competition every year. The competition, which is the same for all participants, poses 12 multiple choice questions, two numerical questions (i.e., exercises, where the answer is an integer) and three open-ended demonstration questions. The first 14 exercises do not require knowledge of any special demonstration techniques, but it is essential to know theorems and procedures in order to correctly perform the calculations and identify the correct answer. In contrast, the last three exercises require writing an actual proof; therefore, it becomes essential to demonstrate good abstraction and generalization skills. These latter exercises alone constitute almost 40% of the total test score (they alone are worth up to 45 points out of a maximum of 115 points). The correct resolution of the latter three exercises therefore becomes discriminating in identifying those who will go on to participate in the next national competition, which is reserved to the best 300 students from the district competition. The competition, lasting up to 4.5 hours, is held every year in May and participants have to solve six open-ended demonstration exercises.

[1] Each Italian region is divided into provinces, which represent portions of territory that include several neighboring municipalities. The districts, 92 in total, coincide approximately with the provinces. Students from different, but geographically close, schools are grouped into the same competition field. The text of the district competition is the same for all Italian districts.

A selection of six students, chosen on the basis of their results in the national competition as well as their performance during a number of internships held during the year, are sent to the IMO (International Mathematics Olympiads), which are held each year, usually in July, in a different host country.

METHODOLOGY

Following the district competition in February 2020, students in the Rome district were given a questionnaire with only eight questions, designed to determine:

1. the level of difficulty perceived during the competition in the different areas that characterize the mathematics Olympiad,
2. whether or not they consider themselves successful in each of these areas,
3. the most difficult moment in solving the problems of the competition,
4. whether or not they have seen exercises at school similar to those in the competition,
5. school performance,
6. possible placement in the ranking, and
7. the level of anxiety during class tests and mathematical competitions.

The questionnaire was administered to all 486 students who participated in the district competition at the end of the competition itself, in paper format and not anonymously, so that each student's answers could be matched to his or her performance in the competition. Subsequently, the data was initially collated in an Excel spreadsheet to gather some initial findings which were then studied in more detail using the SAS software.

In addition to investigating preferences and the level of anxiety experienced in the competition, the questionnaire aimed to understand what perceptions of their own skills were held by the students and which of the areas (among those that distinguish competition mathematics) they considered most difficult and least familiar. Some of the results obtained are discussed in this paper.

In order to understand the performance of students during mathematics competitions, we first analyzed the results obtained by Italian students who competed in the district competitions held from 2018 to 2020. Subsequently, and with reference to the sample restricted to Roman students participating in the district competition in February 2020, the answers given to the questionnaire that was administered and, in particular, to questions 3 and 4 were examined.

The choice to limit the study to the years 2018-20 was unavoidable. In fact, until 2016, the data were not available as they were not systematically collected by the organizing body of the event (the UMI), while in 2017, although the organizers had started to collect them extensively, the data were incomplete, inaccurate in several points and inaccurate (dirty data), which makes their reading and interpretation not fully reliable. Finally, in the years 2021 and 2022, the situation linked to the COVID-19 pandemic led to a remodeling of the district competition. The competition consisted of two distinct phases with the demonstration questions limited to a small band of the total number of participants (about a thousand students, equal to 10% of those who competed in the first phase with only multiple-choice questions). This situation of inhomogeneity would therefore not have allowed a comparison with the situation of the three-year period 2018-20.

The total number of Italian students who took part in the district competition in the years considered amounts to around 10,000. In **Table 1**, we find the exact figures.

DATA ANALYSIS

Descriptive Analysis

The observations below concern the last three exercises of the district competitions, i.e., the open-ended tasks, in the years under consideration. For each of these years, these questions concerned the area of number theory, combinatorics, and geometry, while there were no demonstration questions on algebra.

The first observation made concerns the percentage of answers left blank, shown in **Table 2**.

Table 1. Number of students participating in district competitions for the years 2018-2019-2020

	2018	2019	2020
Exact number of students	9,687	9,905	10,000

Table 2. Percentages of answers left blank in the three demonstration exercises

	Number theory	Combinatorics	Geometry
2018	60%	77%	93%
2019	77%	78%	88%
2020	67%	78%	88%

Table 3. Mean (M), standard deviation (SD), and percentiles (P) of the three demonstration exercises

Variable	2018				2019				2020			
	M	SD	95^{th} P	99^{th} P	M	SD	95^{th} P	99^{th} P	M	SD	95^{th} P	99^{th} P
Combinatorics	1.09	2.64	7.00	13.00	1.76	4.11	14.00	15.00	1.56	3.59	10.00	15.00
Geometry	0.23	1.27	1.00	5.00	0.38	1.73	2.00	13.00	0.69	2.34	5.00	10.00
Number theory	1.56	2.85	7.00	15.00	1.30	3.04	9.00	15.00	1.86	3.59	11.00	15.00

It is clear that the geometry question is the one most often left blank by the students who took part in the competition. It was not possible to ascertain whether the students gave up this type of question a priori (i.e., without even reading the text) or whether they tried to do it, perhaps on rough work sheets (which they did not have to return at the end of the competition), although without success. Understanding the dynamics in the competition could be the subject of future in-depth analysis by means of interviews or targeted questionnaires.

A second aspect to be analyzed concerns the scores given to these exercises. The evaluation grid is provided for each demonstration question to be evaluated with an integer between zero (exercise left blank) and 15 points (exercise completely correct). According to the intentions of the drafters of the test, the idea is to award zero points only in the actual case of a blank exercise. For this reason, the evaluation grid lays down that one single point may be awarded in the case of simple general considerations, if an example of what is required is given (e.g., a particular solution of an equation admitting infinite solutions) or, in the specific case of geometry, a point is awarded even if the figure is merely drawn correctly without any other consideration.

As can be seen from **Table 3**, the geometry problem has a lower mean score than the other questions. Furthermore, the standard deviation of the scores is rather low, indicating that the students are concentrated around the mean score band and therefore present similar scores. This means that the majority of the participants collected zero or very few points in the three different questions; in particular, with regard to the geometry question, it can be observed that, given the low variability, the students are particularly homogeneous with regard to this characteristic and the scores are squashed around the mean score, which was 0.23 points in 2018, 0.38 points in 2019, and 0.69 points in 2020.

In the last three years, we also observe that the top 95% of students (sorted in ascending order with respect to the score obtained in the geometry demonstration question) obtained zero or one point in 2018, at most two points in 2019 and five points in 2020. For the other two areas, the following can be observed: the first 95% of students (sorted in non-decreasing order with respect to the score obtained in the number theory question) obtained at most seven points in 2018, at most nine points in 2019 and at most 11 points in 2020. Similar results were obtained for the combinatorics question: the top 95% of participants (sorted in non-decreasing order of their score in the combinatorics question) scored at most seven points in 2018, at most 14 points in 2019, and 10 points in 2020.

The 99^{th} percentiles confirm that the geometry demonstration question in the last three years has made it so difficult for the participants that few of them scored high marks for combinatorics and number theory: 99% of participants in 2018 scored no more than five marks (no more than 13 marks for combinatorics); in 2019 and 2020, at least 1% of participants scored 15 marks for the combinatorics and number theory demonstration questions, while not even 1% scored the maximum mark for the geometry question.

At this point, we would like to recall the statement made by the President of the Olympiad Commission of the Italian Mathematical Union, Prof. Ludovico Pernazza, in a recent interview on the mathematical

Table 4. Total mean scores of students who scored between 10 and 15 points in each of the three demonstration exercises

	Number theory	Combinatorics	Geometry
2018	55.5	49.2	69.9
2019	49.8	46.1	57.1
2020	50.3	50.4	51.2

divulgation blog "Math is in the air". He said that it is precisely in solving geometry questions, both in district and national and international competitions, that students are most likely to get stuck.

In the same interview, Prof. Pernazza also said that in the IMOs it is customary to include at least two geometry exercises out of six so that in order to achieve a high ranking one must in any case do this type of exercise, suggesting that success in geometry questions can be a discriminating factor for success in mathematical competitions.

It is legitimate to wonder if this hypothesis is also valid in simpler competitions such as the ones we are analyzing, although geometry questions in such competitions have a lower weight (around 25%) on the total score. In this regard, again with reference to the three demonstration exercises of the district competitions of recent years, **Table 4** shows the total mean scores obtained in the competition only by those students with, in the individual demonstration exercises, obtained a score greater than or equal to 10 (out of 15 points):

Consider, for instance, the year 2018. **Table 4** shows that students whose number theory demonstration exercise was marked between 10 and 15 points obtained, on average, an overall score (across all 17 exercises) of 55.5. This value is 49.2 if we consider only those students whose combinatorics question was assessed between 10 and 15 points, and 69.9 for students who scored between 10 and 15 points in geometry. It is only in 2020 that the differences in overall mean scores are less relevant. In any case, **Table 4** suggests that a good performance in the geometry questions is discriminating in order to identify those who, at the end of the competition, will obtain a high final score. To simplify matters, students who do well in geometry also tend to do better in all the questions in the competition, and score higher overall.

Cluster Analysis

In support of what has been observed, referring to the year 2018 and the scores in the three demonstration questions and the total score obtained, we proceed with a hierarchical cluster analysis using the complete link method. The aim is to categories of students in such a way that students belonging to the same group have similar characteristics and students belonging to different groups differ in their results.

The choice of six groups is justified by the fact that it explains 83% of the variability of students. Moreover, in correspondence to the partition into six groups, the pseudo-f index assumes a local maximum (as shown in the graph in **Figure 2**) and the pseudo-t^2 assumes a much smaller value than the value assumed in correspondence to the partition into five groups.

By coordinating the information from these indices, we continue to analyze the solution obtained.

The first three clusters (**Table 5**) are the most numerous: there are respectively 5,512, 2,432, and 1,233 students who obtained a mean score in geometry of 0.05 points, 0.21 points, and 0.34 points; in each of the three clusters, the mean score obtained in the combinatorics and number theory questions is slightly higher than the mean score in geometry. The total mean score of the students in the first group is lower than the global mean score of all the students (about 22 points), so in the first cluster there are students who have done badly overall (and specifically worse in geometry than in the other two disciplines); in the second and third groups, the total score obtained is slightly higher than the global mean score, but in both cases not high (27. 07 points for the students of cluster 2 and 39.38 for cluster 3), although the participants, on average, have at least tried to solve the demonstration exercises of number theory and combinatorics since the mean scores obtained in the two questions are 1.23 and 1.69 points for cluster 2 and 2.31 and 3.34 points for the students of cluster 3. In the fourth group there are 396 students, who obtained a mean score in the geometry question of 0.87 and a mean score in the combinatorics and number theory questions of 6.33 and 7.29, respectively: these students, who have a very low mean score for geometry and a mean score in the 6-10 points range (i.e., a good score) for the other two subjects, obtained a total mean score of 54.21 points. In the fifth group, comprising 87 students, the mean score for geometry is higher than in the previous cases and equal to 3.92;

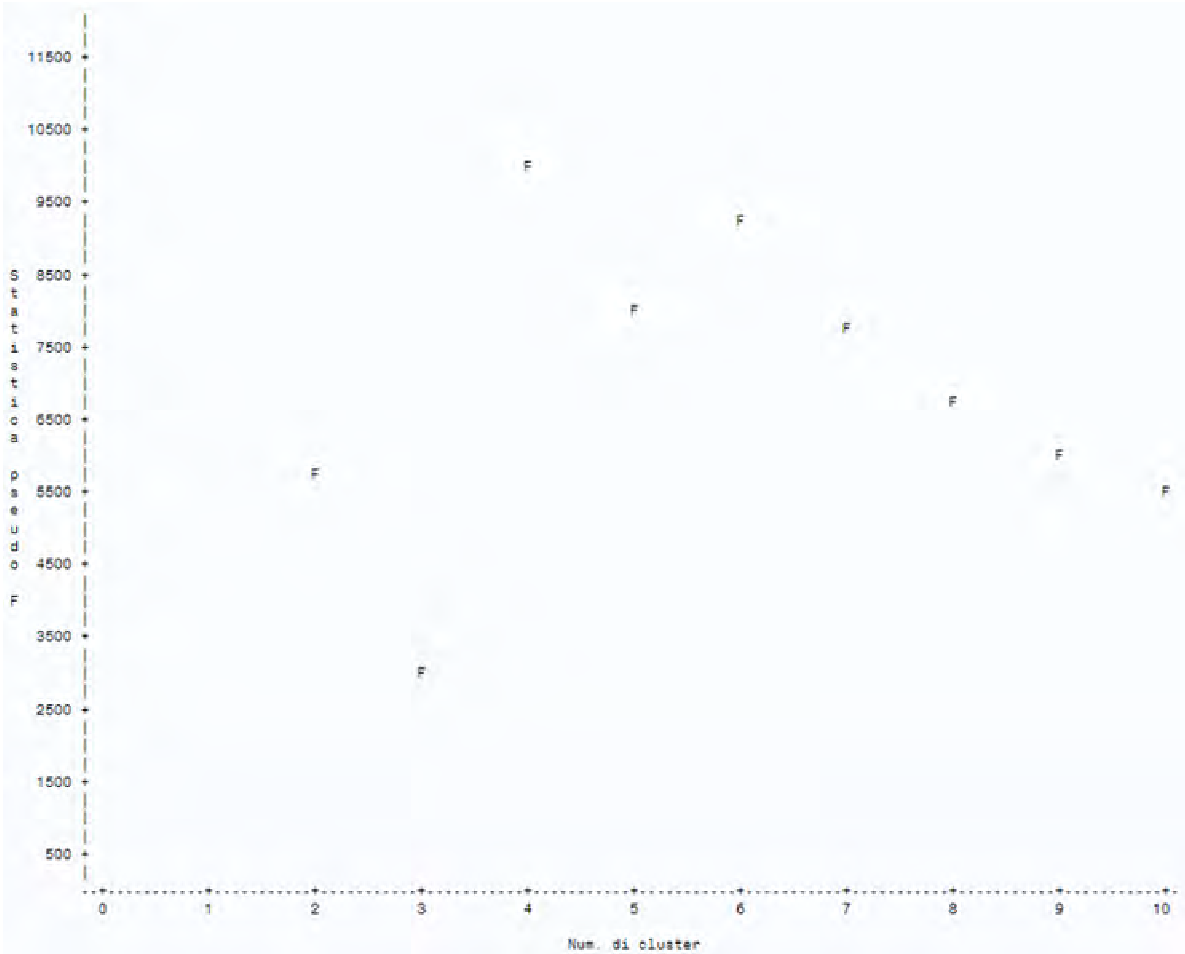

Figure 2. Pseudo-f statistics calculated in the last 10 partitions of the hierarchy (Source: Authors' own elaboration with SAS software)

Table 5. Numerosity (N) and mean (M) scores of the six groups of the chosen partition for the year 2018

Variable	Cluster 1		Cluster 2		Cluster 3		Cluster 4		Cluster 5		Cluster 6	
	N	M	N	M	N	M	N	M	N	M	N	M
Total	5,512	12.70	2,432	27.07	1,233	39.38	396	54.21	87	70.52	27	94.70
Dim_g	5,512	0.05	2,432	0.21	1,233	0.34	396	0.87	87	3.92	27	11.63
Dim_c	5,512	0.27	2,432	1.23	1,233	2.31	396	6.33	87	5.33	27	10.89
Dim_n	5,512	0.46	2,432	1.69	1,233	3.34	396	7.29	87	11.67	27	14.41

it can be observed that the total mean score is also higher than that of the fourth group, although in the fifth group there are students who obtained a lower mean score in the combinatorics question, equal to 5. 33, compared to cluster 4. The last group, with 27 students, is characterized by particularly good students, since they obtain a total mean score of 94.7 points: this group includes students with high scores, and it is the only group in which the geometry mean score is high (belonging to the 10-15 range), equal to 11.63 points.

Each rectangle in **Figure 3** indicates the 25th, 50th (center line), and 75th percentile referring to the group of students belonging to a single cluster (shown on the vertical axis). The square indicates the average score of the students within each cluster (shown on the horizontal axis). The lines extending from the box represent the minimum and maximum scores obtained by students belonging to each cluster. The box plot confirms the previous statements, i.e., that the groups are distinguished by total score and also that they are differentiated by the score obtained in the geometry question; in fact, in the groups in which the students obtained a low mean score in geometry compared to other groups, they have a low total mean score compared to the same groups. In conclusion, students who score high in geometry generally score high overall.

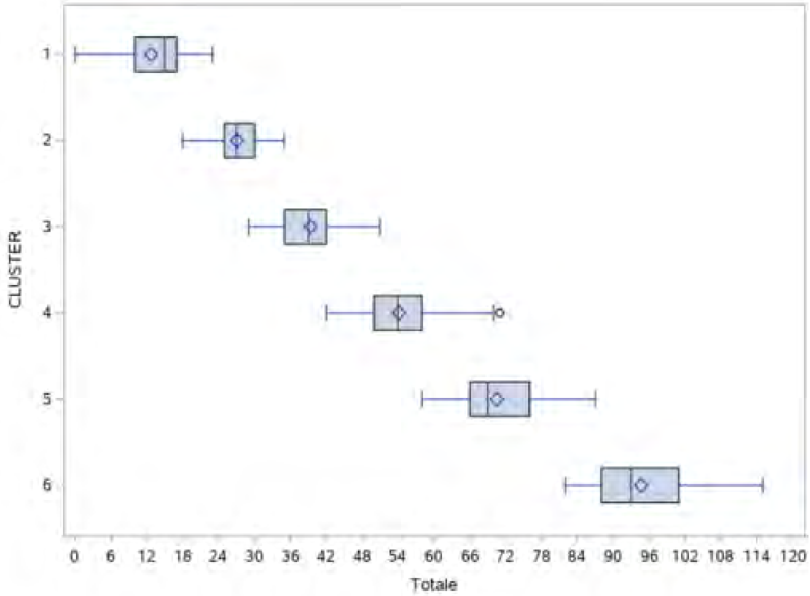

Figure 3. Box plot of the total score for each cluster (Source: Authors' own elaboration with SAS software)

Table 6. Numerosity (N) and mean (M) scores of the six groups of the chosen partition for the year 2020

Variable	Cluster 1		Cluster 2		Cluster 3		Cluster 4		Cluster 5		Cluster 6	
	N	M	N	M	N	M	N	M	N	M	N	M
Total	2,712	8.44	3,592	18.14	2,855	31.06	625	49.63	186	69.17	30	97.00
Dim_g	2,712	0.10	3,592	0.32	2,855	0.70	625	3.07	186	6.23	30	13.27
Dim_c	2,712	0.18	3,592	0.59	2,855	2.28	625	6.49	186	11.02	30	13.90
Dim_n	2,712	0.39	3,592	0.70	2,855	2.33	625	9.11	186	12.22	30	14.17

Thus, geometry would seem to be the most difficult topic but when students manage to achieve a high mean grade in this then they generally also achieve a high grade on the overall test.

For 2020, a similar result is observed. In particular, it can be observed that even in the groups where the score in number theory and combinatorics is in the medium-high range (but where geometry did badly) the total mean score is not very high, whereas in the last group when the mean score in geometry is very high, then the total mean score is also very high.

We proceed with a hierarchical cluster analysis with the method of the complete link. The choice of the 6-group partition is justified by the fact that it reproduces 84% of the variability of students. In addition, the pseudo-f index has a local maximum in the six-group partition and the pseudo-t^2 index has a much lower value than in the five-group partition; the semipartial-R^2 index also indicates this choice because two clusters that are too heterogeneous were merged in the transition from the six-group partition to the five-group partition.

As mentioned above, clusters 4 and 5 (**Table 6**) contain students with a mean score in combinatorics and number theory in the medium-high range but only in cluster 6 (where the mean score obtained in the geometry question is very high) is the total mean score also considerably high, at 97 points.

Similar considerations can be made for 2019: the choice of the five groups is justified by the fact that it reproduces 79% of the students' variability. Furthermore, in correspondence with the partition into five groups, the pseudo-f index assumes a local maximum and the pseudo-t^2 index assumes a much lower value than the value assumed with the partition into four groups. Although in clusters 3 and 4 there are students who have obtained scores relevantly higher than the global average score (1.7 for combinatorics and 1.3 for number theory) in the demonstrative questions of number theory and combinatorics, high average total scores are recorded only in cluster 5, where there is a high average score in the demonstrative question of geometry (**Table 7**).

Referring again to the year 2018, it can also be observed that 20% of the score obtained in the geometry question was cumulated by students who answered all three multiple-choice geometry questions correctly.

Table 7. Numerosity (N) and mean (M) scores of the five groups of the chosen partition for the year 2019

Variable	Cluster 1		Cluster 2		Cluster 3		Cluster 4		Cluster 5	
	N	M	N	M	N	M	3N	M	N	M
Total	6,237	13.10	2,434	29.10	948	44.00	258	63.40	28	87.70
Dim_g	6,237	0.10	2,434	0.40	948	1.10	258	3.10	28	9.70
Dim_c	6,237	0.40	2,434	1.90	948	7.20	258	13.20	28	14.10
Dim_n	6,237	0.30	2,434	1.40	948	5.10	258	8.30	28	12.90

Table 8. Mean and standard deviation scores for question 2 of the questionnaire

	Number theory	Combinatorics	Geometry	Algebra
Mean	6.95	6.30	6.28	6.58
Standard deviation	1.68	1.89	1.90	1.90

Only 4% of the score obtained in the number theory demonstration question was cumulated by those who answered all three multiple-choice number theory questions correctly. This means that, for the year 2018, students who demonstrated the ability to use the theoretical tools to answer the geometry multiple-choice questions actually contributed a high proportion of the overall score in the demonstration question. As far as number theory is concerned, Those who did not answer at least one multiple-choice question well also contributed to the total score in the demonstration question. In conclusion, those who know geometry well and answered all the multiple-choice questions correctly contributed to the total score in the geometry question, while in the number theory question, even those who did not answer at least one of the three multiple-choice questions correctly scored points, thus contributing to the total score even if they do not excel in the subject.

Although the aim of this article is not to analyze the causes and motivations behind such a lackluster performance in geometry by Italian students during mathematics competitions, we can try to provide a possible interpretation of the phenomenon, to be confirmed and further investigated by future research on the subject.

To do this, the starting point is an analysis of the questionnaire administered only to the students of the Roman district (486 students in total) who took part in the 2020 district competition. This questionnaire was carried out by students at the end of the competition; thus, at a time when they were well aware of the difficulties they had encountered.

First of all, it should be noted that the students declared that, on average, they felt as good at geometry as at the other disciplines; this can be seen from the answers given to question 2, which asked how good they felt they were at one of the four areas of Olympic mathematics, regardless of their recent performance in the competition. Students could give a grade from 1 (not at all gifted) to 10 (very gifted). **Table 8** shows the mean answers with the standard deviation.

The differences between the mean scores in the individual areas appear minor, although geometry remains the most difficult subject and number theory the easiest. In each case, the variability is low and therefore the mean score is adequate to describe the individual domains.

At the same time, however, what emerges is that the geometry questions were found to be the most similar to those seen in class; this can be seen from the answers given to question 4 of the questionnaire, in which they were asked to estimate how often they had encountered exercises at school similar to those performed in the competition:

Table 9 shows that one out of three students has often or always encountered exercises similar (if not in difficulty, at least in content and/or structure) to those of the competitions in their normal classroom teaching. In the case of the remaining areas, on average only one student in five has encountered exercises at school that are similar to those in the competitions.

Table 9. Absolute frequencies for question 4 of the questionnaire

	Number theory	Combinatorics	Geometry	Algebra
Never	178	184	94	182
Almost never	202	184	214	182
Often	86	98	147	99
Always	5	4	19	10
No answer	15	16	12	13

Table 10. Answers to question 3 of the questionnaire

	Number theory	Combinatorics	Geometry	Algebra
Formulating hypotheses and working strategies	232	218	194	238
Using knowledge in solving questions	173	194	243	183
Both	2	11	8	10
No answer	83	85	57	75

It can therefore be assumed that the students, even the talented ones, even though they know the main rules of geometry and even though they do it regularly in class, still have difficulties when it comes to competition exercises.

In the above-mentioned interview, Prof. Pernazza stated that in order to solve geometry questions, it is necessary to see or imagine new elements or information in the assigned configuration that are not explicitly written in the text of the problem, and this ability can be acquired with appropriate training and by performing a large number of exercises. Furthermore, he stated that reading and understanding a demonstration is quite a different process from knowing how to produce it, although the study of demonstrations can be a good way to acquire a decent grasp of the technique and ability to produce them. In his opinion, the knowledge of theorems alone is not enough to know how to demonstrate in geometry, but it becomes even more important to know the demonstrations themselves, since the reasoning carried out within the demonstration of a theorem can provide hints, ideas and suggestions about the methodology and procedures to be adopted in solving geometric questions.

All this is confirmed if we read the answers given by the students to question 3 of the questionnaire, in which they were asked to indicate which of the two moments they found most difficult in solving the problems of the competition they had just completed (**Table 10**).

Regarding geometry, students tended to have clearer ideas (it was the item they left least blank). In particular, the major stumbling block they encountered was the ability to use their knowledge to solve the questions, which supports Prof. Pernazza's assertion that it is more useful to know demonstrative techniques than simply stating a theorem.

DISCUSSION AND CONCLUSION

The difficulties that Italian students encounter in studying, and (even more so) in tackling geometry questions, is not homogeneous. Recent data from the TIMSS 2019 survey would seem to suggest that geometry is a strong area for students in grade 8, whereas at upper secondary school level the trend seems to be reversed.

From the analysis of the scores obtained in the competition by the approximately 10,000 Italian students who took part in the district competitions from the year 2018 to 2020, geometry seems to represent the area that creates the most difficulties.

Understanding the motivations behind such alleged difficulties experienced even by those who, by aptitude or commitment, should perform well in all competitive fields remains an open question. The data analyzed in this research provide clear but not fully comprehensive trends. This is an excessively narrow time span, not to mention some variables at play that have not been considered here, such as being able to assess the level of difficulty of the questions, which could in some way influence the results obtained. However, what emerged from the answers to the February 2020 questionnaire provides some interesting indications. The topics that appear in the competition geometry questions are theorems and properties that are studied in Italian schools more than other topics such as averages and inequalities in algebra or Diophantine equations in number theory, aspects that are completely unknown to those who have never attended a competition preparation course or picked up a book dedicated to student competitions. Thus, the geometry exercises that students encounter during student competitions are the most similar to those carried out in the classroom, but they are also those in which the students themselves state that they have most difficulty in using the knowledge they possess. However, the observed trend seems to suggest that geometry questions are the ones which are least attempted by students and, in terms of performance, have lower scores. Those who manage to score well are also those who, at the end of the competition, will tend to be at the higher end of the scale, making geometry a particularly discriminating and predictive marker of competition success.

Further research is needed to better understand not only the factors behind the difficulties analyzed, but also the type of approach and confidence that competing students have towards geometry questions. In any case, the research conducted, although it has provided only broad trends, represents a starting point for trying to frame the phenomenon under study.

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