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Examining Algebraic Habits of the Mind through a Problem Solving: Elementary School Example

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Abstract
In this study, it is aimed to determine the algebraic thinking habits of two eighth grade school students through the answers they gave in the process of solving mathematical problems. The algebraic habits of mind (ZCA) theoretical framework developed by Driscoll (1999) was used to reveal these thinking habits. The research design of this study is a case study and the participants consist of two eighth grade students. The data were analyzed using Driscoll (1999)'s ZCA framework, which is algebraic habits of mind. Descriptive analysis was used in the analysis of the data. When we look at the findings obtained from the research; It is seen that both students can do describing a rule and justifying a rule in the solutions of the problems. In addition, it is seen that computational shortcuts, equivalent expressions and symbolic expressions come to the fore in the solutions of students' problems. On the other hand, the habit of undoing in solving problems was not encountered very rarely in both students. In the light of the findings obtained, the reasons for the existing and non-existent algebraic habits of mind are discussed. As a result of this discussion, it is thought that it is effective to include guide questions to create and develop algebraic thinking habits in students in classroom teaching practices of teachers.

Keywords: Algebraic Thinking, Algebraic Habits of Mind, Elementary Students, Problem Solving

1. Introduction

1.1 Introduce the Problem

Algebraic thinking is a system of thought that includes the use of many mathematical skills (mathematical modelling, reasoning, justification, use of different representations, etc.) (Kaf, 2007). This way of thinking offers the individual a perspective that enables the construction, structuring and use of an algebraic concept, that is, the ability to think mathematically (Hawker & Cowley, 1997; Eroğlu ve Tanışlı, 2017; Sezer & Altun, 2020). In gaining this perspective; Booker & Windsor (2010) first, individuals are experienced with concrete number values, then generalizations through problem solving and finally moving towards abstract thinking come (Vygotsky, 1997; Mason, 2008; Booker & Windsor, 2010; Radford & Sabena, 2015).

The beginning and completion of this process are formally formed in classroom teaching practices. As a matter of fact, when we look at the literature, there are studies showing that it is important to provide students with algebraic thinking skills and to develop them in their mathematics teaching (Herbert & Brown, 1997; Magiera, Van den Kieboom & Moyer, 2013; Magiera, Moyer & Van den Kieboom, 2017). For this reason, it has been a matter of curiosity about the practices that teachers have made in their classroom teaching for their students to gain algebraic thinking skills. Among these applications, it was determined that problem solving is effective in the development of mathematical thinking skills (Herbert & Brown, 1997; Driscoll, 1999; Kaf, 2007; Bilgiç & Argün, 2018). As a matter of fact, when we look at problem solving, it is seen that it constitutes the largest part of the education curriculum by the American National Mathematics Association (National Council of Teachers of Mathematics [NCTM], 2000). Problem solving enables students to make choices about which operations to use and to make some decisions regarding the application steps by using the information contained in the problem (Montague, Applegate & Marquard, 1993). Therefore, as in all academic skills, cognitive and metacognitive strategies and operations are used to perform such operations in order to solve mathematical problems. On the other hand, it has been supported by studies that problem solving plays an important role in the development of mathematics learning and mathematical thinking (Kaput, 2008; Kaput, Blanton & Moreno, 2008; Schliemann, Carraher & Brizuela, 2007; Lins, Rojano, Bell & Sutherland, 2001). In problem solving, it is a process that first requires the determination and analysis of the characteristics of the problem, then the design of a plan for the problem in the problem, and finally, the implementation of the plan (Polya, 1973). Problem solving, which requires a mental process and includes cognitive and metacognitive strategies (Pourdavood, McCarthy & McCafferty, 2020; Polotskaia, Fellus, Cavalcante & Savard, 2022) to fulfill the steps, emerges as an important tool in forming, developing and determining the algebraic thinking habits of the mind.

In the light of the information given above, it is the aim of this research to determine whether students have algebraic thinking habits or not through problem solving. In line with the purpose of the research, the answer to the following question was sought:
- What are the algebraic thinking habits that two 8th grade students use while solving algebra problems?

2. Theoretical Framework

Driscoll (1999) states that making mathematical generalizations, seeing mathematical relationships and interacting with them improve students' mathematical thinking skills. As a matter of fact, one of the tools used while providing this development is the problem solving method. In this context, Schoenfeld (1992) states that students develop a deeper understanding of mathematics while solving mathematical problems, and as a result, it is effective in their conceptualization of the learned mathematics. In addition, Booker & Windsor (2010) stated that problem solving is a powerful way to learn algebraic thinking of students. Thus, students' involvement in mathematical experiences that go beyond routine arithmetic problems creates opportunities for them to develop their algebraic thinking (Silver, Ghousseini, Gosen, Charalambous & Strawhun, 2005, p. 288).

For the last two decades, problems have been created to support the development of algebraic habits of mind (Driscoll, 1999; Kieran & Chalouh, 1993; Lee, 2002; Mason, 1987, 1989; Swafford & Langrall 2000; Eroğlu & Tanışlı, 2017; Caesar, 2020). The problems created should be of a quality that can enable students to think algebraically in order to effectively progress the problem solving stages in classroom teaching practices (Windsor, 2010). This has revealed the necessity for teachers to ask guiding questions about the given problems in order to activate students' algebraic thinking (Blanton & Kaput, 2011; Moss & McNab, 2011; Britt & Irwine, 2011). Blanton & Kaput (2011) stated that the teacher's guide questions should be in a way that directs students from arithmetic operations to algebraic actions. This situation simultaneously develops students' thinking habits about algorithmic and functional processes (Driscoll, 1999). In the light of the aforementioned, it has become an important issue what the thinking habits of the students are that activate their algebraic thinking and what their characteristics are. In the literature, the characteristics of the mind's thinking habits have not been determined to be completely specific. However, there are frameworks to develop students' algebraic thinking. What these frameworks are and the indicators with explanations for the frameworks are shown in Table 1.

Table 1: Theoretical frameworks for algebraic thinking in the literature

Literature	Explanation
Hart et al., (1998)	According to the findings of a project carried out in England by "Concepts in Secondary Mathematics and Science" to reveal students' understanding of algebraic expressions, the development of students' understanding of algebraic expressions can be examined in four consecutive stages.
Kaput (1998)	It presented four sub-themes of algebraic thinking and presented a theoretical framework for analysing students' algebraic thinking abilities. The theoretical framework consists of "generalized arithmetic, functional thinking, modelling, algebraic abstraction and algebraic proof" sub-themes.
"Algebraic Habits of Mind" (ZCA) Driscoll (1999)	In the study of Driscoll (1999), ZCA created three main themes: "doing-undoing, building rules to represent functions and abstraction from computation".
Chimanoi, Pitta-Pattazi & Christou (2018)	They created a theoretical framework in which algebraic thinking can be defined more consistently in 4 dimensions: reasoning styles, processes, concepts and topics.

Looking at the frames in Table 1, it is seen that there is a spiral extending from arithmetic operations to abstraction skills. Within the framework of Driscoll (1999), it also offers guidance questions to teachers that will improve students' algebraic thinking. Thus, it differs from other theoretical frameworks in the literature in terms of revealing students' algebraic thinking. In addition, Driscoll (1999) algebraic habits of mind theoretical framework were used in this study, since it was aimed to reveal what the algebraic thinking habits of two 8th grade students were.

2.1. Algebraic Habits of Mind

Driscoll (1999) conceptualized the Building Rules to Represent Functions and Abstracting from Computation as habits of mind, which are under the umbrella of the Doing-Undoing habit of algebraic thought (Figure 1).

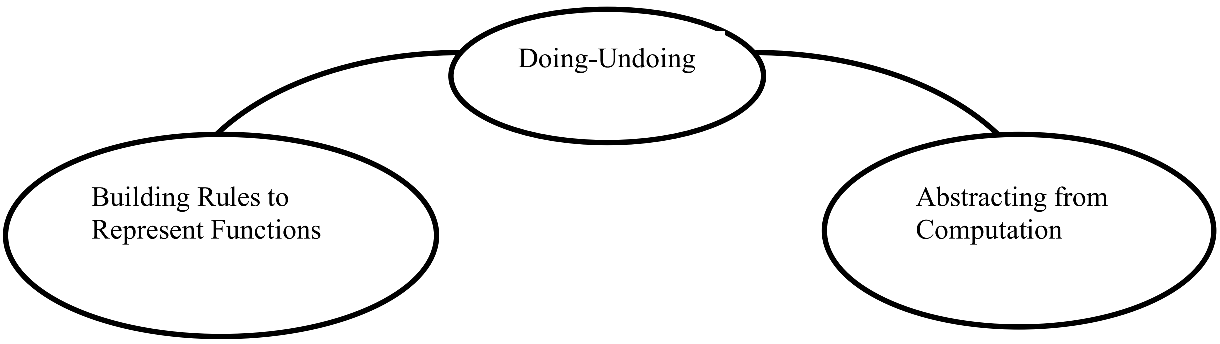


Figure 1: Algebraic Habits of Mind

Doing-Undoing: This algebraic habit of mind acts as a framework for the other two habits. Students should be able to conclude an operation related to algebra and reach the starting point by working backwards from the result of an operation for which they found the result. Thanks to this mental habit, students do not only focus on reaching the result, but also think about the process (Driscoll, 1999). For example, if $4x^2 - 9 = 7$, they should be able to find the solution of this equation as well as form the equation whose roots are $x = 2$ and $x = -2$.

Building Rules to Represent Functions: It is important to recognize patterns from algebraic thinking processes and to generate data to represent these situations. This mental habit involves some thought processes in middle school algebra. These processes are; recognizing and analyzing patterns, investigating and representing relationships, making generalizations beyond specific examples, analyzing how processes or relationships have changed, looking for evidence of how and why rules and procedures work (Magiera, Van den Kieboom & Moyer, 2013). .

Abstracting from Computation: It is the capacity to think about calculations regardless of the numbers used. Abstraction is important for this habit of mind. Abstraction is the process of extracting mathematical objects and relations based on generalization (Lew, 2004).

Thus, in this study, it is aimed to examine the algebraic thoughts of two 8th grade students through problems within the framework of Driscoll's (1999) algebraic habits of mind. In line with this purpose, it is aimed that the teacher will bring out the algebraic habits of the mind given above through algebraic problems.

3. Method

3.1 Research Design

The method of this research is case study. A case study is a qualitative research approach in which the researcher examines one or more limited cases over time with data collection tools (observations, interviews, audio-visuals, documents, reports) that includes multiple sources, and defines situations and situation-related themes (Creswell, 2007). Accordingly, it would not be wrong to claim that the case study is an appropriate method to understand what the algebraic thinking habits of 8th grade students are. As a matter of fact, the case study is the best approach in terms of this research as it is a clear definition of the phenomenon in the study and an effective research method (Dörnyei, 2007) in generating new hypotheses, models and understandings about this phenomenon. Within the scope of the study, two 8th grade students were asked to express aloud their answers, routes and justifications for the given problems.

3.2 Participants

The present study was conducted with two 8th grade students attending a public school in Gaziantep, Turkey. Purposive sampling was chosen as the sample. Purposeful sampling is expressed as a method that will contribute to the in-depth study of selected individuals with the aim of identifying similar situations within the sample group (Creswell, 2007). The academic achievements of the students in the mathematics course are equivalent, and both students are female.

3.3 Data Collection Process

One of the techniques used in the evaluation of cognitive and metacognitive thinking is a-think aloud protocols. AThink-aloud protocols are a method of data collection in which participants verbally state everything they think and do during activities such as reading a given passage or solving a mathematical problem (Ostad & Sorenson, 2007; Rosenzweig, Krawec, & Montague). , 2011; Sweeney, 2010; Veenman & Spaans, 2005). The data obtained through a-think aloud protocol is usually audio recorded or video-recorded, and then the data is transcribed (Van Hout-Wolters, 2000). The data in the transcripts are analyzed and evaluated in line with cognitive and metacognitive thinking (Rosenzweig et al., 2011; Sweeney, 2010). Since the implementation of the a-think aloud protocol takes place in an applied activity such as reading a text or solving a mathematical problem, it is stated as the most important advantage of obtaining detailed information about the mental processes of the participants (Van Hout-Wolters, 2000). In the context of this view, data were collected through a-think aloud protocol within the scope of this study.

3.4. Data Collection Tool

As a data collection tool in the research, 2 problems related to pattern and equations under the algebra learning field and developed by Özdemir (2019) were used. With the first problem, it is aimed to examine the algebraic thinking habits of the participants related to pattern acquisition in the field of algebra learning. It is important to examine the algebraic thinking habits of the students about the pattern, since the concept of pattern forms the basis for the transition from arithmetic to algebra in the pre-algebra period. For this reason, one of the problems that we will examine the algebraic habits of the mind of the students has been selected within the scope of pattern acquisition. With the second problem, it is aimed to examine the algebraic thinking habits of the participants related to the acquisition of equations in the field of algebra learning. Since solving equations is a concept encountered in all learning areas of mathematics, it is important to examine students' algebraic thinking habits about equations. For this reason, one of the problems that we will examine the algebraic habits of the mind of the students has been selected within the scope of equation acquisition.

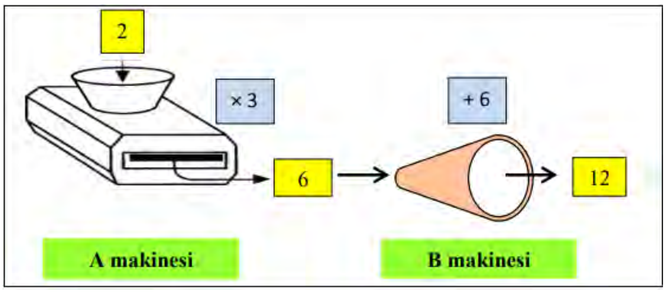
Problem-1: Toothpicks Problem

In the game they play, Ali and Ayşe try to build houses next to each other with toothpicks of equal size. Below are the houses that Ali and Ayşe built while playing this game. According to this,

a) Calculate the number of toothpicks needed for 9 houses.
b) If the number of toothpicks required for 42 houses is 169, how many toothpicks are required for 43 houses?
c) Express algebraically the relationship between the number of houses and the number of toothpicks.
d) Ali and Ayşe want to build houses with a different geometrical shape, again side by side, instead of "pentagon" shaped houses. Identify a different geometric shape to help Ayşe and Ali. Express algebraically the relationship between the number of toothpicks and the number of houses for the houses made of the geometry you have determined.

Problem-2: Equation Machine Problem

As A and B are two mathematical equation machines, when a number is put into machine A, the machine changes this number by multiplying it by 3. Next, the number from machine A enters machine B, and machine B modifies this number by adding 6 to it. Below is an example where the number 2 is inserted into machine A.



As seen in the example, machine A multiplied the inserted number by 3 and the number coming out of machine A became 6. Later this number entered machine B. Machine B changed the number by adding 6, and the number subtracted from machine B as 12. Answer the following questions according to the working principles of A and B machines.
a) If the number 4 is placed in machine A, what will be the number that will enter machine B?
b) When the numbers 7 and 12 are placed in machine A, respectively, what are the numbers coming out of machine B?
c) When the number x is put into machine A, the number coming out of machine B becomes y. Accordingly, express the number y in terms of x.
d) Determine another working rule for machines A and B so that the equality between the number entering machine A and the number leaving machine B does not change.

3.5. Data Analysis

The data obtained from the research were analyzed descriptively. In the descriptive analysis, the data are summarized and interpreted in terms of predetermined themes according to the research questions. In descriptive analysis, direct quotations are frequently used in order to strikingly reflect the discourses of the interviewees. The purpose of this analysis is to present the findings to the reader in a regular and blended manner (Yıldırım & Şimşek, 2011). In this context, the data recorded on video through a-think aloud protocol were analyzed using the theoretical framework of the study, Algebraic Habits of Mind, which was included in the study of Driscoll (1999). In the context of this framework, "Doing-Undoing, Building Rules to Represent Functions and Abstracting from Computation" by Driscoll, Zawojeski, Humez, Nikula, Goldsmith, Hammerman (2003), has been categorized according to the characteristics of the algebraic habits of the mind. The categories and features of the framework are shown in Table 2.

Table 2: Feature of Algebraic Habits of the Mind (Driscoll et. al., 2003)

Categori	Feature Name	Description
Doing-Undoing	Input from output	Finding input from output, or initial conditions from a solution
	Working backward	Working the steps of a rule or procedure backward
Building Rules to Represent Functions	Organizing information	Organizing information in ways useful for uncovering patterns and the rules that define the patterns
	Predicting patterns	Noticing a rule at work and trying to predict how it works
	Chunking the information	Looking for repeating chunks in information that reveal how a pattern works
	Different representations	Wondering what different information about a situation or problem may be given by different representations, then trying the different representations
	Describing a rule	Describing the steps of a rule without using specific inputs
	Describing change	Describing change in a process or relationship
	Justifying a rule	Justifying why a rule works for "any number"
Abstracting from Computation	Computational shortcuts	Looking for shortcuts in computation, based on an understanding of how operations work
	Calculating without computing	Thinking about calculations independently of the particular numbers used
	Generalizing beyond examples	Going beyond a few examples to create generalized expressions, describe sets of numbers, or either state or conjecture the conditions under which particular mathematical statements are valid
	Equivalent expressions	Recognizing equivalence between expressions
	Symbolic expressions	Expressing generalizations about operations symbolically
	Justifying shortcuts	Using generalizations about operations to justify computational shortcuts

A transcript analysis of Driscoll (1999) evaluated within the scope of algebraic habits of mind is as follows:

Table 3: Example Transcript Analysis

Dialogue of Helin and Esra's Algebraic Thinking Processes in the Equation Machine Problem	Analysis
1H: Helin is reading part b of the problem. Let's put the first number 7 in machine A. Then it comes out of machine A as ***7x3=21***. Then, machine B exits from machine B as ***21+6=27, since machine B enters the number 21. Then if the number 12 enters machine A, 12x3=36, 36+6=42.*** 2T: When you enter 7, you say 26 comes out, when you enter 12, you say 42 subtracts. Esra, did Helin's action make sense? 3E: Yes. 4T: So when 100 enters machine A, how many will exit machine B?	Analysis of Helin's Algebraic Thinking Process; 1st line; The expression "7x3=21" contains the definition of the ***equivalent expressions (EE-D)*** code. After the expression, applying the rule of the equation machine to find the output for the number of inputs 12, multiplying by 3 and then adding 6 includes the definition of the ***computational shortcuts (CS-D)*** code.

5H: ***When 100 enters machine A, my teacher (expresses it on the board and then expresses it as 306) 306.*** 6Ö: So when you enter the number n, what is the output? 7H: (Writes on the board) ***3n+6*** 8O: What is n? 9H: n is a number. 10Ö: So, it is the number we put in machine A. So what is the result? 11E: It is the number that comes out of machine B.	5th line; It contains the definition of ***generalizing beyond examples (GBE-D)*** code to reach generalization according to the condition of the number entering the equation machine being 100. In addition, expressing the answer for 100 as "306" as a result of mental operations includes the definition of the ***calculating without computating (CWC-D)*** code. 7th line; Expressing the rule as 3n+6 for the number of n entries includes the definition of the ***describing a rule (DaR-D)*** code.

In the context of the reliability of the analysis of the data, two researchers independently analyzed the written student dialogues in the context of the algebraic habits of mind. According to the dimensions in the framework used in the descriptive analysis, the answers to the analyzes made by the researchers were compared and the questions with "Agreement" and "Disagreement" were determined. If the researchers expressed the same dimension in the relevant section, it was accepted as consensus, if they marked different options, it was accepted as disagreement. The reliability of the study was carried out using the formula and the average reliability was calculated. The "Percentage of Concordance" for this study was found to be 87% on average for algebraic habits of mind. This rate is considered reliable (Miles & Huberman, 1994). Although the rate obtained was accepted as reliable, the researchers came together and discussed the points of disagreement until they reached a consensus. In this way, the reliability of data analysis is increased.

4. Results

In line with the research problems determined in the research, "What are the algebraic thinking habits that two 8th grade students use while solving algebra problems?" The descriptive analyzes of students' a-thinking aloud protocol applications for each problem according to Driscoll et al (2003) are included in the following headings.

4.1 Algebraic thinking habits of students about the toothpick problem

The answers given by the students Helin and Esra to the toothpick problem and their algebraic thinking habits were observed as follows:
1H: How many toothpicks are needed for 9 houses, since ***5 toothpicks are needed for each house, 9x5=45.*** **CS-D**
2R: Are you sure about this answer? Esra, do you think your friend thinks right?
3E: It is as if an edge becomes common when combining. That's why we need to get 4.
4R: Can you show me? **DC-D**
5E: So it goes like this. While forming ***the first house and the second house in their conjunction, one side becomes common. From this we need to divide by 4.***
6H: But it's 5 toothpicks for a house.
7R: Let's think about it this way. How many houses are here? (Teacher shows the drawn house on the board)
8H: 3 of them.
9R: There are 3 houses. How many toothpicks were used in total for 3 houses?
10H: (counting toothpicks) 13
11E: But normally, according to Helin's calculation, there should have been 15. But 2 toothpicks are missing.
12H: Should we multiply by 3 then?
13R: How many houses did we create with 13 toothpicks?
14E: Then ***let's divide 13 by 3***. **CS-D**
15R: All right, let's split it up, Esra.
16E: (Dividing) For 3 houses, then we multiply by 4. **1 more remains. Let's add that too.** **PP-D**

17R: Let's examine it now. Chapter 4 and the remainder 1. How many toothpicks did Esra use for the second house?
18E: (Counting) 4 of them.
19P: For the 3rd house?
20E: 4 of them.
21R: For the 1st house?
22E: 5 of them.
23R: What does this situation mean to you? (No answer) How many toothpicks do we need to build 4 houses?
24E: 17.
25R: Let's write that there are 17 toothpicks for 4 houses.
26E: (Writes the number of toothpicks on the board)
27R: How many toothpicks do we need for 5 houses?
28E: ***We add 4 over 17, 21***. **CS-D**
29R: 21 for 5 houses.
30E: (Writing on the board)
31R: Does what Esra wrote make sense to you?
32H: No, it doesn't.
33R: Does Esra make sense to you?
34E: Yes.
35R: Well, can you tell Helin?
36E: I think so. When we use 5 toothpicks in the 1st house, ***we add 4 toothpicks since one side is common*** when creating the second house. **JS-D**
37R: So after we build the first house, we add 4 toothpicks since there is a common edge in other houses. Is not it? Then we add 4 toothpicks at each stage. (Students approve)
38R: So how many toothpicks do we use for 100 houses?
39E:***401.***
40R: Did you use a rule for this?
41E: ***Increases by 4 each time. That's why I multiplied by 4 and then added 1***. **JR-D**
42R: Helin 43 How many toothpicks do we use for the house?
43H: ***Then I multiply 43 by 4 and add 1. Result 173.*** **CS-D**
44R: Let's move on to the next question.
45H: Helin reads the question "Express the relationship between the number of houses and the number of toothpicks algebraically".
46E and H: Both say ***4n+1***. **DaR-D/SE-D**
47R: How did you find the rule?
48E: ***Since it increases by 4 each time, we need to multiply the number of houses by 4 and add 1.*** **JR-D**
Helin approves of Esra.
49E: writes 4x+1 on the board.
50R: So what is this x?
51H: x is the number of houses. The result we get is the number of toothpicks.
Esra also approves of Helin.
52R: So, is this rule valid for every number?
53H: Yes.
54E: Valid for all houses from the 2nd house.
55H: No. It is valid in the 1st house. ***Let's substitute 1 for x, we get 5.*** **CS-D**
56R: Well, this rule is valid for every number.
Students say yes.
57R: How many houses would we build if we had 37 toothpicks?
58E: This one is different from the others. It becomes ***4x+1=37. From here we find 9 houses.*** **EE-D/CWC-D**
59Ö: Helin How many houses would we build with 117 toothpicks?
60H: ***4x+1=117 (doing the operations), we find 29 houses.*** **EE-D/CWC-D**
61R: So guys, does this rule work?
62E and H: Students say yes.
63E: Here we find the general term.

64R: So, what is this concept about?
65E and H: Students express as patterns.
66R: So you found a pattern. In particular, you actually found a series. So what kind of series is this?
67E and H: Students express it as an arithmetic sequence.
68T: So how do we show what we have done in the form of a table?
69H: Draws a rectangle on the board and splits it in half. It is written as the house number in the first column. Write the number of toothpicks in the second column. Then it refers to the toothpick numbers for the 1st, 2nd and 3rd houses.
70R: So how do we express it for what?
71H: ***4n+1*.** **DaR- D**
72E: ***Exactly. It will be in the form of 4n+1.*** **DaR- D**
73R: Well, can we show it another way?
74E: ***Draws graphs. It expresses the number of houses on the x-axis and the number of toothpicks on the y-axis. Writes numbers on the axes and connects the dots to show them as a line graph.*** **DR- D**
75R: Okay, let's look at our last question.
76H: Helin "Ali and Ayşe want to build houses with a different geometric shape, again side by side, instead of houses in the shape of a pentagon. Identify a different geometric shape to help Ayşe and Ali. Express the rule between the number of toothpicks and the number of houses for houses made of the geometric shape you have determined".
77R: What do you think about this question?
78E: Since the geometric shape will change, the rule changes.
79R: Helin, can you tell me a geometric figure?
80H: Triangle.
81R: Okay, friends, let's create triangular houses with toothpicks.
82H: Helin creates triangular houses on the board.
83R: Well, Esra, create a house from a geometric shape.
84E: Esra builds square houses.
85R: Well, how many toothpicks do you need for Helin, 3 houses?
86H: ***The rule for this is 2x+1*.** **DaR- D/SE-D**
87R: How did you find that?
88H: ***Madam, when constructing houses from pentagons, it has 5 sides, but we found the rule as 4x+1. In this one, if we substitute 1 for x in the rule to create 1 house, there are 3 toothpicks. He finds the toothpick numbers for the 2nd and 3rd houses.*** **JR- D**
89R: Can we make a table about it?
90H: ***Draws a rectangle on the board and splits it in half. It is written in the first column as the House number. It is written as the number of toothpicks in the second column. Then it refers to the toothpick numbers for the 1st, 2nd and 3rd houses. Writes the expression 2n+1 for n houses.*** **DR- D**
91R: Well, let's look at Esra square houses.
92E: Esra, our ***general term here is 3x+1. When we put it in its place, 4 for 1 house, 7 for 2 houses, 10 for 3 houses, …*** **DaR- D/SE-D**
93R: It's beautiful. So what is the relationship between the number of houses and the number of toothpicks?
94E: ***We take 1 less the number of sides of the geometric shape of our house. Then we add 1.*** **DC- D**
95H: No.
96E: ***We take 1 minus the number of sides as a coefficient.*** **JR- D**
97R: So beautiful…

Table 4: Findings Concerning Algebraic Thinking Habits of Students Regarding Toothpick Problem

		Doing (D)		Undoing (UD)	
Categori	**Participants Features Name**	**Esra (frequency)**	**Helin (frequency)**	**Esra (frequency)**	**Helin (frequency)**
	Organizing Information (OI)	1			

Building Rules to Represent Functions	Predicting Patterns (PP)				
	Chunking the Information (CtI)	1			
	Different Representations (DR)		1		
	Describing a Rule (DaR)	3	3		
	Describing Change (DC)	2			
	Justifying a Rule (JR)	3	1		
Abstracting from Computation	Computational Shortcuts (CS)	2	2		
	Calculating Without Computing (CWC)	1	1		
	Generalizing Beyond Examples (GBE)				
	Equivalent Expressions (EE)	1	1	1	1
	Symbolic Expressions (SE)	1	1		
	Justifying Shortcuts (JS)	1			

When Table 4 is examined, a total of 26 codes were obtained for the doing habits of both students in the answers given by the students to the problems, while only 2 codes were obtained for the undoing habit. In the context of building rules to represent functions habit, 10 codes from Esra and 5 codes from Helin were obtained. Similarly, the category named describing a rule stands out in both students. In the context of Abstracting from computation habit, 7 codes were obtained from Esra and 6 codes from Helin. Thus, when the answers they gave to the problems in the context of building rules to represent functions and abstracting from computation habits are examined, it can be said that Esra has more habits than Helin.

4.2 Algebraic Habits of Students' Mind on the Equation Machine Problem

The answers given by Helin and Esra to the equation problem and their algebraic thinking habits were observed as follows:

OI-D

1H: Helin is reading part a of the problem. ***If the number 4 is entered in machine A, the number that will be entered into machine B will be 12, since machine A has tripled it. (He expresses the operation "4x3=12" by writing on the board.)***

CS-D

2R: Esra, did Helin's action make sense?
3E: Yes. It made sense.
4R: Let's look at the other question, Helin.
5H: Helin is reading part b of the problem. Let's put the first number 7 in machine A. Then it comes out of machine A as ***7x3=21***. Then, machine B exits from machine B as ***21+6=27, since machine B enters the number 21. Then if the number 12 enters machine A, 12x3=36, 36+6=42.*** **CS-D/EE-D**
6R: When you enter 7, you say 26 comes out, when you enter 12, you say 42 subtracts. Esra, did Helin's action make sense?
7E: Yes.
8R: So when 100 enters machine A, how many will exit machine B? **CWC-D/GBE-D**
9H: ***When 100 enters machine A, my teacher (expresses it on the board and then expresses it as 306) 306***.
10R: Well, when the number n is entered, what is the output?
11H: (Writes on the board) ***3n+6*** **DaR-D**
12R: What is n?
13H: n is a number.

14R: So, it is the number we put in machine A. So what is the result?
15E: It is the number that comes out of machine B.
16R: Well, what number do we need to put in machine A to get 42 from machine B?
17H: ***We solve the equation 3n+6=42. 3n = 36. So the number that comes out of the number A is 36. When we divide each by 3, we get n=12.*** **EE-D/CS-UD**
18R: What is 12? Is it entering or exiting machine A?
19H: 12 is the inbound.
20R: Esra, did Helin's action make sense?
21E: Yes.
22H: Helin is reading part d of the problem.
23R: What is the relationship between the number entering machine A and the number leaving?
24H: ***Equation is 3n+6.*** **SE-D**
25R: What does 3n+6 mean?
26H: Sir, then we will do it as 3(n+2).
27R: Then tell me the rule, what should we say in the rule?
28H: ***In the rule, we say that 2 more than n is 3 times.*** **DaR-D D**
29R: How can we express this in terms of input and output from the machine?
30H: The number n is the number entering A. The result of this is the number from B.
31R: Well, can you describe it as in the question?
32H: Yes, sir. Multiply the number entering machine A by 3 and then adding 6 to get the number coming out of machine B.
33R: Esra, can you say the 3rd (n+2) equation we created above in this way?
34E: ***If we say n to the number entering machine A, I first add it by 2 and then multiply it by 3.*** **DaR-D**
35R: Then we will multiply the resulting number by 3 in machine B?
36E: Yes.
37R: Did Helin make sense?
38H: Yes.
39R: Well, can you find a different working principle? Of course he will keep 3n+6. If I multiply both sides of this expression by 2, both sides of the equation will not break, right?
40H: Helin writes the expression 2x(3n+6)=2x(3n+6) on the board.
41R: What is 3n+6?
42H: is the input.
43R: Let's call this expression 2x(3n+6) the number coming out of machine B. What kind of expression is that as if it's a function isn't it?
44E: Esra shakes her head.
45R: Normally it was 3n+6. This expression was equal to the number that came out of machine B. Can I give a letter to this expression even if I don't call it "the number coming out of machine B"?
46E: Esra shakes her head.
47H: ***Let's say y*** **SE-D**
48R: Now, if I multiply both sides of the equation by 2 in the expression 3n+6=y, will the equality be broken?
49H: Helin writes the expression ***2.(3n+6)=2.y.*** **EE-D**
50R: Well, can you distribute 2?
51H: 6n+12=2y.
52R: Well, can you express this verbally? What was our?
53E: It was the number that went into machine A.
54R: Well, what is y? How can we express this verbally?
55H: ***12 more than 6 times the number entering machine A is equal to 2 times*** the output number. **DaR-D**
56R: So how can we say that? Let's say the number that goes into machine A...
57E: I'll multiply by 6. I will add 12 to the resulting number. When I divide the resulting number by 2...
58R: We will have found the result, right?
59E: yes.
60R: Well, friends (showing the board) how can we show what we're doing here differently?
61H: We showed it here (pointing to the board).
62R: Different from this question, can you give a different representation of the rule you found?

63H: I can ***draw a table***. (draws a table on the board) Draws a rectangle on the board and divides it in half. Writes in the first column as the number entering machine A. It writes in the second column as the number from machine B. Then it becomes 9 if he enters 1, 12 if he enters 2, and 15 if he enters 3. For the number n, it becomes 3n+6. **DR-D**
64R: Did Esra make sense?
65E: Yes.
66R: Well, nice..

Table 5: Findings on Algebraic Thinking Habits of Students Regarding the Equation Machine Problem

		Doing (D)		**Undoing (UD)**	
Categori	Participants Features Name	Esra (frequency)	Helin (frequency)	Esra (frequency)	Helin (frequency)
Building Rules to Represent Functions	Organizing Information (OI)		1		
	Predicting Patterns (PP)				
	Chunking the Information (CtI)				
	Different Representations (DR)		1		
	Describing a Rule (DaR)	1	2		
	Describing Change (DC)				
	Justifying a Rule (JR)				
Abstracting from Computation	Computational Shortcuts (CS)		4		
	Calculating Without Computing (CWC)		1		
	Generalizing Beyond Examples (GBE)		1		
	Equivalent Expressions (EE)		4		
	Symbolic Expressions (SE)		1		
	Justifying Shortcuts (JS)				

When Table 5 is examined, a total of 16 codes were obtained for the doing habits of both students in the answers given by the students to the problems, while the code for the undoing habit could not be obtained. In the context of the building rules to represent functions habit, 1 code from Esra and 4 codes from Helin were obtained. In the context of Abstracting from computation habit, 11 codes were obtained from Helin, while it could not be coded from Esra. Thus, when the answers given to the problems in the context of building rules to represent functions and abstracting from computation habits are examined, it can be said that Helin has more habits than Esra.

5. Discussion and Conclusion

When examined in the context of Doing-Undoing habit, the problem solutions of both students were found very rarely. However, in the literature, Kieran, Pang, Schifter & Fong Ng (2016) stated that the habit of undoing in the Singapore education system is one of the thinking processes emphasized by the curriculum. For this reason, it can be said that problems and guidance questions should be included in the course process to help students gain this habit. In addition, it is possible to say that the acquisitions aimed at developing this habit should be increased in the mathematics curriculum.

When examined in the context of the habit of building rules to represent functions, it is seen that both students have a similar number of codes. Max & Welder (2020) concluded that the features in the subcategories of algebraic habits of the mind are interrelated. He stated that one of the related sub-categories is the describing change with predicting patterns. Similarly, in our study, it is seen that there is no finding about two characteristics of a student named Esra. Thus, it can be said that the predicting patterns in the habit of building rules to represent functions and the describing change features are similar to the findings obtained. In addition, it is seen that the habit of building rules to represent functions is frequently encountered when the answers given by the students to both problems regarding describing a rule are examined. Similarly, Magiera, Kieboom and Moyer (2017) and Max & Welder (2020) stated that pre-service teachers and students did not have any difficulties in describing a rule and determining the problems made in the context of the habit of building rules to represent functions. However, it is similar to the findings of the study, which they found insufficient to confirm and justify the rule.

When analyzed in the context of Abstracting from computation habit, when the answers given by both students to the problems were examined, the feature of computational shortcuts was frequently encountered. On the other hand, there was less finding about justifying shortcuts made by students. Similarly, Max & Welder (2020) justifying shortcuts stated that they should make generalizations about the operations performed to verify computational shortcuts. For this reason, in studies in the literature (Simon & Blume, 1996; Herbst & Chazan, 2011; Cai, Morris, Hohensee, Hwang, Robison, Cirillo, Kramer, Hiebert & Bakker; 2020) similar to the findings. In addition, when the answers given by the students to the problems in the context of the calculating without computing feature, which is included in the abstracting from computation habit, are examined, there is very little finding about calculating without computing. Rubestein (2001) stated that students' use of calculating without computing feature facilitates their learning in many important structural subjects. In this respect, it is important to include guidance questions in the teaching process for the development of students' calculating without computing feature.

This research is limited to examining the problem solutions of only two 8th grade students regarding pattern and equation outcomes. In future studies, the use of guidance questions by teachers in teaching processes can be examined in the development of algebraic thinking habits of students at different grade levels. In addition, in-service training can be given to mathematics teachers on the use of guidance questions in teaching processes. In the teaching practices to be carried out to develop the algebraic habits of the mind of the students, the lesson introduction problems should be carefully prepared. Thus, the problems covered in the course should serve both to develop the algebraic habits of the mind of the students and to teach the relevant outcome.

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Notes

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