European Journal of Science and Mathematics Education, 2023, **11**(1), 105-122
ISSN: 2301-251X (Online)

OPEN ACCESS

Research Article

Knowledge for teaching mathematical problem-solving with technology: An exploratory study of a mathematics teacher's proficiency

Hélia Jacinto [1*]
0000-0003-4768-2080

Susana Carreira [1,2]
0000-0002-0746-7258

[1] Instituto de Educação, Universidade de Lisboa, Lisbon, PORTUGAL
[2] FCT, Universidade do Algarve, Faro, PORTUGAL
[*] Corresponding author: hjacinto@ie.ulisboa.pt

Citation: Jacinto, H., & Carreira, S. (2023). Knowledge for teaching mathematical problem-solving with technology: An exploratory study of a mathematics teacher's proficiency. *European Journal of Science and Mathematics Education, 11*(1), 105-122. https://doi.org/10.30935/scimath/12464

ARTICLE INFO

Received: 31 Dec 2021

Accepted: 8 Sep 2022

ABSTRACT

This paper presents the results of an exploratory case study that examined a veteran secondary teacher's knowledge for teaching non-routine mathematical problem-solving with digital technologies. Data was collected through the observation of a veteran mathematics teacher, in real time, solving a mathematical problem with the digital tools of his choice. The descriptive model Mathematical Problem-Solving with Technology (MPST) was used to analyse the teacher's utterances and actions while solving a mathematical problem, with GeoGebra and a spreadsheet, and expressing his reasoning also with technology. Our findings reveal the complexity of expert problem-solving with technology, through regulation processes and several micro-cycles that mainly involve the processes integrate and explore. Mathematical problem solving with technology entails relevant mathematical knowledge as well as knowledge about the mathematical affordances of the digital tools available, and the ability to combine them to develop a conceptual model of the solution to the problem. Thus, the teacher's techno-mathematical fluency seems crucial to successfully solve the problem and express the reasoning with technology. Based on the findings, we discuss the teachers' knowledge for teaching mathematical problem-solving with technology as including a particular kind of proficiency, techno–mathematical fluency for solving-and-expressing problems with technology. The limitations of the study, further research topics and implications for professional development and teacher education programmes are discussed.

Keywords: mathematical problem-solving knowledge for teaching, mathematical problem-solving, mathematics teachers, techno-mathematical fluency, technology

INTRODUCTION

Digital technologies have been offering new opportunities for mathematical teaching and learning, namely in representing and visualizing, exploring, manipulating, modelling, identifying variants or invariants, triggering conjectures, or even supporting justifications and generalizations (Santos-Trigo & Reyes-Martínez, 2019; Yao & Manouchehri, 2019). Despite these opportunities and even though technological resources have been gradually making their way into the classroom, many technology interventions in mathematics education research still replicate "traditional approaches, with some functional or conceptual improvement" lagging "behind its perceived potential to enhance the learning experience" (Bray & Tangney, 2017, p. 265).

Concomitantly, mathematical problem-solving has been a fertile research field in mathematics education (Liljedahl & Cai, 2021). Still, the role of digital tools in solving non-routine mathematical problems and their impact in expressing mathematical thinking remains an under researched topic. A few research teams have been adding evidence on the problem-solving strategies and ways of reasoning developed by students and teachers by means of technological tools (e.g., Jacinto & Carreira, 2017a, 2021; Koyuncu et al., 2014; Santos-Trigo, 2019; Santos-Trigo & Reyes-Martínez, 2019). General studies on mathematical problem-solving expertise have proposed idealised prescriptive models and, more recently, descriptive models anchored in empirical data (e.g., Rott et al., 2021). Actual mathematical problem-solving seldom occurs in a linear, straightforward way; instead, as shown with young or adult problem solvers, the process takes place in a cyclic way (Carlson & Bloom, 2005). This is also the case when problem solvers make use of digital tools in carrying out the process (Carreira & Jacinto, 2019; Jacinto & Carreira, 2021; Rott et al., 2021). However, the role of technology in triggering this cyclic activity and pushing the solver to move forward is still insufficiently understood.

As with other kinds of mathematical enriched tasks, the integration of technology in problem-solving brings new challenges to the teacher, either novice or experienced, especially arising from the "access to new habits of mind and to the new environments resulting from a serious presence of digital technologies" (Hegedus & Moreno-Armella, 2009, p. 397). As pointed out by Hervey (2015), even veteran teachers must make considerable changes to their technological and professional knowledge when integrating digital tools into their teaching practices. Those teachers, unlike beginners, have many years of experience and expertise in their subject matter and a solid didactic knowledge, are committed to their professional development and accomplishment, and can reflect on the complexity of their practices (Carrillo & Flores, 2018; Hervey, 2015; Monteiro et al., 2020). Nevertheless, the veteran mathematics teacher's activity of problem-solving with technology remains understudied, while the impact of technological tools in problem-solving with pre-service teachers has had some expression in the literature (e.g., Hernández et al., 2020; Silva et al., 2021).

Thus, this study aims to explore the uses that a veteran mathematics teacher makes of digital tools when solving a non-routine mathematical problem and expressing its solution. This descriptive and exploratory study was guided by the following research question: what is the role of technology on the mathematical problem-solving process of a veteran mathematics teacher and how does it support his mathematical thinking? The following section presents the key ideas on mathematical problem-solving with technology and empirical research on teacher's use of technology in solving problems, which will subsidize the study's theoretical framework.

LITERATURE REVIEW

Cognition in digital environments has been conceptualized as stemming from the interactions between individuals, technology, and the surrounding media, hence, *humans-with-media* entails the transformational and reorganizational power of the digital tools with which one thinks and acts (Borba & Villarreal, 2005). Sinclair (2020) agrees in that it is not possible to establish a clear boundary between one's actions and the thoughts prompted by digital tools in addressing those possibilities of interaction.

Empirical research on mathematical activity with technology has advanced metaphors for the role of digital tools in classroom settings. For instance, Geiger (2005) elaborated on three metaphors: in the *technology as master* metaphor the student blindly consumes the outputs produced, limited by their mathematical knowledge, subservient to the tool; in the *technology as servant* metaphor, the individual gives instructions to the tool that carries them out obediently, the tool is used to replace pen-and-paper work as it is reliable and time-saving; in the *technology as partner* metaphor, the student develops a bond with the tools, almost as if it is a human partner whose messages prompt further action; in the *technology as extension-of-self* metaphor, the students "incorporate technological expertise as an integral part of their mathematical repertoire", the boundary between the student and the tool becomes blurry in the sense that technology supports mathematical thinking "as naturally as intellectual resources" (Geiger, 2005, p. 371). More recently, Kuzle (2017) referred to digital tools as *governors* of human cognition, as cognitive and creative *partners*, or *extensions* of the cognitive self.

Technology, thus, plays a significant role in the development of mathematical thinking also allowing innovative ways of accessing information. It affords new styles of thinking and knowing, producing a reorganization of cognitive activity (Borba & Villarreal, 2005), namely in problem-solving.

Key Ideas on Solving Mathematical Problems with Technology

We have been focusing our research on *non-routine mathematical problems* that elicit the development of conceptual models of the solutions. By non-routine mathematical problem, we refer to a challenging situation for which the solver does not have a straightforward mathematical process to reach the solution (Saadati & Felmer, 2021). Solving a non-routine mathematical problem with technology entails a mathematisation activity where the *solver-with-media* must devise a productive way of thinking to address such challenging situation (Lesh & Zawojewski, 2007). Thus, the adoption of a mathematical point of view includes to call upon technological knowledge and skills, so that the solver-with-media develops mathematical thinking by means of digital tools.

While solving a problem, the solver-with-media develops a conceptual model of the solution often entailing a progressive mathematisation activity, where a *model of* a particular situation evolves into a *model for* mathematically explaining or justifying that solution (Gravemeijer, 2005). According to Lesh and Doerr (2003), the conceptual model usually displays the solvers' mathematical thinking developed throughout the approach. But, as mathematical thinking is being produced by a solver-with-media, the boundary between the solving phase and the reporting phase of the activity becomes blurred. In fact, they may be so intertwined that the concept of *problem-solving-and-expressing* highlights the synchronous process of mathematisation and expression of mathematical thinking (Carreira et al., 2016).

Key Ideas on Teacher's Use of Technology in Mathematical Problem-Solving

Teachers' digital competence has been under the spotlight in recent years and several frameworks have been proposed not only for its assessment, but also to promote its improvement either in pre-service or in-service training programs. Teachers' experiences with digital tools are crucial to the success of technology integration in mathematics teaching and learning (Clark-Wilson & Hoyles, 2019; Drijvers et al., 2013; Leung, 2017). They must be aware of the affordances and limitations of specific technological tools that make them suitable, or not, to carry out a particular task (Koehler & Mishra, 2008). In what concerns teachers' knowledge needed to successfully integrate digital technologies into their classrooms, Mishra and Koehler (2006) proposed the theoretical concept of Technological, Pedagogical And Content Knowledge (TPACK) which, in the field of mathematics education, brings together mathematical knowledge, technology, and pedagogy.

A few recent studies have been examining prospective mathematics teachers dealing with problems by means of digital tools. Silva et al. (2021) discussed how the media that future teachers used in tackling a problem shaped the strategies and solutions devised. Their study reports on the ways of thinking-with-technology developed by the collectives of teachers-with-media, concluding that the technologies brought to the fore–GeoGebra and the spreadsheet–influenced the exploration of the problem visually, numerically, and experimentally. Hernández et al. (2020) have also studied pre-service mathematics teachers engaged in problem-solving with GeoGebra, with a particular focus on the mathematical understanding they developed. The future teachers proved to be able to work with different representations, to conjecture on the solution and, throughout the activity, they resorted to various resources and skills, demonstrating their understanding of mathematics.

In the same line, Santos-Trigo et al. (2021) analysed in-service mathematics teachers' use of GeoGebra and other online tools in the development of problem-solving approaches. GeoGebra allowed participants to use a set of heuristics that were helpful in modelling dynamically the challenging situation (e.g., using GeoGebra to solve a simpler but related problem). Teachers made tactical and strategic decisions in the devising of a plan to reach the solution, which supports the importance of metacognitive control strategies in solving mathematical problems with technology. One of their study's implications refers to the need to consider "the coordinated use of technology affordances to represent and work on mathematical tasks or problems" (p. 17).

THEORETICAL FRAMEWORK

Mathematics Teachers' Proficiency in Problem-Solving with Technology

Teachers have a critical role in their students' performance. For instance, teachers' dispositions towards and experiences with non-routine mathematical problems influence their overall enjoyment and engagement with mathematics learning (Liljedahl, 2014) as well as their students' problem-solving performance (Saadati & Felmer, 2021). Furthermore, the teachers' perspectives about the nature of technology in the classroom not only includes beliefs about availability or purposes of technology, but also about their own knowledge regarding the (pedagogical) use of technological tools (Leatham, 2010).

Claiming that problem-solving should be an integral part of the mathematics knowledge necessary for teaching, Chapman (2015) developed a framework to analyse the knowledge that mathematics teachers need to teach mathematical problem-solving efficiently and to develop problem-solving skills in their students. The mathematical problem-solving knowledge for teaching (MPSKT) comprises six components: knowledge about problems; knowledge of problem-solving; knowledge of problem-posing; knowledge of students as problem-solvers; knowledge of problem-solving instruction; and affective factors and beliefs. In this study we are particularly interested in analysing an under researched component of that framework: mathematics teachers' *proficiency* in solving mathematical problems, particularly with technology.

Successful problem-solving requires, amongst others, the use of suitable mathematical resources (Schoenfeld, 1985). However, to characterize the proficiency of a solver-with-media whilst solving and expressing a problem with technology requires to consider digital tools equally indispensable (Jacinto & Carreira, 2017a). Although the pervasive use of digital tools has an impact on the nature of mathematical skills and ways of knowing and understanding (Borba & Villarreal, 2005), it does not diminish the need for mathematical knowledge (Hoyles et al., 2010). So, it is timely to consider the proficiency of the solver-with-media regarding both these sets of resources: mathematical and technological ones.

Hoyles et al. (2010) labelled the ability to use both technological and mathematical knowledge to solve every day or work-related problems as 'techno-mathematical literacies.' As witnessed in the workplace settings they studied, the workers' techno-mathematical literacies were not being properly developed on the job, which made the researchers aware of the need to develop those skills explicitly. Hence, the idea of techno-mathematical literacies is relevant to address mathematics teachers' proficiency with digital tools to solve mathematical problems. As a significant feature of such activity, the notion of fluency, as brought by Papert and Resnick (1995), seems more appropriate to describe the ability to articulate a complex idea by resorting to a digital tool, and to be able to do or construct relevant things with it. Thus, the concept of *techno-mathematical fluency* aims to capture the ability to combine mathematical and technological knowledge for solving-and-expressing mathematical problems (Jacinto & Carreira, 2017a, 2017b). As with digital fluency (Barron et al., 2007), techno-mathematical fluency entails to be able to select useful resources from a pool of possibilities, either mathematical or technological, to recognize useful affordances or constraints in those resources, and to know how a particular tool can be used to create a techno-mathematical solution to a problem (Jacinto & Carreira, 2017b). Still, mathematical knowledge guides the use of the technological tool as it allows the recognition of affordances that may determine the problem-solving approach and the corresponding conceptual model (Yao & Manouchehri, 2019).

A Descriptive Model of Mathematical Problem-Solving with Technology

In an earlier study (Jacinto & Carreira, 2017a), we have synthesised a descriptive model of MPST. It illustrates the intertwining between mathematical and technological knowledge along the processes undertaken by the solvers when solving-and-expressing non-routine mathematical problems with the use of the digital tools of their choice. A keystone of the MPST model is the inseparability between the subject and the digital tool in solving the problems and expressing the solver's techno-mathematical thinking.

The lack of a theoretical tool to describe the processes of merging together mathematical and technological knowledge, and understand the role of technology in devising a path to the solution of a non-routine problem (Santos-Trigo & Camacho-Machín, 2013) led to the combination of two frameworks: one that addresses the activity of an individual approaching a technological task or problem (Martin & Grudziecki,

Table 1. Processes of mathematical problem-solving with technology (Jacinto & Carreira, 2017a)

MPST process	Description	
Grasp	First encounter with the problem, by reading or stating it; appropriation of the situation and conditions, and early ideas on what it involves.	Communicate: Comprises interactions with others of relevance dealing with problem.
Notice	Initial attempt to understand what is at stake, the mathematics that may be useful and the digital tools that may be necessary.	
Interpret	Place affordances in the technological resources to ponder mathematical ways of approaching the solution.	
Integrate	Combine technological and mathematical resources within an exploratory approach.	
Explore	Use technological and mathematical resources to explore and analyse conceptual models that may enable the solution.	
Plan	Outline an approach to achieve the solution based on the analysis of the conjectures previously explored.	
Create	Carry out the outlined approach, recombining resources in new ways to enable the solution and synthesizing new knowledge objects that will contribute to solve-and-express the problem.	
Verify	Engage in activities to explain and justify the solution based on the mathematical and technological resources available.	
Disseminate	Present the solution or outputs to relevant others and ponder on the success of the problem-solving process.	

2006), and another model that allows to describe the processes involved in mathematical problem-solving (Schoenfeld, 1985).

The MPST model comprises ten processes (**Table 1**) obtained by the comparison between the two frameworks which allowed us to identify prominent actions, merge processes within the Martin and Grudziecki's (2006) model and decompose some stages of Schoenfeld's (1985) mathematical problem-solving model, with the support of a paradigmatic case (Jacinto & Carreira, 2017a). For instance, we realized resemblances between Schoenfeld's (1985) stage of exploration and Martin and Grudziecki's (2006) processes of organisation, integration, and analysis. We merged organisation and integration under the designation of *integrate*, which refers to the organised combination of technological and mathematical resources that will enable an exploratory approach. The next stage, to *explore*, is that of the use of these resources to analyse the outcome of the integration, to investigate conceptual models that will enable finding the solution of the problem. Another example is the fact that Martin and Grudziecki's (2006) model did not consider a *verification* process, which is vital to the success of mathematical problem-solving, as proposed by Schoenfeld (1985). Hence, we included the process called *verify* that refers to the activity of explaining and justifying the solution using the available mathematical and technological resources.

The literature discussed above informed the present study in two essential ways. Firstly, it provides a way of thinking about the phenomenon by discussing the key ideas and concepts that will support the data collection for the study and assist in the organization of the findings. Secondly, the theoretical foundation allows to explore the possibility of elaborating on Chapman's (2015) framework of MPSKT, specifically by extending the component of teachers' knowledge about solving problems with technology. We argue that, to efficiently integrate digital technologies in mathematical problem-solving, teachers need to develop a particular kind of proficiency that includes mathematical problem-solving knowledge entwined with technological knowledge, useful in developing a conceptual model of the solution and in expressing the techno-mathematical thinking associated, i.e., teachers' techno-mathematical fluency.

METHOD

To find out the role of technology in the mathematical problem-solving activity developed by a veteran mathematics teacher and to investigate his perspectives regarding the use of digital tools in such activity, the producing of in-depth descriptions of the thinking and actions in mathematical problem-solving with technology is required. Hence, a case study methodology (Stake, 1995) is appropriate as it enables the production of the in-depth knowledge we seek. In particular, an exploratory case study allows to develop new assumptions or explanations about the teacher's knowledge of mathematical problem-solving with technology that can be further studied and used to provide a more comprehensive view on the previous theoretical discussion.

A qualitative approach was adopted in data collection, organization, and analysis. For this study, we intentionally selected a veteran mathematics teacher, Mr. Pereira (pseudonym). He is Portuguese, has over 20 years of experience in teaching mathematics in middle and secondary school, regularly integrating digital tools in his mathematics lessons, and he often engages in offering professional development courses on the use of digital tools in mathematics teaching and learning.

Data collection firstly included a semi-structured interview focusing on the teacher's experiences with and perspectives about the use of digital tools in the teaching and learning of mathematics in a broader sense, and in problem-solving tasks. As data collection took place during the Covid-19 pandemic, the interview was carried out through a Zoom meeting which enabled its video recording. In a second stage of the research, we invited Mr. Pereira to solve a non-routine problem with the use of the digital tools of his choice. To assure that we were proposing truly non-routine mathematical problems, we selected and adapted four problems that could be addressed using techno-mathematical tools, such as a calculator, a spreadsheet or a dynamic geometry environment, but did not require advanced mathematical knowledge. The researcher explained that the research aim was on the use of the digital tools to solve a non-routine mathematical problem, hence asked the teacher to look at the problems displayed and choose the one he felt most challenged about, reassuring that the focus was on the process rather than on the product. Hence, his contribution to the research was not based on achieving a final solution to the problem, but on the process of devising a possible approach.

The teacher was instructed to verbalize his thinking and actions while solving the problem, following a think-aloud protocol (Bookman, 1993), which has been used for studying mathematical problem-solving (Kuzle, 2017; Montague & Applegate, 1993). It aims to gain insights into thinking processes, so it allows to retrace processes otherwise hindered, such as failed attempts, hypothesis posed, the mathematical notions or the digital tools used along the process of solving and expressing the problems. Since the problem-solving session also took place through a Zoom meeting, the teacher was asked to share his computer screen with the researcher, which made possible to record his actions and utterances. The files produced by the participant were also collected. The interviews and the problem-solving session were transcribed and translated, and the actions and utterances were then coded using a qualitative data analysis software, as described in the following section.

Data Analysis

The coding and the analysis of the data were based on the descriptive model of MPST (Jacinto & Carreira, 2017a). To investigate the processes of solving-and-expressing with technologies, we analysed the teacher's utterances based on the transcription, and his actions based on the video recording, looking for and identifying critical events (Powell et al., 2003) that would allow to segment and organize the solving-and-expressing activity. During transcript, with the support of NVivo, relevant non-verbal actions were also annotated, such as gestures, transitions between computer windows, formulas inputted, or dragging, deleting, or colouring objects.

The first author analysed the data, in light of the theoretical framework, and coded the segments independently. Afterwards, the codification was reviewed by both authors looking for inconsistencies in the interpretation and coding until full agreement was reached. **Table 2** presents examples of actions expected to take place so that a data segment could be categorized in a particular MPST process. We note that, although the processes are clearly defined, appear in a sequential list, and have seemingly distinct boundaries, in fact as admitted by the original models, in the MPST model they are rather flexible. For instance, while in studies with middle grade students the process communicate would emerge throughout the activity (Jacinto & Carreira, 2021), it is expected that it will not be patent in the activity of the participant in this study, due to the nature of the research setting (online observation of individual activity).

Finally, we carried out the writing of the case of Mr. Pereira solving-and-expressing on the screen, which also required the revisiting of the video file and the transcript to create snapshots of the computer screen containing relevant information for the descriptive analysis. This within-case analysis and report affords rich, detailed descriptions of the processes in which Mr. Pereira engaged in while resorting to GeoGebra, a spreadsheet and a text editor to develop his solution to the problem he selected.

Table 2. Description of the coding scheme

MPST process	Examples of critical events
Grasp	Skims the statement; reads out loud; identifies the mathematical topic; realizes own familiarity with the problem; looks for similar situations previously solved.
Notice	Tries to clarify the understanding of the conditions, anticipates useful mathematical notions or procedures, identifies technological tools, at reach, that may be useful.
Interpret	Thoroughly analyses the conditions in the statement, assesses what he/she can do with the mathematical and technological resources available.
Integrate	Organizes resources to carry out concrete actions in a particular technological environment from a mathematical point of view (e.g., makes a geometrical construction with GeoGebra, creates a list with a spreadsheet).
Explore	Engages in an exploratory activity about a conceptual model that may lead to the solution; examines the outcomes of the use of the techno-mathematical resources to test and to generate conjectures and analyses the results of the experimentations; reformulates the approach until finding a solution or a feasible path to reach it.
Plan	Sets an approach to obtain the solution, based on the analysis of the explorations and conjectures; ponders the techno-mathematical resources useful to express the solution (e.g., adds a new object that reveals how the solution is being envisioned, tries to use an algebraic approach).
Create	Recombines the resources in new ways to obtain the solution that may originate new understandings of the situation; new strategies, representations, or conceptual models; includes a mathematisation that reveals abstraction and generalization.
Verify	Engages in explaining and justifying the solution, using techno-mathematical resources; reports the techno-mathematical thinking developed during the activity.
Disseminate	Reflects on the success of the mathematical problem-solving with technology activity, presents the solution to relevant others for inspection including the solution, outputs or digital files produced.
Communicate	Interacts with relevant others whilst the problem-solving-and-expressing activity with technology (e.g., a colleague, a teacher, the researcher, online search).

FINDINGS

In this section, we present the main findings from the exploratory case study of Mr. Pereira's problem-solving with digital technologies. The results are organized under four themes:

1. his viewpoints on the use of technology and problem-solving in mathematics classes,
2. a report on how he solved the problem he chose with technology, using the MSPT model as the analytical framework,
3. the way in which he reflected on that activity, and
4. our interpretation of the process of integrating mathematical and technological knowledge to obtain the solution.

Perspectives on Technology Use and Problem-Solving in the Classroom

The role of digital technologies in mathematics teaching and learning was, in Mr. Pereira's own words, "*simply functional*" as it must utterly serve a mathematical purpose. In his view, mathematics teachers should not teach how to use a particular digital tool on its own, but rather when it is useful to achieve some mathematical learning goal.

> "It's not that I could not solve some problems without technology. But, if by using technology, I can do it faster, or better, or in another way, or present the solution easily or in a more creative way, then I should use that tool" (interview).

He also believed that technology plays several important roles in the learning of mathematics: to carry out computations, to produce a large amount of data, to explore options, or to support mathematical reasoning and communication. These roles for digital tools in mathematics classroom activity range from *servants* to *cognitive and creative partners* (Geiger, 2005; Kuzle, 2017). Technology has yet another role, that the teacher

> Leonor borrowed the video camera from her mother to film the general rehearsal of the play she is preparing with her colleagues at the theatre club. She knows that the camera's battery lasts 2 hours if it is in recording mode and it lasts 3 hours in playback mode. Leonor wants to record the rehearsal and immediately watch that video with her colleagues, but cannot re-charge the battery. What is the maximum amount of time of rehearsal that she can record, in minutes, to be able to view everything she recorded, right after?
>
> **Don't forget to explain your problem-solving process!**

Figure 1. The problem chosen by Mr. Pereira

considered a by-product of classroom learning, which is that of the development of certain digital skills that may be useful latter in work related settings, such as the ones associated with the use of a spreadsheet.

Mr. Pereira was quite enthusiastic about dynamic geometry software, such as GeoGebra, especially for middle grade classes where he usually resorted to its mobile app to work on the curriculum topics, except for statistics. In that case, Mr. Pereira preferred to use a conventional spreadsheet, although he assumed not to use it as often as he would like. The preference for GeoGebra's mobile app is related to its accessibility, as most of his students have a smartphone. When referring to his secondary students, his choice was on the graphing calculator because they are required to use it on national exams. Hence, Mr. Pereira seemed to recognize that although these tools may have a similar set of affordances, to use them efficiently for mathematical purposes requires different knowledge that must be taught and learnt within those different technological contexts. This suggests that he was quite aware of and intentionally promoted the development of his students' techno-mathematical fluency.

> "I feel sort of compelled to use the graphing calculator because there's an exam context in which they [students] must be familiar with it… for instance, to define a visualization window in the mobile app you use the pinch movement… in the graphing calculator it must be an algebraic formatting, there is a minimum and a maximum, and so on" (interview).

Regarding the implementation of mathematical problem-solving tasks, Mr. Pereira stated that he really enjoys solving challenges, but he admitted that non-routine problems end up not being part of his classes, although he considers them important. He identified several obstacles to the implementation of this kind of tasks: i) the students, who are not accustomed to problem-solving heuristics, or keep asking if that kind of problems will be on the test; and ii) the (former) curriculum, where it was stated that problem-solving is about applying a set of well-known rules or procedures previously learnt.

> "Occasionally, on particular situations, I think that a non-routine problem creates an excellent context where to use the mathematical contents. In that case, yes, but not as often as it should be or as I would like to" (interview).

In his view, there may be more non-routine problem-solving in the earlier grades, but it fades away as the students proceed and the learning of mathematics becomes "*more algebraic, more formal, more abstract, hence, the opportunity for a problem to create an excellent context … decreases*". This creates a snow-ball effect, as the lesser the students are exposed to non-routine problem-solving, "*the lesser the habits, the routines, and the heuristics are established*", as declared. Interestingly, he acknowledged that providing opportunities for students to interact regularly with techno-mathematical tools contribute to this skill's improvement which, in turn, facilitates the introduction and use of new tools for mathematical learning; even though they may not be allowed in examinations or explicitly endorsed by the curriculum at the time. Hence it seems that while Mr. Pereira believed students should develop non-routine problem-solving and technological skills, these skills were not equally valued and incorporated into his teaching practices.

Mr. Pereira Solving-and-Expressing on the Screen

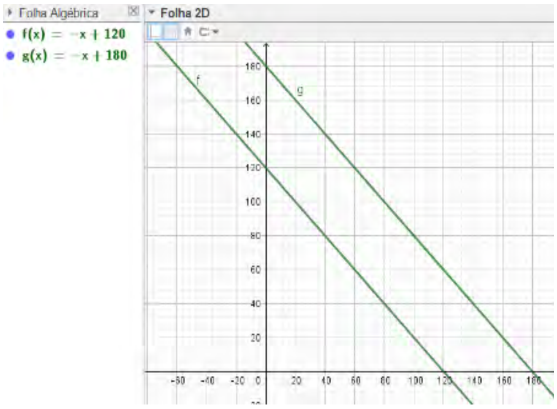
In this section we describe and analyse Mr. Pereira's problem-solving with technology by presenting segments of his activity and summarizing, in the form of tables, the processes that were carried out at each

Table 3. Mr. Pereira's actions and utterances while selecting the problem

What Mr. Pereira said or did	MPST process
Reads the problems, choses the battery problem "*This is the one that challenges me the most.*"	Grasp
"*I think I will manage with a spreadsheet*", says, splitting his desktop screen reads the problem for clarification and mentally eliminates the unnecessary parts of the problem.	Notice
Starts to organize the information on a table (record: 120; play: 180). Hypothesis 1: discounting the minutes in a "*more or less balanced way*" will lead to the solution.	Interpret

Table 4. Mr. Pereira actions and utterances while experimenting with GeoGebra

What Mr. Pereira said or did	MPST process
Moves to GeoGebra as it seems a "*better*" tool where he may "*try to see the lines*". In the input bar of the algebra view he types $f(x) = -2x + 120$.	Integrate
He is surprised by the line on the screen: "*It can't be… because I want it to take 2h only.*" "*I have to fix this, the slope must be steeper… 120, could it be?*"	Interpret
"*I'm trying not to compute, just experimenting*" types in $f(x) = -120x + 120$.	Integrate
Expects the line to intersect *Ox* at 2h, so he decides that the slope must be -60.	Interpret
Inputs the function $f(x) = -60x + 120$. "*This line models the battery time [available] to recording.*"	Integrate
Analyses the line and realizes he has been misusing minutes and hours for the values inputted.	Interpret
Decides to go back to a previous hypothesized function $f(x) = -2x + 120$ and graphs it.	Integrate
Analyses the graphical representation and concludes it is not possible to reach zero after 60', because the battery lasts 2h when recording.	Interpret
After changing the function to $f(x) = -x + 120$ inputs another one that represents the battery power when playing back, $g(x) = -x + 180$.	Integrate
Analyses the two parallel lines: "*I want to use both, so somewhere here I have to start going up*" concludes he will not reach a solution through this approach, moves back to the spreadsheet.	Interpret



stage, based on the MPST model. From the four problems given, he chose to solve the one presented in **Figure 1**.

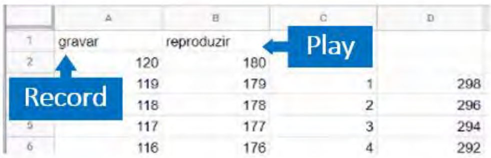
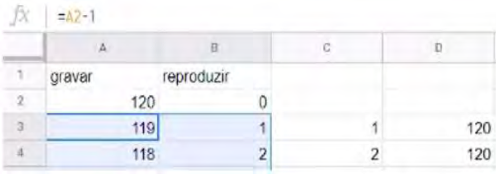
Selecting the problem and grasping the conditions

Mr. Pereira started by *grasping* the statement of the problems available, reading them carefully, while offering some reasons for declining three of them: some had familiar situations that he recalled having dealt with earlier in his career, and one mentioned constructing a square with some cubes, which he disliked. He chose the problem he felt most challenged about. He *noticed* what the problem was about and guessed he would "*manage with a spreadsheet*", such as Google Sheets. As he read the statement, slowly and out loud, he mentally eliminated the unnecessary parts of the story, trying to understand the situation, seeking clarification of the conditions in the statement and the goal of the problem. Then, he started to *interpret* the situation as he evaluated the affordances of the tabular organization, realizing the usefulness of using minutes rather than hours, and considering a mathematical approach based on the hypothesis that discounting the minutes evenly could lead to the solution (**Table 3**).

Experimenting with functions in GeoGebra

At this moment, Mr. Pereira decided to move to GeoGebra (**Table 4**) since he thought to be a better tool for constructing and analysing graphical representations of functions that could model the situation. His actions involved several cycles of *integration* of mathematical and technological knowledge (knowing

Table 5. Mr. Pereira's actions and utterances while selecting the problem

What Mr. Pereira said or did	MPST process
Considers the spending of 1′ in recording and in playback mode ($A3 = A2 - 1$). Creates a list of positive integers in C. Changes the previous formulas so that the time spent depends on C ($A3 = A2 - C3$). Represents the sum of the remaining minutes to record and playback on D. 	Integrate
Highlights the last row with the mouse, analysing: "*but they must be equal, right?... I have to find a way to make the recording and playback times the same.*"	Explore
Deletes the information on column B for each minute spending while recording ($A3 = A2 - 1$), 1′ is considered in the playing mode ($B2 = 0$; $B3 = B2 + 1$). 	Integrate
Analyses D (that still presents the sum 120): "*hum,... I'm not saying it lasts 2h... I have 2h but, in playback, it lasts 3h, [curses]! So the complete battery lasts... Its 120 units... This is cool! This is good! [laughs].*"	Explore
"*I have to compute 180 dividing by something... it must be x... I have to divide the 180′... 180 dividing by the playback [time]...*" "*It may be handy to convert the 2h to 3h, exactly, and this is some sort of decay, a spending of something per minute... this is hard... this is stressful...*"	
Identifies that this spending per minute must be different in each case concludes that the slopes of the lines built with GeoGebra must be different as well.	Interpret
"*I will subtract a minute at a time... no, because they don't run at the same time...*" "*I have to find a way to compare them,... let's say, if I have a unit of battery, I will spend half of a unit per hour.*"	Explore

GeoGebra's commands, tools and display allowed him to test several parameters in the functions), and of mathematical *interpretation* of the visual feedback offered by GeoGebra (when analysing the intersection of the line with Ox, its slope, or the two parallel lines obtained on the screen).

Testing approaches with the spreadsheet

Mr. Pereira then went back to the spreadsheet and continued by testing out his initial hypothesis as he considered the spending of 1 minute in recording (*A3=A2-1*) and in the playback mode (**Table 5**).

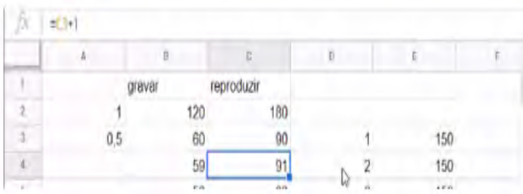
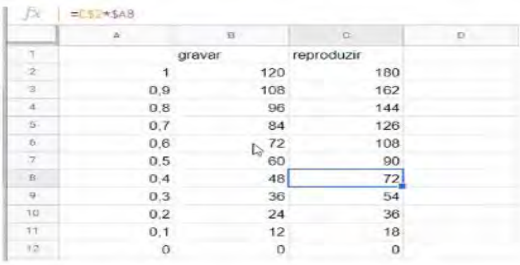
His conceptual model developed as he conjectured that the spending rate of the battery level should be different in each case, leading Mr. Pereira to realize that the comparison between both rates may be achieved by considering the unit as the full capacity of the battery. His work revealed a progressive mathematisation characterized by a set of cycles entailing the processes *integrate* and *explore*. The combination of mathematical knowledge with several affordances of the spreadsheet (such as the use of formulas, locking rows or columns, using the handle fill), with the mathematical exploration and analysis of the outputs (expecting the results to be equal, concluding the sum of A and B cannot be always 120′, exploring a mathematical way to relate the 2h and the 3h, or deciding on the use of a unit), led Mr. Pereira into developing a more robust conceptual model of the situation.

Finding the solution with the spreadsheet

Following the prior conjecture analysis, Mr. Pereira seemed confident in implementing a refined approach: he *planned* to use the full battery power as a unit. He inserted a new column on the left that he would use in the testing of several particular cases, again forming a cycle of *integration* and *exploring*. In between these processes, he went back to *interpret* the meaning of the conditions as he considered that the amount of time spent in playing and recording did not have to be the same, although he mentioned that he has "*to spend a bit more in playing*", influenced by the idea that the battery lasts longer in the playback mode.

After the testing of particular cases (such as 0.5, 0.1 and 0.9), he abandoned the mental experimentations and moved to the recombination of mathematical and technological affordances to *create* the solution by formalizing a dependency between the recording and the playback times for each percentage of the battery considered. His subsequent steps involved the use of formulas and the analysis of the spreadsheet outputs

Table 6. Mr. Pereira's actions and utterances while selecting the problem

What Mr. Pereira said or did	MPST process
"*Let me try this out!*" Considers a unit to be the full battery power and the spending of half a unit per hour inserts a column on the left of the previous table, which now becomes A.	Plan
Inserts 1 in A2 (corresponding to the full charge of the battery, i.e., 120' for recording or 180' for playing back something recorded) and then 0.5 in A3.	Integrate
"*So, if I spend half of a battery unit, following the rule of 3, a proportion, half unit will correspond to 60'… [types 60 directly in B3], and half unit will correspond to playback… 90' [typing 90 directly in C3]. . . . Maybe I'll manage this way*." Inserts 0.1 in A3 and, mentally computing the results, inserts 12 and 18 in cells B4 and C4. 	Explore
Says that he needs those results to be equal "*No, I don't have to spend the same amount of battery in playing and recording, I have to spend a bit more in playing…*"	Interpret
"*How am I doing this?*", focusing on the numbers he typed in row 3. Realizes that, when considering half of a unit, he has divided the previous row by 2.	Explore
Inserts 0.9 in A3, and decides on how to compute B3 ($B3 = B2 * 9/10$) and C3 ($C3 = C2 * 9/10$)	Integrate
Says confidently: "*this might be true!*" and abandons the mental experimentations with 0.5, 0.1, and the spreadsheet formula to test 0.9. Aims to create a dependency between the recording and the playback times for each percentage of the battery considered.	Create
Rewrites the formulas in B and C by locking the column A in those expressions ($B3 = B2 * \$A3$, $C3 = C2 * \$A3$). Sets a decreasing increment in the battery level of 0.1 ($A4 = A3 - 0.1$).	Integrate
Drags the fill handle and looks at the results (decimals): "*no, this is not working well.*" Realizes he should have locked the cells B2 and C2 that contain the amount of time that the battery holds for recording and playing when completely charged (120' and 180').	Explore
Corrects the formulas, locking row 2 ($B3 = B\$2 * \$A3$ and $C3 = C\$2 * \$A3$). Redoes the auto fill.	Integrate
Analyses the row corresponding to 0.5 to see if the result matches the previous hypothesis tested (0.5 corresponds to 60' recording or to 90' playing). "*Now, this is cool!*" Searches the data in the spreadsheet looking for identical values in the recording and in the playing mode's lists and observes that 72 minutes appears in both lists.	Interpret
Confirms they refer to complementary percentages of the battery. "*So, if I spend 72' recording, I will spend 60% of the battery. No, I will spend 0.4, as I still have 60%. If I spend here [points to C8] … 40%. So, this is the answer. Its 72'. It corresponds to 40% of the battery, hum, 40% of the battery while recording… and 60% in playback mode. This makes sense because the playback mode uses more [battery power].*" "*No… It's the other way around! It is 0.6 to record and 0.4 to reproduce.*" 	Verify

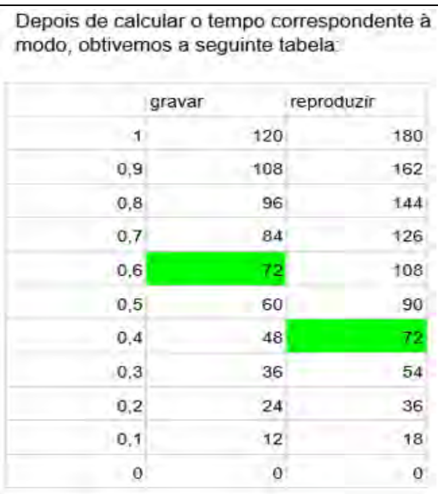
in another micro-cycle of *integrate*-*explore* until he listed all possibilities for the spending of the battery in decreasing increments of 10%. Once the sheet presented all results from full to empty battery, under those conditions, Mr. Pereira *interpreted* the values obtained looking for similar amounts of time and found the solution: 72 minutes.

Next, he engaged in initial *verification* processes by confirming that the two cells containing the answer, 72', corresponded to complementary percentages of the battery spending. He tried to make sense of the solution by reviewing the reasoning, although the meaning of the values on column A was not completely clear, yet, as he referred to them as "*battery left*" and, later, as "*battery spent*" (**Table 6**).

Expressing the solution

Mr. Pereira created a new Google Docs file and started to describe his procedures (*verify*) (**Table 7**). He reported solely on the last approach that led him to the solution, not acknowledging a considerable amount of effort put in assessing other possibilities. He used the table created in the spreadsheet, where he highlighted the cells with the solution in green. He pasted the table in the text editor and continued to verbalize his writing and thinking. While explaining that the 72' corresponded to 60% of battery usage in

Table 7. Mr. Pereira's actions and utterances while selecting the problem

<table>
<tr><th>What Mr. Pereira said or did</th><th></th><th>MPST process</th></tr>
<tr><td>Assumes he is pleased with his solution and decides to create a new file using Google Docs. Initially aims to explain that he started by testing the battery power but decides to mention only that he "calculate[ed] the time corresponding to the percentage of battery use in each mode." Colours in green the cells that contain the solution "72", in the spreadsheet, copies the table and inserts it in the text editor. Verbalizes what he is writing: "In the table it was possible to identify that 72 minutes occur in both modes of use with complementary percentages (the sum is 100%). Thus, we can conclude that the recording and playback time is 72 minutes, corresponding to 60% of usage in recording mode." Stops writing and thinks out loud: "it's the opposite, 40% in recording mode and 60% of the load in playback mode. Maybe it is not true, it was correct. It only lasts 2h, so it spends more, it's 60% recording and 40% playing. It lasts longer, exactly."</td><td>
Depois de calcular o tempo correspondente à modo, obtivemos a seguinte tabela:
<table>
<tr><th></th><th>gravar</th><th>reproduzir</th></tr>
<tr><td>1</td><td>120</td><td>180</td></tr>
<tr><td>0,9</td><td>108</td><td>162</td></tr>
<tr><td>0,8</td><td>96</td><td>144</td></tr>
<tr><td>0,7</td><td>84</td><td>126</td></tr>
<tr><td>0,6</td><td>72</td><td>108</td></tr>
<tr><td>0,5</td><td>60</td><td>90</td></tr>
<tr><td>0,4</td><td>48</td><td>72</td></tr>
<tr><td>0,3</td><td>36</td><td>54</td></tr>
<tr><td>0,2</td><td>24</td><td>36</td></tr>
<tr><td>0,1</td><td>12</td><td>18</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table></td><td>Verify</td></tr>
<tr><td colspan="2">Says he no longer recalls exactly what was being asked initially and goes back to reading the problem statement. Completes his written answer by adding information: "Thus, we can conclude that the <u>maximum</u> recording and playback time is 72 minutes <u>(since the time must be equal)</u>, corresponding to 60% of usage in recording mode and 40% of the load in playback mode."</td><td>Interpret</td></tr>
<tr><td colspan="2">"This can be improved… it depends on our goal… [but] this my most natural resolution!"
Sends the files (Google Docs and Sheets) via e-mail to the researcher.</td><td>Disseminate</td></tr>
</table>

recording mode, he stopped to think out loud: "*it's the opposite, 40% in recording mode and 60% of the power in playback mode. Maybe it is not true, it was correct. It only lasts 2 hours, so it spends more, it's 60% recording and 40% playing. It lasts longer, exactly*." This was when this fluctuation, which had been around throughout the several processes, was finally resolved.

About to submit his answer, Mr. Pereira still went back to the statement of the problem, aiming to clarify his *interpretation* of what was being asked. He realized that the problem requires to determine the maximum amount of time for recording and playback, so he included the word maximum in his previous written answer and decided to clarify that the answer is 72' because the "*time must be equal*". Even though he considered this as his final solution, he stated it could still be improved depending on the next goals but decided to submit his answer sending the spreadsheet and the text file to the researcher by e-mail (*disseminate*).

Finally, we note that while the *communication* process is considered in the original MPST model, it was not evidenced in the teacher's actions as he was asked to engage in a think-aloud protocol.

Looking Back to His Own Activity

After submitting the solution, Mr. Pereira reflected on particular episodes of his activity. He clarified that he chose this problem because he felt intrigued by "*this conflict of not spending [the battery] at the same pace*" and confirmed that, at first, he noticed the mathematical and technological tools that would be helpful: "*I was thinking about the lines, the slope, the intersection, so I was somehow modelling and looking for the tools that would be useful*".

Mr. Pereira did not feel the need to use paper-and-pencil, since "*the spreadsheet is a sort of paper-and-pencil, in fact, what I wrote here in the side is paper-and-pencil* [work]". Still, he recognized several affordances in this tool such as dragging the fill handle, or the easy and timesaving computations which allowed him to analyse several results.

> "I was not sure whether if using 0.1 would take me to the solution or not, but I was prepared to move on to 0.95 or some similar value. I was able to expand my computation, as it felt necessary. With paper-and-pencil, it would be more difficult to believe I would reach the solution at first or

second attempt... I believe that with paper-and-pencil I would have chosen my computations more wisely. Like this [in the spreadsheet] it is brute force" (interview).

The fact that he had to explain his problem-solving process, drove him back to the statement several times throughout the activity, as he knew that sometimes one may lose track of what is being asked: "*it could happen that I would focus on the 60-40 and not mention the 72, for instance*".

Integrating Mathematical Knowledge and Technology Throughout the Activity

The digital tools used by Mr. Pereira supported particular instances of his conceptual model. At first, GeoGebra afforded an analytical and graphical approach to convey the covariation between the variables at play: the *used time* (x), and the *remaining time* (y). The functions corresponding to recording and playback mode are both decreasing functions: $y=-ax+b$, ($a>0$). If $b=120$, in recording mode, and $b=180$, in playback mode, the consumption rate is $a=1$ in each case, thus the two parallel lines in the graph. GeoGebra provided graphs of the linear functions, afforded reading the graphs and obtaining the zeroes of the functions, and accentuated the relative position of the lines. This initial approach with GeoGebra led him to conclude that finding the expressions of the linear functions was not helpful.

He carried on by using a spreadsheet to explore the covariation between the variables through a numerical approach. A recursive method was used, where the two columns represent the remaining battery time decreasing by a constant rate of one minute (in both cases), starting with 120 and 180 (**Table 5**). Then, he replicated the previous reasoning, shifting to discrete variables and introducing the sum of the available time in both cases. The sum did not deliver the expected results, so he abandoned this approach and the idea of different decreasing rates for the two cases started to build up. The spreadsheet provided a "tabular calculator", where attempts to define the rate of decrease in each case were performed.

The integration process continues with the spreadsheet being used to outline a system of linear functions. A new independent variable was introduced–the battery spent (in percentage), whilst the two dependent variables were the *used battery time*, in each mode. The spreadsheet columns for the used battery time, r and p, can be defined by the equations $r(u)=120u$, where u is the battery spent in recording mode, with $u=1-\alpha$, $0\leq\alpha\leq1$; and $p(v)=180v$, where v is the battery spent in playback mode, with $v=1-\beta$, $0\leq\beta\leq1$. The variables u and v were represented in one single column containing a linear sequence of numbers, starting in 1 and ending in 0, which allowed him to "see" the two other columns as representing the used battery minutes depending on the battery spent (in percentage). A common value for both columns occurs when the time used in recording is equal to the time used in playing back, i.e., $r(u)=p(v)$. Additionally, the whole battery must be spent, which is given by the condition $u+v=1$. Hence, the solution can be obtained by the values of u and v such that $r(u)=p(v) \wedge u+v=1 \Leftrightarrow 2u=3v \wedge u+v=1 \Leftrightarrow (u, v)=(0.6, 0.4)$. The spreadsheet provided a quick way of solving the system of simultaneous equations, thus resulting in an effective integration of technological and mathematical knowledge.

DISCUSSION AND CONCLUSIONS

This exploratory case study of a veteran mathematics teacher solving a non-routine mathematical problem and expressing its solution with the digital technologies of his choice portrays his proficiency in the use of techno-mathematical resources. In the following sections we, firstly, present and discuss our main results and, secondly, hypothesise about the nature of teachers' knowledge regarding mathematical problem-solving with technology based on this exploratory case.

The Cyclic Nature of Mathematical Problem-Solving with Technology

The MPST model has been developed to address a gap in the literature since there was no explanatory tool to support the analysis of the processes taking place in mathematical problem-solving mediated by digital tools. Although this model was designed based on the processes carried out by middle school students (Jacinto & Carreira, 2017a), this case suggests its effectiveness in modelling the processes of a veteran teacher and in analysing the role of digital and mathematical resources throughout his activity.

In what concerns the role of the tools in the problem-solving activity, two inter-related findings stand out:

1. technological tools played major, but different, roles throughout the teacher's problem-solving activity and
2. the problem-solving with technology activity progressed through micro-cycles composed of several processes.

Initially, GeoGebra was used to get a better clarification of the conditions of the problem, leading to a micro-cycle composed of the processes *integrate-interpret*. Later, a different cycle emerged, comprising the processes *integrate-explore-interpret*, as the teacher used the spreadsheet to consider a different approach, to test its feasibility and, after disregarding it, to decide on another approach that would lead to the solution.

Additionally, this case provides evidence that the teacher's conceptual model of the situation evolves as the cycles *integrate-explore* render more sophisticated relationships between the variables. The testing with particular cases (model of) led to a confirmation that the approach seemed to work, so the teacher decided to inscribe a more formal character to his solution (model for), within a progressive mathematisation approach (Gravemeijer, 2005).

This case shows the nature of the activity of solving-and-expressing-with-technologies. The outputs produced in the spreadsheet were included in the final answer, so they were used both as solving and as expressing resources. Moreover, the justification of the solution only becomes completely disclosed as he expressed his reasoning within the verification process. Even though the exploratory activity in the spreadsheet has induced a particular plan, the solution only became clear to the teacher when he engaged in the explanation of his procedures. This reinforces the idea that the 'solving' is closely related to the 'expressing'; they are simultaneous activities of mathematisation that culminate in obtaining a techno-mathematical solution of the problem.

These results are in line with our previous findings obtained with middle grade students (Carreira & Jacinto, 2019; Jacinto & Carreira, 2017a, 2021) and with other studies that report on the non-linearity of problem-solving processes (Carlson & Bloom, 2005, Rott et al., 2021). In fact, these results not only contribute to contradict a view of mathematical problem-solving activity as a straightforward progression from the givens to the goals, as they support the claim that it develops around "iterative cycles of expressing, testing, and revising current ways of thinking" (Lesh & Zawojewski, 2007, p. 772), also when technology plays a significant role in the activity. This is particularly clear in the process of *integrating technological and mathematical knowledge and procedures*, which is at the heart of these micro-cycles. Effective integration is, thus, a central action that supports several other fundamental processes for the advancement of the solution, for instance, *exploration* and *creation*.

Teachers' Knowledge About Mathematical Problem-Solving with Technology

This study portrays the case of a mathematics teacher that is familiar with a diversity of digital tools useful in mathematics teaching and learning, who creates opportunities for his students to use them in the classroom and is able to use them to solve-and-express a non-routine mathematical problem. Mathematics teachers' knowledge about teaching problem-solving requires, according to Chapman (2015), to be proficient in solving problems and in understanding the nature of problem-solving both as a process and as a way of teaching. Our exploratory study contributes with the case of a veteran mathematics teacher who regularly uses technologies for mathematics teaching and learning, enjoys non-routine problem-solving, and exposes his proficiency in mathematical problem-solving with technology.

Teacher's techno-mathematical fluency

The teachers' proficiency is characterized by the recognition of particular affordances in several technological tools that can be useful in developing a solution to the non-routine problem. At start, GeoGebra afforded algebraic and graphical representations of the functions for the battery spending in each mode. He did not pursue this approach, yet it allowed him a deeper understanding of the situation. By resorting to a spreadsheet, he elaborated conjectures and explored them by placing specific affordances in this tool, namely, to organize sequences of related values and easily testing the effects of changing the relations. By incorporating the formatted table on the text editor, to present his reasoning, he created a techno-mathematical answer to the problem, thus representing his conceptual model of the solution. The teacher's techno-mathematical fluency includes the recognition of affordances in the digital tools with several purposes: to interpret the situation from a techno-mathematical point of view, to explore a conceptual model, and to produce a techno-mathematical solution.

Technological tools played a paramount role throughout the mathematical problem-solving-and-expressing activity of the teacher. His proficiency lies in the intertwining of technological and mathematical resources, within micro-cycles comprising the processes of integration and exploration. These have preceded major advancements in the development of a conceptual model of the situation, thus having a clear effect on achieving a techno-mathematical solution. This suggests that being techno-mathematically fluent (Jacinto & Carreira, 2017b) is an essential skill in productively solving-and-expressing problems with digital technologies.

Furthermore, this exploratory case seems to account for the complexity of the activity of successful mathematical problem-solving with technology as it reveals that metacognitive skills, namely control and regulation strategies, are of paramount importance to progress. It has been extensively documented in the literature that productive mathematical problem solvers take their time in understanding the problem and its goal structure, in identifying the mathematical resources that may be necessary, they constantly monitor their solution, and, since they are able to evaluate the efficiency of their strategies they also know when to abandon a certain approach (Bookman, 1993; Chiu et al., 2013; Hanin & Van Nieuwenhoven, 2020; Schoenfeld, 1985). The findings of this exploratory case study on mathematical problem-solving activity with technology are consistent with these features. However, the case shows that not only does this veteran teacher have the relevant mathematical knowledge, but also the technological knowledge, and he is able to efficiently combine and use them in developing his conceptual model of the solution.

Knowledge for teaching mathematical problem-solving with technology

As different technological tools have their own potentials and limitations and provide different affordances to work on mathematical problems (Koehler & Mishra, 2008), it seems important that teachers know those possibilities and guide their students to use them and thus develop their techno-mathematical fluency in rich mathematics classroom experiences. Thus, this exploratory study opens a new perspective regarding teachers' knowledge to teach mathematical problem-solving with the use of digital technologies. Knowledge for teaching mathematical problem-solving with technology includes mathematical problem-solving knowledge that must be efficiently intertwined with technological knowledge to develop a conceptual model of the solution and to express the techno-mathematical thinking produced.

Limitations and directions for further research

This study has achieved its main goal by expounding the role of digital tools in the processes of mathematical problem-solving-and-expressing as well as the techno-mathematical thinking of a veteran teacher. Still, it has some limitations that are worth discussing. First, the research was designed as a single case study, thus the findings are not generalizable nor transferable, as anticipated. Furthermore, the participant in which this study rests was intentionally chosen using a set of predetermined characteristics, which included being an in-service teacher with long experience in using digital technologies in teaching mathematics to middle grade and secondary students. We believe this study can be replicated, hence more explorations with respect to participants with different experience backgrounds may add empirical evidence on the two main topics addressed.

But, most importantly, this exploratory study's findings point towards two promising directions within mathematics education research. One of them is the need to further develop the conceptualization of techno-mathematical thinking as conveying the particular features of mathematical thinking mediated by digital tools and entailing the skills needed to efficiently solve-and-express mathematical problems. Additionally, it may be timely to inquire what techno-mathematical thinking looks like when digital tools are used to address modelling problems. Finally, and despite the limitations discussed above, these and further findings may be the necessary trigger to rethink professional development and teacher education programmes so that mathematics teachers, either pre-service or in-service, may develop their techno-mathematical thinking and fluency to teach their students to solve non-routine mathematical problems with digital technologies.

Author contributions: **HJ:** original idea, design of the study, data collection, writing of the original draft. **HJ & SC:** conceptualization, formal analysis, review and editing of the final version. Both authors approve the final version of the article.

Funding: This work was partially supported by Fundação para a Ciência e a Tecnologia, under the Stimulus of Scientific Employment Program (CEECIND/01584/2017) and within the scope of Unidade de Investigação e Desenvolvimento em Educação e Formação [Research and Development Center on Education and Training, UIDB/04107/2020].

Ethical approval: The authors state that the study was approved by the Ethical Commission of the Institute of Education of the University of Lisbon, on the 16th January 2020, with the approval code nr. 186.

Acknowledgements: The authors would like to thank to the teacher who participated in this study, for his kindness and willingness to participate in the research even though being aware of the stress and anxiety, but also pleasant aha moments, that non-routine mathematical problem-solving can bring.

Declaration of interest: Authors declare no competing interest.

Data availability: Data generated or analysed during this study are available from the authors on request.

REFERENCES

Barron, B., Martin, C., & Roberts, E. (2007). Sparking self-sustained learning: report on a design experiment to build technological fluency and bridge divides. *International Journal of Technology and Design Education*, *17*(1), 75-105. https://doi.org/10.1007/s10798-006-9002-4

Bookman, J. (1993). An expert-novice study of metacognitive behavior in four types of mathematics problems. *PRIMUS*, *3*(3), 284-314. https://doi.org/10.1080/10511979308965710

Borba, M., & Villarreal, M. (2005). *Humans-with-media and the reorganization of mathematical thinking*. Springer. https://doi.org/10.1007/b105001

Bray, A., & Tangney, B. (2017). Technology usage in mathematics education research–A systematic review of recent trends. *Computers & Education*, *114*, 255-273. https://doi.org/10.1016/j.compedu.2017.07.004

Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent problem-solving framework. *Educational Studies in Mathematics*, *58*(1), 45-75. https://doi.org/10.1007/s10649-005-0808-x

Carreira, S., & Jacinto, H. (2019). A model of mathematical problem solving with technology: The case of Marco solving-and-expressing. In P. Liljedahl & M. Santos Trigo (Eds.), *Mathematical Problem solving* (pp. 41-62). Springer. https://doi.org/10.1007/978-3-030-10472-6_3

Carreira, S., Jones, K., Amado, N., Jacinto, H., & Nobre, S. (2016). *Youngsters solving mathematics problems with technology.* Springer. https://doi.org/10.1007/978-3-319-24910-0

Carrillo, C., & Flores, M. A. (2018). Veteran teachers' identity: What does the research literature tell us? *Cambridge Journal of Education*, *48*(5), 639-656. https://doi.org/10.1080/0305764X.2017.1394982

Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT*, *3*(1), 19-36. https://doi.org/10.31129/lumat.v3i1.1049

Chiu, M. M., Jones, K. A., & Jones, J. L. (2013). Building on Schoenfeld's studies of metacognitive control towards social metacognitive control. In Y. Li, & J. Moschkovich (Eds.), *Mathematical proficiency and beliefs in learning and teaching–Learning from Alan Schoenfeld and Günter Toerner* (pp. 69-88). Sense Publishing. https://doi.org/10.1007/978-94-6209-299-0_6

Clark-Wilson, A., Hoyles, C. (2019). A research-informed web-based professional development toolkit to support technology-enhanced mathematics teaching at scale. *Educational Studies in Mathematics*, *102*, 343-359. https://doi.org/10.1007/s10649-018-9836-1

Drijvers, P., Tacoma, S., Besamusca, A., Doorman, M., & Boon, P. (2013). Digital resources inviting changes in mid-adopting teachers' practices and orchestrations. *ZDM–The International Journal on Mathematics Education*, *45*(7), 987-1001. https://doi.org/10.1007/s11858-013-0535-1

Geiger, V. (2005). Master, servant, partner and extension of self: a finer grained view of this taxonomy. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice*. MERGA.

Gravemeijer, K. (2005). What makes mathematics so difficult, and what can we do about it? In L. Santos, A. P. Canavarro, & J. Brocardo (Eds.), *Educação matemática: Caminhos e encruzilhadas* [*Mathematics education: Paths and crossroads*] (pp. 83-101). Associação de Professores de Matemática [Association of Mathematics Teachers].

Hanin, V., & Van Nieuwenhoven, C. (2020). An exploration of the cognitive, motivational, emotional and regulatory behaviours of elementary-school novice and expert problem solvers. *Canadian Journal of Science, Mathematics and Technology Education*, *20*(2), 312-341. https://doi.org/10.1007/s42330-020-00092-9

Hegedus, S. J., & Moreno-Armella, L. (2009). Introduction: The transformative nature of "dynamic" educational technology. *ZDM–The International Journal on Mathematics Education*, *41*, 397-398. https://doi.org/10.1007/s11858-009-0201-9

Hernández, A., Perdomo-Díaz, J., & Camacho-Machín, M. (2020). Mathematical understanding in problem solving with GeoGebra: a case study in initial teacher education. *International Journal of Mathematical Education in Science and Technology*, *51*(2), 208-223. https://doi.org/10.1080/0020739X.2019.1587022

Hervey, L. G. (2015). Between the notion and the act: Veteran teachers' TPACK and practice in 1: 1 settings. In C. Angeli, & N. Valanides (Eds.), *Technological pedagogical content knowledge* (pp. 165-189). Springer. https://doi.org/10.1007/978-1-4899-8080-9_8

Hoyles, C., Noss, R., Kent, P., & Bakker, A. (2010). *Improving mathematics at work: The need for techno-mathematical literacies*. Routledge. https://doi.org/10.4324/9780203854655

Jacinto, H. & Carreira, S. (2017a). Mathematical problem solving with technology: The techno-mathematical fluency of a student-with-GeoGebra. *International Journal of Science and Mathematics Education*, *15*(6), 1115-1136. https://doi.org/10.1007/s10763-016-9728-8

Jacinto, H., & Carreira, S. (2017b). Different ways of using GeoGebra in mathematical problem-solving beyond the classroom: Evidences of techno-mathematical fluency. *Bolema, 31*(57), 266-288. https://doi.org/10.1590/1980-4415v31n57a13

Jacinto, H., & Carreira, S. (2021). Digital tools and paper-and-pencil in solving-and-expressing: How technology expands a student's conceptual model of a covariation problem. *Journal on Mathematics Education*, *12*(1), 113-132. http://doi.org/10.22342/.12.1.12940.113-132

Koehler, M. J., & Mishra, P. (2008). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education*, *9*(1), 60-70.

Koyuncu, I., Akyuz, A., & Cakiroglu, E. (2015). Investigating plane geometry problem-solving strategies of prospective mathematics teachers in technology and paper-and-pencil environments. *International Journal of Science and Mathematics Education, 13*, 837-862. https://doi.org/10.1007/s10763-014-9510-8

Kuzle, A. (2017). Delving into the nature of problem solving processes in a dynamic geometry environment: Different technological effects on cognitive processing. *Technology, Knowledge and Learning*, *22*(1), 37-64. https://doi.org/10.1007/s10758-016-9284-x

Leatham, K. (2007). Pre-service secondary mathematics teachers' beliefs about the nature of technology in the classroom, *Canadian Journal of Science, Mathematics and Technology Education, 7*(2-3), 183-207. https://doi.org/10.1080/14926150709556726

Lesh, R., & Doerr, H. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh, & H. Doerr (Eds.), *Beyond constructivism–Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3-33). Lawrence Erlbaum Associates. https://doi.org/10.4324/9781410607713

Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763–804). Information Age Publishing and National Council of Teachers of Mathematics.

Leung, A. (2017). Exploring techno-pedagogic task design in the mathematics classroom. In A. Leung, & A. Baccaglini-Franck (Eds.), *Digital technologies in designing mathematics education tasks* (pp. 3-16). Springer. https://doi.org/10.1007/978-3-319-43423_1

Liljedahl, P. (2014). Approaching professional learning: What teachers want. *The Mathematics Enthusiast*, *11*(1), 109-122. https://doi.org/10.54870/1551-3440.1294

Liljedahl, P., & Cai, J. (2021). Empirical research on problem solving and problem posing: A look at the state of the art. *ZDM Mathematics Education*, *53*, 723-735. https://doi.org/10.1007/s11858-021-01291-w

Martin, A., & Grudziecki, J. (2006). DigEuLit: Concepts and tools for digital literacy development. *Innovation in Teaching and Learning in Information and Computer Sciences*, *5*(4), 249-267. https://doi.org/10.11120/ital.2006.05040249

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*, *108*(6), 1017-1054. https://doi.org/10.1111/j.1467-9620.2006.00684.x

Montague, M., & Applegate, B. (1993). Mathematical problem-solving characteristics of middle school students with learning disabilities. *The Journal of Special Education*, *27*(2), 175-201. https://doi.org/10.1177/002246699302700203

Monteiro, A., Mouraz, A., & Dotta, L. T. (2020). Veteran teachers and digital technologies: Myths, beliefs and professional development. *Teachers and Teaching, 26*(7-8), 577-587. https://doi.org/10.1080/13540602.2021.1900809

Papert, S., & Resnick, M. (1995). *Technological fluency and the representation of knowledge.* MIT Media Laboratory.

Powell, A., Francisco, J., & Maher, C. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, *22*(4), 405-435. https://doi.org/10.1016/j.jmathb.2003.09.002

Rott, B., Specht, B., & Knipping, C. (2021). A descriptive phase model of problem-solving processes. *ZDM–Mathematics Education*, *53*, 737-752. https://doi.org/10.1007/s11858-021-01244-3

Saadati, F., & Felmer, P. (2021). Assessing impact of a teacher professional development program on student problem-solving performance. *ZDM–Mathematics Education*, *53*, 799-816. https://doi.org/10.1007/s11858-020-01214-1

Santos-Trigo, M. (2019). Mathematical problem solving and the use of digital technologies. In P. Liljedahl, & M. Santos-Trigo (Eds.), *Mathematical problem solving* (pp. 63-89). Springer. https://doi.org/10.1007/978-3-030-10472-6_4

Santos-Trigo, M., & Camacho-Machín, M. (2013). Framing the use of computational technology in problem solving approaches. *The Mathematics Enthusiast, 1-2,* 279-302. https://doi.org/10.54870/1551-3440.1268

Santos-Trigo, M., & Reyes-Martínez, I. (2019). High school prospective teachers' problem-solving reasoning that involves the coordinated use of digital technologies. *International Journal of Mathematical Education in Science and Technology*, *50*(2), 182-201. https://doi.org/10.1080/0020739X.2018.1489075

Santos-Trigo, M., Barrera-Mora, F., & Camacho-Machín, M. (2021). Teachers' use of technology affordances to contextualize and dynamically enrich and extend mathematical problem-solving strategies. *Mathematics*, *9*(8), 793. https://doi.org/10.3390/math9080793

Schoenfeld, A. (1985). *Mathematical problem solving*. Academic Press.

Silva, R., Barbosa, L., Borba, M., & Ferreira, A. (2021). The use of digital technology to estimate a value of pi: Teachers' solutions on squaring the circle in a graduate course in Brazil. *ZDM–Mathematics Education*, *53*, 605-619. https://doi.org/10.1007/s11858-021-01246-1

Sinclair, N. (2020). On teaching and learning mathematics–technologies. In Y. Kolikant, D. Martinovic, & M. Milner-Bolotin (Eds.), *STEM teachers and teaching in the digital era* (pp. 91-107). Springer. https://doi.org/10.1007/978-3-030-29396-3_6

Stake, R. E. (1995). *The art of case study research*. SAGE.

Yao, X., & Manouchehri, A. (2019). Middle school students' generalizations about properties of geometric transformations in a dynamic geometry environment. *The Journal of Mathematical Behavior*, *55*, 1-19. https://doi.org/10.1016/j.jmathb.2019.04.002