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Research Article

Drawing on a computer algorithm to advance future teachers' knowledge of real numbers: A case study of task design

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ABSTRACT

In our investigation of university students' knowledge about real numbers in relation to computer algebra systems (CAS) and how it could be developed in view of their future activity as teachers, we used a computer algorithm as a case to explore the relationship between CAS and the knowledge of real numbers as decimal representations. Our work was carried out in the context of a course for university students who aim to become mathematics teachers in high schools. The main data consists of students' written responses to an assignment of the course and interviews to clarify students' perspectives in relation to the responses. The analysis of students' work is based on the anthropological theory of the didactic (ATD). Our results indicate that simple CAS-routines have a potential to help university students (future teachers) to apply their university knowledge on certain problems related to the decimal representation of real number which are typically encountered but not well explained in high school.

Keywords: CAS, ATD, real numbers, future teachers

INTRODUCTION

Klein (2016, p. 1) pointed out that young university students find it hard to use the mathematical knowledge, which they learnt at school as they deal with university level mathematics problems. Then, after they graduate from the university and go back to upper secondary school as mathematics teachers, it is also hard for them to find connections between teaching there, and what they learnt at university. This constitutes Klein's (2016) classical "double discontinuity" problem.

Durand-Guerrier (2016) studied the first discontinuity in relation to the specific case of real numbers. The author identified several gaps between the ways real numbers are conceived and taught in high school and at university. Along the lines of Durand-Guerrier (2016), González et al. (2019) proposed two challenging situations for the teaching of real numbers at high school. In this paper we show a proposal for university teaching that aims to address the second discontinuity in the special case of student's relation to real numbers.

So, the question is what should future teachers know about the real numbers and especially, what should they learn about real numbers at university? This question is almost as old as mathematics education research itself. Already, Klein (2016, p. 34-35) suggested that decimal notation was historically decisive to lead mathematicians onto the general arithmetic of "irrational numbers". Nevertheless, research on real numbers teaching or learning we can find is not much. González-Martín et al. (2013) believe that one of the difficulties in teaching or learning real numbers lies in the relevant definitions. For example, Zazkis and Sirotic (2004) found the obstacle to learn irrational numbers for students is their understanding of the equivalence of definitions. There is a "missing link" between the fraction representation and decimal representation of a real

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number. This kind of "missing link" is not only the negligence in teaching but also the unclear definition of real numbers in high school textbooks. González-Martín et al. (2013) studied the introduction of irrational and real numbers in Brazilian textbooks for secondary education. They found that the definition of real numbers always appears after the definition of irrational numbers and is expressed as the union of the set of rational numbers and the set of irrational numbers, i.e., $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$, where $\mathbb{R}$, $\mathbb{I}$, and $\mathbb{Q}$ are the sets of real, irrational, and rational numbers respectively. However, the definition of the irrational number is based on the assumption of the existence of real numbers, i.e., $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q}$. This seems to make sense but in fact, an independent definition of real numbers did not appear at all. This problem is easily ignored in textbooks by teachers and students, and they accept the definition in textbooks as "a transparent rule which does not need justification" (González-Martín et al., 2013, p. 239). Enhancing teachers' knowledge of real numbers might assist them in going beyond circular and unsatisfactory definitions as found in the textbooks mentioned above. Our present work is on the teacher knowledge about decimal representations of real numbers and focuses on those university students who intend to become teachers at secondary school.

Another question arises: How could we support them to develop that knowledge? Maybe today's widespread use of digital technology (in mathematics and other school subjects) offers and requires modified answers to this question. We can find some studies related to the use of computer algebra systems (CAS) in mathematics education (e.g., Gyöngyösi et al., 2011; Lagrange, 2005). In this paper we take into consideration an aspect of secondary school mathematics which has so far been neglected in most of the literature on the teaching of real numbers there: the use of calculators and computers, including more advanced uses such as graphing and programming, which is common at this level in many countries. In Denmark, CAS like *Maple*, *TI Nspire*, and *Geogebra* are commonly used in upper secondary schools. In schools and in society at large, real numbers and functions are increasingly accessed and handled through CAS and other mathematics software. However, in the teaching of mathematics at university, such use of tools appears at most in introductory Calculus courses and not in later, more theoretical courses. Future teachers of mathematics often take such more theoretical courses and find that they are far from what is taught at secondary school. In this paper, we particularly investigate the use of simple programming as one strategy to bridge the gap.

The paper is structured as follows. In the next section, we will recall how the anthropological theory of the didactic (ATD) works as a theoretical framework to reformulate Klein's (2016) second discontinuity problem and based on this formulate our research questions. Then, the mathematical context will be introduced, including the background on infinite decimal model of real numbers, the elaboration of the given computer algorithms and addition of infinite decimals. After mathematical context, we will introduce the empirical context for the study, and the methodology used to analyze the research questions will also be shown in this section. Then, we will present the results based on data collected in a "capstone course" called UvMat. In the end we will draw up conclusions along with perspectives for further research.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

ATD, initiated by Chevallard (2019), was used by numerous authors (Barquero & Winsløw, 2022; Winsløw, 2013; Winsløw & Grønbæk, 2014) to model Klein's (2016) "double discontinuity". Following Chevallard (2019), we denote by $R_I(p, o)$ the relation between a position p within the institution I, and the knowledge object o. In ATD, a knowledge object is modelled as a *praxeology*. A praxeology contains a *praxis* block and a *logos* block. There are two parts in the praxis block, a type of tasks and techniques used to solve the tasks. The logos block is composed by the technology part, which is a discourse about the techniques, and the theory part which justifies the technology part and explains its relation to praxis block. These notions are more thoroughly introduced in Chevallard (2019) or Winsløw (2011).

The praxeologies we study in this paper are related to real numbers as these appear in Danish high school and at university. Using the theoretical model of Klein's (2016) second discontinuity proposed by Winsløw (2013) we can represent the passage we are interested in, as

$$R_U(\sigma, \omega_{\mathbb{R}}) \to R_S(t, o_{\mathbb{R}}), \tag{1}$$

where $\omega_{\mathbb{R}}$ is any mathematical praxeology about real numbers worked by students σ in the university U, while $o_{\mathbb{R}}$ is any mathematical praxeology about real numbers supposed to be taught in the institution secondary school S by teachers t (the $\to$ of the passages above can be considered as a gap between university and high school). Future teachers' knowledge could be important for narrowing this gap. In particular, we can sometimes select some elements related to logos blocks from $\omega_{\mathbb{R}}$ to justify the praxis blocks from $o_{\mathbb{R}}$. In other cases, we need to add some mathematical elements to bridge the gap, ending up with a slightly larger object which we denote by $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$. Therefore, for those future teachers in university (represented by σ_{ft}), we aim at a new relation $R_U\left(\sigma_{ft}, \overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}\right)$, where $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$ is really connected by links known to σ_{ft}.

The first question is what kind of content should be included in $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$. Barquero and Winsløw (2022) have considered the decimal representations of real numbers such as constructing a decimal representation for a given number. They particularly investigated students' work on the graphs of a function by using different representations of $\sqrt{3}$ on *Maple* (a CAS the students know from the first semester courses). In this paper, we continue to consider real numbers as decimal representations, particularly about addition of infinite decimals operated on *Maple.*

In university, mathematics students who intend to become future teachers will acquire a certain amount of knowledge of real numbers. How students select the suitable theoretical elements from $\omega_{\mathbb{R}}$ to explain the real number problems in secondary school is a sub-discontinuity under the Klein's (2016) second gap, which can be formalized as the following transition:

$$R_U(\sigma, \omega_{\mathbb{R}}) \rightarrow R_U\left(\sigma_{ft}, \overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}\right). \tag{2}$$

How to support this transition is our second question, which is also the main work in this paper. We formulate these two questions as the precise research questions:

RQ1: How could the idea of 'infinite decimal' be related to university mathematics and taught to future secondary school teachers?

RQ2: In particular, how can working with a given computer algorithm support university students' use of (university level) mathematical knowledge to address secondary school level questions related to the infinite decimal model of real numbers? What new mathematical and didactical knowledge on decimal representations of real numbers can such work enable students to develop?

Note that RQ1 is a theoretical research question, asking for specific links between a school mathematical notion ("infinite decimals") and appropriate undergraduate mathematics, thus it is *about* the knowledge to be taught to future teachers. RQ1 is in general answered by several classical and newer texts (e.g., Sultan & Artzt, 2018), and we summarize one answer at the beginning of the next section. We address RQ2 by investigating students' work on a concrete assignment, also presented in the next section. Our answers to RQ2 are derived from analyzing students' written reports in response to the assignment, as well as interviews with selected students.

MATHEMATICAL CONTEXT

Real Numbers Represented as Infinite Decimals

In university, students will meet the completeness property of real numbers in one of the first courses in analysis. They may also be given various explanations of what real numbers are, from the number line to Cauchy sequences or Dedekind cuts (Bergé, 2010), although these constructions are seldomly treated in detail (so, they do not study equivalence classes of Cauchy sequences, etc.). The elements actually covered, particular various consequences of completeness related to convergence and compactness, then form part of $\omega_{\mathbb{R}}$, but with little direct connection to the earlier praxeologies $o_{\mathbb{R}}$ learnt at school. The completeness property, however, was coined at the end of the 19th century in order to formalize the idea that any real number can be represented by an infinite decimal (Bergé, 2010), an idea already met in school. Completeness–or, more intuitively, the decimal representation–can be used to explain that the set of real numbers satisfies the Archimedean axiom, i.e., for any $x \in \mathbb{R}$, there is an $N \in \mathbb{Z}$ such that $x \leq N$. It follows from this that, for any $x \in \mathbb{R}$, there is a unique $N \in \mathbb{Z}$ such that $N \leq x < N + 1$.

This kind of knowledge is thus formally related to logos block in $\omega_{\mathbb{R}}$, but will not be explicit at the secondary school level, although it is entirely compatible with the "number line" metaphor used there. It may not even be taught to university students although it is crucial to the connected extension $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$ (a metaphorical notation we use to designate the coherent praxeology aimed at future teachers). For RQ1, our aim is to elaborate from $\omega_{\mathbb{R}}$ the meaning of an infinite decimal as a more or less unique representation of a real number. An infinite decimal is denoted as $\pm(N.c_1c_2c_3\ldots)$, where $N \in \mathbb{N} \cup \{0\}$ and $c_i \in \{0,1,2,\ldots,9\}$. We mainly consider nonnegative infinite decimals here, as a negative infinite decimal can be defined as the additive inverse of a positive one. We use a part of chapter 8 in the book *The mathematics that every secondary school math teacher needs to know* (Sultan & Artzt, 2018) to outline a possible answer to RQ1.

An infinite decimal $0.c_1c_2c_3\ldots \in [0,1]$, where $c_1, c_2, c_3 \ldots \in \{0,1,2,\ldots,9\}$ can be rigorously defined by $0.c_1c_2c_3\ldots = \sum_{i=1}^{\infty} c_i \cdot 10^{-i}$. It is sufficient to consider the interval $[0,1]$ because we can get numbers in other intervals by adding some integer. The two main results that should be established are (a) $\sum_{i=1}^{\infty} c_i \cdot 10^{-i}$ always converges and (b) every $x \in (0,1)$ can be written as a unique infinite decimal which does not terminate in $\bar{9}$. Result (a) means that any infinite decimal makes sense because the infinite series $\sum_{i=1}^{\infty} c_i \cdot 10^{-i}$ always has a finite sum, due to convergence properties known from $\omega_{\mathbb{R}}$ (geometric series). Result (b) begins with the converse: any real number in (0, 1) can be written as an infinite decimal. The last part of (b) amounts to prove, again from $\omega_{\mathbb{R}}$, that $0.c_1\ldots c_k000\ldots$ and $0.c_1\ldots c_{k-1}(c_k-1)999\ldots$ represent the same number. The two results are divided into three theorems (Sultan & Artzt, 2018) and the corresponding proofs based on $\omega_{\mathbb{R}}$ can be found there too. After this, we can define irrational numbers as infinite decimals that do not represent a fraction of integers. It is also proved that irrational numbers correspond exactly to infinite, non-periodic decimals. Several examples and exercises related to secondary school mathematics are provided by Sultan and Artzt (2018), concerning, for instance, how to find a fraction representation of a periodic infinite decimal representation. These praxeologies can, to some extent, eliminate the "missing link" of the two representations mentioned before. With this we have outlined the central elements of $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$, which forms the basis of our answer to RQ1. We now consider how to elaborate on this answer by considering RQ2.

Computer Algorithms

Before we answer RQ2, we need to introduce two things: computer algorithms, and more knowledge on infinite decimals related to (but exceeding) what is presented by Sultan and Artzt (2018). These two things are concretely developed in an assignment (see **Appendix A**) designed for students and the answer to RQ2 is based on students' answers to the assignment. This assignment contains two parts. The first part is the understanding of a given routine. This routine is implemented in *Maple* and can be used to find the first 10 digits of $\sqrt{2}$, one by one.

It is not a new idea that programming could be a possibly central tool to introduce in undergraduate mathematics. As early as 1959, Forsythe (1959) proposed to integrate coding in most of all introductory university mathematics. Our aim with the assignment was not quite as ambitious. The aim of introducing the *Maple* based routine is to show how to compute the decimals of certain well-known irrational numbers by elementary computations (relying only on the four operations with finite decimals, and on evaluating inequalities of rational numbers) that could in principle be carried out manually. The routine merely allows us to speed up the calculation. Producing the decimal representation of $\sqrt{2}$ is a secondary school task, carried out there with calculators, but seeing how it could be done concretely is not a common experience in secondary school. Thus, the new technique contributes to the praxis block in $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$, while considering its justification and consequences contributes to its logos block.

No matter which way we use, computer or pen-and-paper, we usually cannot specify all of the decimal representation of an irrational number. Therefore, we usually use an approximate decimal to represent an irrational number and this approximate decimal is a rational number. Let $x = N.c_1c_2c_3\ldots$ be an infinite decimal where $N \in \mathbb{N}$ and $c_i \in \{0,1,2,\ldots,9\}$ for all $i \in \mathbb{N}$. We denote by $x(n)$ the n digits approximation of x, so $x(1) = N.c_1 = N + \frac{c_1}{10}$, $x(2) = N.c_1c_2 = N + \frac{c_1}{10} + \frac{c_2}{100}$,..., $x(n) = N.c_1c_2c_3\ldots c_n = N + \sum_{i=1}^{n} c_i \cdot 10^{-i}$. Obviously $\{x(1), x(2), \ldots, x(n), \ldots\}$ is a monotone bounded sequence, so the limit of this sequence exists and is in fact equal to x. This is connected to the content in the previous subsection and justifies the praxis block.

In this routine we use the polynomial $f(x) = x^2 - 2$ whose unique positive root is $\sqrt{2}$. Since f is increasing on $(0,\infty)$, and we have $f(1) = -1 < 0$ and $f(2) = 2 > 0$, it follows from the intermediate value theorem ($\omega_{\mathbb{R}}$) that $\sqrt{2}$ is located in $(1,2)$. Similar basic reasoning shows that $\sqrt{2}$ is in $(1.4, 1.5)$. Therefore, the first decimal of $\sqrt{2}$ is 4. The routine uses two loops to repeat this process until it finds all first 10 decimal digits of $\sqrt{2}$. First of all, the initial value (which is called K here) should be set to 1, the integer part of $\sqrt{2}$ according to the above.

In the routine, i is used to number the decimals that we aim to find, and this is the "external" loop. The number of the decimal digits to be produced can be modified by the users (in the assignment, we set it so as to find the first 10 decimal digits of $\sqrt{2}$). Considering that our focus is on the decimal part, we simplified the routine. Now, let us turn to the "inner" loop, where $j \in \{0, 1, 2, \ldots, 9\}$ and $j \cdot 10^{-i}$ represents a potential ith decimal contribution to the sum. When $\left(K + j \cdot 10^{-i}\right)^2 - 2 \leq 0$, the computer will save the last value $p = K + j \cdot 10^{-i}$ and continue to increment j until $\left(K + j \cdot 10^{-i}\right)^2 - 2 \geq 0$. After the *if*- condition is satisfied, the value of the sum K will be updated for the next i and the computer will print the value $x(i)$ which is equal to $p = K + j \cdot 10^{-i}$ (*Maple* by default gives a fraction form of p, so we have to use the *evalf*-command to transform p to a (finite) decimal form; no "rounding" is done here. The number of digits produced by the *evalf*-command includes the integer part, so we need to use $i + 1$). In general, for each i, the "inner" loop will produce a new K and an $x(i)$ (In order to only show $x(i)$ in the final result, we use ":" to make K invisible).

The main part of this routine is the "inner" loop (the "external" loop is mainly used to determine the position of the decimal digits). In this loop, we can get that for each $x(i)$, one has $x(i)^2 - 2 \leq 0$ and $(x(i) + 10^{-i})^2 - 2 \geq 0$. Therefore, by results known from $\omega_{\mathbb{R}}$, $\lim_{i \to \infty} x(i) = \sqrt{2}$. The proof of the limit is a part of what students could produce as an answer to question b) in the assignment. We also hope students could associate with the intermediate value theorem for continuous functions as they explain the working of this routine. A complementary visualized explanation from students to this routine is also asked for (question c)); indeed, using *Maple* to "visualize" difficult points forms part of what could reasonably be expected from teachers' relationship to instrumented techniques.

Addition of Infinite Decimals

The real numbers are not simply to be computed digit by digit; we also need to consider operations with real numbers, which leads to difficulties in the case of decimal representations. To add two integers or finite decimals numbers, the algorithm learnt in primary school is to add digits from the last position on the right (possibly "carrying over" exceeding digits). For some infinite decimals like irrational numbers, this way does not work. Can we add with approximate decimals of numbers? To be concrete, two irrational numbers, 0.1234 … and 0.8765 …, the sum of their first four decimal digits is 0.9999. If the sum of their 5th decimal digits is more than nine, the first four decimal digits of the new sum will turn out not be 9's but 0's. Some students may have become aware of this problem while studying decimals, or their teachers could have mentioned this in secondary school. The investigation about how students use the computer algorithms above to address this question from the new university level construction is also a part of our answer to the first part of RQ2.

We need to introduce some notation from the first part of the assignment. Let $\mathbb{D}_n = \{10^{-n}y : y \in \mathbb{Z}\}$ for $n \in \mathbb{N}$ and $\mathbb{D} = \cup_{n \in \mathbb{N}} \mathbb{D}_n$. We denote by $\mathbb{D}_\infty$ the set of formal expressions $\pm(N.c_1c_2c_3 \ldots)$ where $N \in \mathbb{N} \cup \{0\}$ and $c_i \in \{0, 1, 2, \ldots, 9\}$, and by $\mathbb{D}_0$ be the set of formal expressions $\pm(N.c_1c_2 \ldots c_j0000 \ldots)$ where $N \in \mathbb{N} \cup \{0\}$ and $c_1 \ldots c_j \in \{0, 1, 2, \ldots, 9\}$. Clearly we can interpret numbers in $\mathbb{D}_\infty \setminus \mathbb{D}_0$ as real numbers based on the above (the representation being furthermore unique). Let $x, y \in \mathbb{D}_\infty \setminus \mathbb{D}_0$. Through calculation (by computer or pen-and-paper), we can only get n digits approximation of x and y, denoted by $x(n)$ and $y(n)$ where $x(n), y(n) \in \mathbb{D}_n$. For each n we can form $z(n) = x(n) + y(n)$. When n goes to infinity, one has that $\lim_{n \to \infty} x(n) + \lim_{n \to \infty} y(n) = \lim_{n \to \infty} z(n)$ determines some number w in $\mathbb{D}_\infty \setminus \mathbb{D}_0$. But we have not, thereby, reduced addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$ to the addition on $\mathbb{D}$; but we cannot know if "carry overs" will lead to a failure of the following equation:

$$x(n) + y(n) = w(n), \text{ for any } n \in \mathbb{N}. \tag{3}$$

This quandary is not treated in any depth by Sultan and Artzt (2018). We hope this quandary would be discovered by students through working with the example in e) and f) of the assignment in **Appendix A**. This will then develop students' technical and theoretical knowledge related to the non-trivial addition of two infinite decimals.

In the question e) of the assignment in **Appendix A**, students were concretely asked to discuss the sum of $\sqrt{2}$ and $\sqrt{3}$. If we regard $\sqrt{2} + \sqrt{3}$ as one number, which is still an irrational number, the above routine can be adapted to produce the first 10 (or more) decimal digits of $\sqrt{2} + \sqrt{3}$. The main job for students is to find a polynomial with $\sqrt{2} + \sqrt{3}$ as root, and with integer coefficients (in order to remain with finite decimals in the algorithm). The simplest candidate, found by many students (following a technique known from exercises in the textbook) is $p(x) = x^4 - 10x^2 + 1$. Now, let $x = \sqrt{2}$, $y = \sqrt{3}$ and $w = \sqrt{2} + \sqrt{3}$. Students were expected to use the above routine to find $x(n)$, $y(n)$ and $w(n)$ for n from 1 to 10. When $n = 6$, we will get $x(6) + y(6) = 3.146263$ and $w(6) = 3.146264$, which means $x(6) + y(6) \neq w(6)$. This shows concretely that the equation (3) does not always hold, and that is the point which students are asked to make in question f).

It is clear that the use of computers to answer the last question greatly saves time and in practice enables getting to the point above. Students' didactical knowledge gained from this assignment will be analyzed later.

CONTEXT AND METHODOLOGY

We conducted an experiment to investigate the two research questions presented above in a course called UvMat, which is taught for future mathematics teachers in secondary school at the University of Copenhagen (Denmark). UvMat is not a mandatory course, and it involves 20-30 students each year, a professor who plans the course and gives the lectures, and a teaching assistant (TA) who is in charge of exercise classes and of grading assignments. In Denmark, high school teachers have to qualify in two subjects, one is called "major" and is studied for about three years, the other is the "minor" and is studied for about two years. Most participants in UvMat study mathematics as a minor, in addition to their major subject (such as physics) and will then be authorized to teach these subjects in high school. Due to the shortage of authorized teachers, some of them already have some experience with part-time high school mathematics teaching. The aim of this "capstone course" is to help future teachers to consider high school mathematics from an advanced standpoint, according to Klein's (2016) expression. This kind of "capstone course" thus aims to strengthen the connections between praxeologies met in high school and at university. Throughout the course, students attend lectures and exercise classes (in Danish), following the textbook (Sultan & Artzt, 2018), and they also do mandatory written assignments in groups (in Danish), every week.

The fourth week of UvMat is based on the second part of chapter 8 of the textbook, dealing with decimal representation of real numbers. We designed a group assignment (called WA4) with six questions for this week (the full text is included here as an appendix which the original version is in Danish). Our study is designed to investigate RQ2 by analyzing data collected from students' written answers to WA4 and the interviews with students. Our answer to RQ1 (see the previous section) outlines what students could learn from lecture and textbook and is part of the background for WA4.

The first question in RQ2 is analyzed from the logos blocks that were used by students to explain the given computer algorithm (question b) and c) in WA4 and their observations related to the addition of infinite decimals with the help of this computer algorithm [question f] in WA4). Eight groups' answers to WA4 from students were received. First, we reviewed their answers to question b) and c) which asked students to explain the given routine in two ways: mathematical proof, and visualization. Our analysis focused on two aspects: what university-level knowledge the students applied, and how students made use of this knowledge. Secondly, we reviewed students' answers to question f). The analysis of students' answers considered their extent to which they can apply the mathematical theory of the course to explain observations from using the routine on the given concrete case. In addition, we also interviewed five students from five different groups who had volunteered to participate. The purpose of the first part of the interviews is to further understand the written answers given by the students. For example, some groups post similar visualizations to question c), but without a clear description, so the rationales and intentions behind those visualizations could still differ. In the interview, they got another chance to explain what they want to express through the visualizations and how it is, for them, connected to the given routine. In addition, there were also some errors that appeared in students' written answers to question f), where we could not decide whether they are due to calculation mistakes or more conceptual problems, as many groups did not show how they get to their answers. So, in the beginning of the interview, students were asked to elaborate on their answers to question c) and f), which was used to complement our previous analysis for the first question of RQ2. The last part of the interviews relate to more general questions relating to the purpose of the assignment and its relation to the rest of the course.

UvMat does not aim to impart didactical knowledge, but we still expected that students could develop some didactically relevant knowledge from WA4. Question f) was designed not only to investigate the addition between two infinite decimals, but also to identify the theoretical perspectives that students could develop from exploring this question (the second question of RQ2). To investigate the perspectives and views of students emerging from their work with WA4, we asked relatively broad questions in the interviews such as: what did you learn from working with WA4? and what is the relation between WA4 and mathematics teaching in high school? Our analysis for the second question of RQ2 was based on their answers to such questions (and follow up questions) raised during the interview.

RESULTS

Students' Work With the Given Computer Algorithm

In question b), we asked students to solve the tasks: Explain what the routine does, why $x(n) \in \mathbb{D}_n$, and why $x(n) \to \sqrt{2}$. The main challenge of question b) is explaining the convergence of $x(n)$. Therefore, in students' answers, the university-level notions involved were "sequence" and "limit of sequences". However, students use these notions in similar and rather informal ways in their answers. We found that seven groups did not give a formal argument (as hoped for) but explained the convergence informally, like merely summarizing what they observed from the numbers produced by running the routine. Such empirical reasons are common in secondary school but are not acceptable in most undergraduate mathematics courses. A typical example of an explanation is this answer from one group:

> In this way, the routine finds, for each decimal added, the decimal number that is closest to $\sqrt{2}$ from below. We thus form a sequence of numbers $x(n)$, where n is the number of decimals, and $x(n)$ converges towards $\sqrt{2}$. That is $\lim_{n\to\infty} x(n) = \sqrt{2}$ (translated from Danish).

Only one group solved this task as we explained in the former section by using the definition of limit:

> We will now show that $\lim_{n\to\infty} x(n) = \sqrt{2}$. By construction, $x(n) \leq \sqrt{2}$ for all n, as in the code, we only add decimals as long as the number is smaller than $\sqrt{2}$. Furthermore, the decimal which is added in the nth decimal is the largest number for which the output remains under $\sqrt{2}$, so if you add 10^{-n}, you will get above $\sqrt{2}$. Therefore, $x(n) + 10^{-n} \geq \sqrt{2}$. If we put this together, we can get $x(n) \leq \sqrt{2} \leq x(n) + 10^{-n}$, which is the same as $0 \leq \sqrt{2} - x(n) \leq 10^{-n}$. That means $|\sqrt{2} - x(n)| \leq 10^{-n}$. So, for any $\varepsilon > 0$, we have $\left|\sqrt{2} - x(n)\right| < \varepsilon$, for all $n > log_{10}(\frac{1}{\varepsilon})$. So, for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\left|\sqrt{2} - x(n)\right| < \varepsilon$ when $n > N$. This means exactly $\lim_{n\to\infty} x(n) = \sqrt{2}$ (translated from Danish).

In general, students do have a strong tendency to prefer informal explanations based on secondary school mathematical conceptions even though the concept of sequence only appears at university in Denmark–not in high school. This situation can be attributed to students not putting themselves in the right position, as university students about to deepen their knowledge of high school mathematics. When they were facing what they perceive as a secondary school task, without being given specific directions for what to do, most students act as high school students s rather than preservice students σ_{ft} at university. They did not spontaneously consider that it is necessary or useful to draw on university level methods. In addition, this also reflects the lack of coherence, perceived by students, between logos blocks (from $\omega_{\mathbb{R}}$) and praxis blocks (from $o_{\mathbb{R}}$). Therefore, this type of exercise helps with the construction of $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$ and thus the transition (2).

We also asked students to explain the routine in another way (question c): use *Maple* to produce a visual explanation of how the routine works. Similar with question b), the pertinent university level knowledge in question c) is about sequences and continuity, and we focus on whether students' answers draw on such knowledge,. All groups used the point plot to show first 10 elements of the sequence $\{x(n)\}_{n=1}^{\infty}$, which can be done easily after running the routine. Students used this way to illustrate the convergence $x(n) \to \sqrt{2}$, as visualization of their explanations in question b). To explain this convergence, some groups combined the point plot with a function plot for $x \mapsto x^2 - 2$. Another common way is to superpose the point plot with a plot of the line $y = \sqrt{2}$ where we can see the 10 points are closer and closer to this line. One group superposed the graph of the function $y = x^2 - 2$ and the ten points $\big(x(1), y(x(1))\big), \ldots, \big(x(10), y(x(10))\big)$, showing how they get closer and closer to the zero ($\sqrt{2}$, 0) of y. Even though this group had plotted the function $y = x^2 - 2$ on the interval (0, 1.42), after the fourth point it is no longer possible to see the changes between points. This problem was also indicated in the group's answer. In fact, this problem also occurred on figures with the added line $y = \sqrt{2}$. One of the groups solved this problem with three local zooms of the figure.

The visualizations above emphasize the sequence produced by the routine, rather than how the routine actually works (the two loops) and why (intermediate value theorem). Two of the groups made up for this by also presenting another kind of point plot (e.g., **Figure 1**), to explain how the "inner" loop works. **Figure 1** shows how the "inner" loop determines $x(1)$, by stopping as it "overshoots" and returning the previous value. In fact, **Figure 1** only compared $(K + j * 10^{-i})$ with $\sqrt{2}$ when $i = 1$, which simplified from the *if-command* in the routine (which compares, in fact, $\left(K + j * 10^{-i}\right)^2 - 2$ with 0). The "inner" loop stops when it finds some j such that $\left(K + j * 10^{-i}\right)^2 - 2 > 0$ and that is why in **Figure 1**, the sequence stops at $j = 5$.

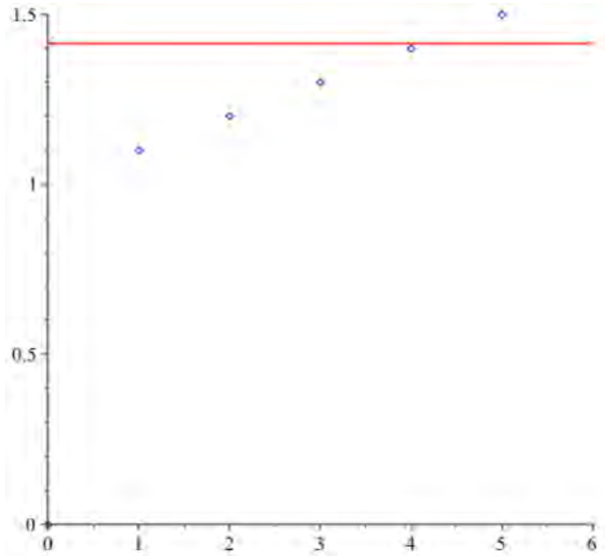

Figure 1. The point plot that shows how the "inner" loop of the routine determines $x(1)$ (Source: Author's own elaboration)

Although we think most visualizations fail to substantially address "how the routine works", students were quite confident about their work, when asked to explain it during interviews. They believed the figures match their explanation of the routine. One student described their figure in the interview as: *"… it made good sense…"* Many groups also, verbally, explained details in the coding of the routine, like why K starts from 1, but this was not visualized. From the explanations both in their answers and interviews, students thought the routine is used to create the sequence and thus, the convergence is, to them, the most important aspect of the routine. Therefore, the point plot which shows the subsequence of $\{x(n)\}_{n=1}^{\infty}$ appears sufficient for students. Still, we think that merely showing a visualization of the outcome or product of the routine does not really correspond to the normal understanding of "visualizing how the routine works" (the process that produces the outcome or product).

In addition to the students' capacity of reading the routine, question c) also requires students to use simple codes from *Maple* by themselves. Students' competence to use computer tools influenced their creation of the visualizations. In the interviews, all participants were satisfied with the content of their figures, but they still thought the figure itself could be improved. One interviewee said his group considered to use an animation to show the "inner" loop and the final subsequence, but they failed to realize that idea. Though all students had some experience with *Maple* before UvMat, the interviewees still recognized that they spent more time on tedious details of how to make a figure than on how to understand the routine. One student described the main difficulty in answering question c) as related to names of colors in *Maple*:

> "… I was not sure which colors *Maple* has. That was something I could not find easily, and I was not sure how to program the colors …"

Questions d) and e) serve to extend students' work with the routine, and to prepare question f). The first half of question d) prompts students to activate their knowledge of the intermediate value theorem, which clearly belongs to university mathematics. This theorem, in fact, guarantees the success of the routine. The second part of question d) and e) asked students to use the routine to find approximate decimals of $\sqrt{3}$ and $\sqrt{2}+\sqrt{3}$ respectively. All groups were able to modify the routine to get correct answers. Question e) also examined students' knowledge of polynomial which typically presents at university. We will use the results obtained in question d) and e) by students and analyze their answers for question f) in the next subsection.

Students' Work on the Addition of Infinite Decimals

The last question of the assignment is: f) investigate what the results from b), d), and e) tell you about addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$. This is a very open question. It does not require new applications of the routine, but rather to reflect on previous results. As explained before, the point of this question is that addition and approximation to finite decimal do not commute: with $x=\sqrt{2}$, $y=\sqrt{3}$ and $w=\sqrt{2}+\sqrt{3}$, we may have $x(n)+y(n)\neq w(n)$. In fact, students computed $x(1)$ to $x(10)$ using the given routine in b), $y(1)$ to $y(10)$ in d) and $w(1)$ to $w(10)$ in e). Comparing, they could notice that $x(6)+y(6)\neq w(6)$.

To explain this phenomenon, the students can combine knowledge from $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$. On the one hand, we know from practice with calculators that something which ought to be 0 sometimes end up as something which is not zero. On the other hand, the theoretical model of real numbers as infinite decimals does not offer an easy way to describe the basic operations like addition: we cannot add "from the right" and adding from the left may lead to errors (as in the case above).

Six groups pointed out the "unusual" addition. Three of them showed the comparison of $x(n)+y(n)$ and $w(n)$ on *Maple* and found the special case when $n=6$ as we expected (e.g., **Figure 2**). However, the other three groups did not show any examples, but they all pointed out that two infinite decimals cannot be added decimal by decimal. We interviewed two students from these groups. One of them was not happy with this answer because they did not see what the question required from them. According to this student they compared the data obtained from the previous questions both by hand and *Maple*, and noticed $x(6)+y(6)\neq w(6)$, but then struggled to explain the phenomenon, considering if it might be related to the possibility of equivalent decimal representations which terminate in infinite numbers of 9's or 0's.

Fra b) har vi følgen $(x_n)\to\sqrt{2}$, fra d) har vi følgen $(y_n)\to\sqrt{3}$, og fra e) har vi følgen $(z_n)\to\sqrt{2}+\sqrt{3}$. Her er de fundende følger.

$x_1 = 1.4$	$y_1 = 1.7$	$z_1 = 3.1$
$x_2 = 1.41$	$y_2 = 1.73$	$z_2 = 3.14$
$x_3 = 1.414$	$y_3 = 1.732$	$z_3 = 3.146$
$x_4 = 1.4142$	$y_4 = 1.7320$	$z_4 = 3.1462$
$x_5 = 1.41421$	$y_5 = 1.73205$	$z_5 = 3.14626$
$x_6 = 1.414213$	$y_6 = 1.732050$	$z_6 = 3.146264$
$x_7 = 1.4142135$	$y_7 = 1.7320508$	$z_7 = 3.1462643$
$x_8 = 1.41421356$	$y_8 = 1.73205080$	$z_8 = 3.14626436$
$x_9 = 1.414213562$	$y_9 = 1.732050807$	$z_9 = 3.146264369$
$x_{10} = 1.4142135623$	$y_{10} = 1.7320508075$	$z_{10} = 3.1462643699$

Men bemærk at $x_6+y_6=3.146263\neq 3.146264=z_6$, og at vi har lignende uoverensstemmelser for senere n. Vi har altså at $(x_n+y_n)\neq(z_n)$

Figure 2. An example from one group that how to find the "unusual" addition by *Maple* (reprinted with permission of the students)

In fact, this concern is irrelevant because the question was restricted to the set $\mathbb{D}_\infty \setminus \mathbb{D}_0$, which avoids decimals with nothing but 0's in the end. The other student was very satisfied with the given answer and thought there was no need to put more effort into question f) because the conclusion they gave was obvious. In this group they did not compare the results from previous questions because they believed cases like $x(6)+y(6)\neq w(6)$ were not a surprise. As the student from this group said

> "… it makes sense, but we did not think we have to do it …"

In addition to the above-mentioned six groups, there were also two groups that produced entirely misleading answers. One of the groups presented their conclusion after comparing $x(n)+y(n)$ and $w(n)$ thus:

After looking at the different decimal numbers, it is the same whether you first take the different decimal expansions separately or you take it together …" (translated from Danish).

One possible reason for this answer is that students did not carefully compare the data. Another group did not pay attention to the comparison between $x(n) + y(n)$ and $w(n)$. They gave the following answer:

"… If we combine the three observations, these examples indicate that addition in $\mathbb{D}_\infty \setminus \mathbb{D}_0$ is a closed operation, since the sum of two elements from $\mathbb{D}_\infty \setminus \mathbb{D}_0$ is again an element in $\mathbb{D}_\infty \setminus \mathbb{D}_0$. This is consistent with that $\mathbb{D}_\infty \setminus \mathbb{D}_0$ is isomorphic with $\mathbb{R}$ (question a)), which is closed during addition …" (translated from Danish).

We can see this group was trying to answer question f) at a theoretical level using ideas (about properties of operations) learnt at university, but there are two problems in their answer. Firstly, all results from the routine are finite decimals which means the addition between these decimals happened in $\mathbb{D}_n$. Students could not get any results about addition in $\mathbb{D}_\infty \setminus \mathbb{D}_0$ directly from such observations but they could consider the "cut off" map $\varphi: \mathbb{D}_\infty \setminus \mathbb{D}_0 \to \mathbb{D}_n$ given by $\varphi(x) = x(n)$. Then the question would turn into looking at whether $\varphi(x + y)$ equal to $\varphi(x) + \varphi(y)$. Obviously, from the same case ($n = 6$), the equation does not always hold, so that (in university language) φ is not a homomorphism. Secondly, the operation on $\mathbb{R}$ can be transferred to $\mathbb{D}_\infty \setminus \mathbb{D}_0$ because there exists a bijection between $\mathbb{D}_\infty \setminus \mathbb{D}_0$ and $\mathbb{R}$ (proved in question a)) so that, trivially $\mathbb{D}_\infty \setminus \mathbb{D}_0$ is isomorphic with $\mathbb{R}$ when endowed with this addition. But the results to be considered for question f) suggest that students should think about whether the addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$ could be defined directly using decimals. This, however, is far from straightforward, as the examples show. From the interview of one student of this group, we know that they did not really reflect on this difficulty. The way they thought about this question falls short of university level standards and also lacks links to praxis block of secondary school mathematics (as we will now detail).

Students' Mathematical and Didactical Knowledge From the Assignment

Finally, how could the knowledge have developed from students' work on this assignment support teaching–as integrated mathematical and didactical knowledge? One of the purposes of this assignment is to help students think about how computers and calculators handle infinite decimals. We all understand that infinite decimals cannot be completely displayed by a computer, so the computer has to somehow convert infinite decimals into finite decimals. In particular, irrational numbers are handled as a special kind of rational numbers. This transformation could cause computers to make apparent errors when they operate on irrational numbers (e.g., question f). Indeed, secondary school teachers in Denmark need to manage pupils' use of CAS in relation to real numbers. How can a teacher deal with infinite decimals on CAS when they teach real numbers? The routine given in the assignment allows students to take a look into a possible procedure for directly calculating the first 10 decimal digits of $\sqrt{2}$ (and, in fact, a wide range of zeros of other given functions). Although we do not consider the actual algorithms behind commands such as "sqrt" (square root) or "solve", the routine opens up the "black box" to some extent. This could help future teachers reflect on how computers may more generally handle real numbers. In addition, the discussion about addition on infinite decimals could also help teachers understand why this sometimes leads to strange-looking results.

However, not all students saw the didactical relevance of WA4 as we would like them to. When we asked students about the relation between this assignment and secondary school teaching during interviews, only one student's response directly involved the representation of infinite decimals on computers and suggested that secondary school teachers

"should be able to figure out when *Maple* is good to use and when it is not."

Whereas other interviewees gave a neutral answer like

"… no matter what you teach, it is always a good thing to know more than you actually need …"

or claimed that this assignment is beyond the secondary mathematics level,

"… the curriculum of high school students is very far from this …"

Students did not mention question c) about visualizing the algorithm when asked to identify the main points of the assignment. We consider this question may have been too open or technically demanding to really contribute to their relation of type $R_U\left(\sigma_{ft}, \overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}\right)$.

Students' responses in the interview did not reflect much awareness about the didactical implications of the assignment. It would require further research to evaluate if and how working with such interface tasks could nevertheless leave an impact on students' subsequent teaching practice.

DISCUSSION

How can a course like UvMat support the development of secondary school mathematics? More specifically, to what extent can achieved new relationships $R_U(\sigma, \overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}})$ improve their future teaching? Our data does not shed light on this, but the relation between teachers' knowledge and school teaching is not a new topic. For example, Hill et al. (2005) found teachers' mathematical knowledge (close to the mathematics they teach) has a significant effect on their students' achievement at primary level in USA. From a more global point, the teacher education and development study in mathematics (TEDS-M) conducted surveys about teacher education within 17 countries, which included future teachers at secondary level (Krainer et al., 2015). Schmidt et al. (2011) reexamined the data from 2010 TEDS-M and focused on middle school teachers' course taking. Their results indicate that high performing teacher education programs include both general undergraduate mathematics courses and also courses which, like UvMat, focus more specifically on mathematics for teaching. But these studies all consider mathematical knowledge in very broad categories, and it is a completely open question how work with specific school mathematical themes like the real number system in practice contribute to teaching related to those themes.

CONCLUSION

Our brief analysis of the students' answers and interviews with them, suggests that problem situations involving simple CAS-routines could be a promising setting for applying university mathematics on high school level problems related to real numbers, but that this does not necessarily develop new, didactically relevant knowledge. Indeed, we found that when we do not clearly indicate the direction of problem solving, some students do not draw properly on university knowledge, but resort to informal or misguided explanations rooted in their high school experience. Further research is needed to explore how problems could be designed in order to reduce or eliminate this kind of regression.

To summarize our answer to RQ1, the main idea is to explore the definition of real numbers as 'infinite decimals' in terms of infinite, convergent sums of fractions. This turns out to be a challenging approach for students, in particular when confronted with special cases where these sums are computed step by step using a computer routine, and do not add up as expected. Indeed, our observations in relation to RQ2 suggest (as developed in more detail in previous sections) that students struggle to explain the subtle difficulties that arise with addition and, more generally, with the operations on real numbers in this approach. Nevertheless, this approach is highly relevant to understand the shortcomings of computers and calculators when it comes to numerical computations with real numbers.

Indeed, with the increasing use of digital tools in high school mathematics, it becomes problematic if teachers have no idea of the connections–and differences–between theoretical mathematics (in particular, real numbers and their operations) and the representations and operations which such tools offer. We should also note that the first answers by students, considered here, were not the end of their work with the assignment: incorrect or incomplete answers had to be reworked in order for the assignment to be accepted. Confronting students with inadequacies or insufficiencies in their initial answers, and prompting them to submit acceptable ones, certainly leads the students to realize how advanced mathematical viewpoints can be used to think about subtleties in what appears, initially, to be elementary and somewhat trivial. To make students realize the need to question and analyze outputs from digital tools–as well as to look into how they are or may be produced–is one important goal which assignments, such as the one considered, may contribute to achieve.

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REFERENCES

Barquero, B., & Winsløw, C. (2022). *Preservice secondary school teachers revisiting real numbers: a striking instance of Klein's second discontinuity* [Manuscript submitted for publication]. University of Copenhagen.

Bergé, A. (2010). Student's perception of completeness property of the set of real numbers. *International Journal of Mathematical Education in Science and Technology, 41*(2), 217-227. https://doi.org/10.1080/00207390903399638

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71-114.

Durand-Guerrier, V. (2016). Conceptualization of the continuum, an educational challenge for undergraduate students. *International Journal of Research in Undergraduate Mathematics Education, 2*, 338-361. https://doi.org/10.1007/s40753-016-0033-2

Forsythe, G. E. (1959). The role of numerical analysis in an undergraduate program. *The American Mathematical Monthly, 66*(8), 651-662. https://doi.org/10.2307/2309339

González, M. F., Kuzniak, A., Delgadillo, E., & Vivier, L. (2019). Two situations for working key properties of R. In U. T. Jankvist, M. Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education* (pp.2492-2493). ERME.

González-Martín, A. S., Giraldo, V., & Souto, A. M. (2013). The introduction of real numbers in secondary education: An institutional analysis of textbooks. *Research in Mathematics Education, 15*(3), 230-248. https://doi.org/10.1080/14794802.2013.803778

Gyöngyösi, E., Solovej, J. P., & Winsløw, C. (2011). Using CAS based work to ease the transition from calculus to real analysis. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Congress of the European Society for Research in Mathematics Education* (pp. 2002-2011). ERME.

Hill, H. C., Rowan, B., & Ball, D.L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406. https://doi.org/10.3102/00028312042002371

Klein, F. (2016). *Elementary mathematics from a higher standpoint: Volume I: Arithmetic, algebra, analysis* (G. Schubring, Trans.). Springer. https://doi.org/10.1007/978-3-662-49439-4

Krainer, K., Hsieh, F. J., Peck, R., & Tatto, M. (2015). The TEDS-M: Important issues, results and questions. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 99-121). Springer. https://doi.org/10.1007/978-3-319-12688-3_10

Lagrange, J. B. (2005). Using symbolic calculators to study mathematics: The case of tasks and techniques. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 113-135). Springer. https://doi.org/10.1007/0-387-23435-7_6

Schmidt, W. H., Houang, R., & Cogan, L. S. (2011). Preparing future math teachers. *Science, 332*(6035), 1266-1267. https://doi/10.1126/science.1193855

Sultan, A., & Artzt, A. F. (2018). *The mathematics that every secondary school math teacher needs to know*. Routledge. https://doi.org/10.4324/9781315391908

Winsløw, C. (2011). Anthropological theory of didactic phenomena: Some examples and principles of its use in the study of mathematics education. In M. Bosch, J. Gascón, A. Ruiz-Olarría, M. Artaud, A. Bronner, Y. Chevallard, G. Cirade, C. Ladage, & M. Larguier (Eds.), *Un panorama de la TAD* [*An overview of the TAD*] (pp. 117-140). Centre de Recerca Matemàtica [Center for Mathematical Research].

Winsløw, C. (2013). The transition from university to high school and the case of exponential functions. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of 8th Congress of the European Society for Research in Mathematics Education* (pp. 2476-2485). ERME.

Winsløw, C., & Grønbæk, N. (2014). Klein's double discontinuity revisited: Contemporary challenges for universities preparing teachers to teach calculus. *Recherches en Didactique des Mathématiques* [*Research in Didactics of Mathematics*]*, 34*(1), 59-86.

Zazkis, R., & Sirotic, N. (2004). Making sense of irrational numbers: Focusing on representation. In M. Høines, & A. B. Fugelstaed (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (pp. 497-504). PME.

APPENDIX A: WEEK ASSIGNMENT 4

For $n \in \mathbb{N}$ we define $\mathbb{D}_n = \{10^{-n}x : x \in \mathbb{Z}\}$, and we define $\mathbb{D} = \cup_{n\in\mathbb{N}}\mathbb{D}_n$. Also define $\mathbb{D}_\infty$ to be the set of formal expressions $\pm N.c_1c_2...$ where $N \in \mathbb{N} \cup \{0\}$ and $c_k \in \{0, 1, ..., 9\}$ for all $k \in \mathbb{N}$, and finally let $\mathbb{D}_0$ be the set of formal expressions $\pm N.c_1c_2...c_k000...$ where $N \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$ and $c_1, ..., c_k \in \{0, 1, ..., 9\}$.

a) Prove that there exists bijections $\varphi : \mathbb{D}_0 \to \mathbb{D}$ and $\psi : \mathbb{D}_\infty \setminus \mathbb{D}_0 \to \mathbb{R}$, but that no bijection exists between $\mathbb{R}$ and $\mathbb{D}$.

b) Consider the following routine in Maple (try it out!):

```
K := 1 :;
for i from 0 to 10 do
      for j from 0 to 9 do
           if (K + j * 10^(-i))^2-2 <= 0 then
                p := K + j*10^(-i);
           end if ;
end do;
K := p :;
print(x(i) = evalf(p, i + 1));
end do :;
```

Explain what the routine does, why $x(n) \in \mathbb{D}_n$, and why $x(n) \to \sqrt{2}$.

c) Use Maple to produce a visual explanation of how the routine from b) works.

d) Explain how a similar routine can be made for any continuous function f, to find a zero between $a \in \mathbb{Z}$ and $a + 1$, when $f(a)f(a + 1) < 0$. How does the intermediate value theorem come into play? How can you use this idea to approximate $\sqrt{3}$ by numbers from $\mathbb{D}$?

e) Find a polynomial p such that $p(\sqrt{2} + \sqrt{3}) = 0$, and use the idea from d) to approximate $\sqrt{2} + \sqrt{3}$ by numbers from $\mathbb{D}$.

f) Investigate what the results from b), d) and e) tell you about addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$.

(Source: Author)