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TEACHING GEOMETRY THROUGH DIDACTICAL SITUATIONS: THE CASE OF THE TRIANGLE INEQUALITY

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Abstract: This paper aims to discuss the use of Brousseau's Theory of Didactical Situations in mathematics (TDS) for eighth-graders to explore the concept of triangle inequality in Euclidean geometry. Data sources included observation notes, video recordings of lessons, and students' written work. Data analysis was done through a deductive content analysis approach that utilized the conceptual framework based on Brousseau's notion of didactical situations. Findings revealed that student behaviors that were expected to take place through situations occurred at every phase as stated in the theory. Mathematical ideas leading up to the construction of new knowledge were gradually formulated and justified as stages progress. Students worked out different methods of evaluation to solve an open-ended exploratory task and defended them in a way that invited other students to implement their chosen strategies. They developed their implicit informal knowledge by building on their thinking and the thinking of others through situations. All these results highlight how important it is to use didactical situations to pave the way for learning to learn, as it not only facilitates the purposeful exchange of ideas through whole-class discussion that ensures a common understanding of mathematical ideas but also allows students to create their own learning adventures.

Keywords: theory of didactical situations in mathematics, a-didactical situations, triangle inequality

1. Introduction

The issue of creating appropriate learning environments in which students have a leading role in their own learning becomes more and more important in today's education. Since learning mathematics is considered both a process of active individual construction of knowledge and a process of acculturation (Cobb, Wood & Yackel, 1991; Cobb, 1999), modern approaches to mathematics teaching have drawn attention to social constructivist paradigms that allow meaningful learning in the planned teaching environments. A social constructivist orientation requires learners to share and develop mathematical meaning by taking into account their existing knowledge, abilities, and feelings (Bishop, 1985; Ernest, 1994). With the social constructivist approach to teaching, the expectations from all major stakeholders that make up the education system have also been rearranged. While the student becomes the main actor in her learning process, the teacher is expected to be the moderator who guides this process. In this sense, much more emphasis is placed on learners taking responsibility for socially constructing knowledge through discourse and interaction related to challenging problems (National Council of Teachers of Mathematics, 2014). The trend towards constructivism has also led to a new research focus in mathematics education that brings with it various educational theories. One of these theories is the Theory of Didactical Situations in mathematics (hereafter denoted as TDS) (Brousseau, 1997), which ensures the rigorous construction of mathematical knowledge in a particular teaching and learning situation designed and utilized to provide students with opportunities to achieve a better understanding of mathematical concepts.

TDS is mainly influenced by Piaget's theorization of cognitive development as a constructive adaptation process (Brousseau & Warfield, 1999). Constructivism underlines that knowledge is not transferred from one person to another but is actively constructed by the learner (Kilpatrick, 1987). It is the person herself who has all power and authority over the process of constructing knowledge (Lerman, 1989). Like the constructivist approach, the theory of didactical situations also involves the construction of

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knowledge by the learner rather than just passively receiving and memorizing it in the learning context what is called the milieu, which represents the whole environment and circumstances interacting with the learner. The emphasis of TDS is the realization of meaningful learning by bringing together all components necessary for the individual to learn. The theory focuses on solving the structure of the complex relationship between students and the teacher by identifying their fundamental roles and responsibilities, and defining the milieu and the conditions that this milieu should have. Learners have to work autonomously without relying on the teacher's guidance, which is defined as an a-didactical situation. According to TDS, every concept arises from a problem situation, and knowledge is constructed with independent adaptation to a milieu conceptualized through an a-didactical situation (Brousseau, 1997). Hence, the theory of didactical situations in mathematics aims to better understand, interpret and improve teaching by providing in-depth explanations about what happens in the learning process of mathematical concepts. Since many students find it difficult to learn some important concepts in geometry, we think it is essential to elaborate on the teaching situations with the help of the theory of didactical situations that can provide opportunities for students to explore and reflect on these concepts.

A triangle is one of the basic geometric concepts that students are exposed to quite early in the curriculum and they make use of it when creating other geometric shapes and studying their properties and areas (Fey, 1982). Triangles are of great importance for students of different age levels to learn other topics in geometry, but they also have difficulties in learning the concept of triangles (Damarin, 1981; Gutierrez & Jaime, 1999; Vinner & Hershkowitz, 1980). Although students recognize a triangle has three sides and three corners, they do not fully realize the relationships among the lengths of its sides (Damarin, 1981). However, they need to be aware of the fact that three random line segments do not always form a triangle and that there are some relationships between the lengths of the sides to draw a triangle. It is therefore important to provide appropriate learning situations for students to make enough sense of the relationships between side lengths of a triangle, i.e., the triangle inequality. These learning situations should allow students to question and construct the relevant concept in geometry by active participation and synthesis of the target mathematical knowledge through critical thinking and communication skills (Clements & Battista, 1992). One way a teacher achieves this is to design a situation that includes both a problem whose optimal solution requires the concept in question and a milieu as in TDS (Hersant & Perrin-Glorian, 2005). The fundamental argument of the TDS is that "every mathematical concept is the solution of at least one specific system of mathematical conditions, which itself can be interpreted by at least one situation" (Brousseau & Warfield, 2020, p. 209). Accordingly, the present paper aims to analyze the use of the theory of didactical situations in mathematics (TDS) for teaching eighth-grade students the concept of triangle inequality in Euclidean geometry. The following research question has guided the study: How is the exploration of triangle inequality performed by eighth-grade students in a milieu designed according to TDS?

2. Background and literature review

2.1. Triangle inequality

The triangle inequality principle states that the length of any side of a triangle is less than the sum of the lengths of the other two sides and greater than the absolute value of the difference between these lengths (Wallace & West, 2015), which can be proven in the following. Let's take a point S on the extension of the line segment PQ outside of a triangle PQR such that $\overline{SQ} = \overline{QR}$, and draw the line segment SR to form a triangle PSR. Since the triangle QRS becomes an isosceles triangle, $m\angle QRS$ and $m\angle QSR$ are equal to each other, and $m\angle PRS$ becomes greater than $m\angle QRS$= $m\angle QSR$. Besides, $\overline{PQ} + \overline{QS} = \overline{PS}$ and $\overline{QR} = \overline{QS}$, then $\overline{PQ} + \overline{QR} = \overline{PS}$. On the other hand, if one angle in a triangle is larger than another angle, then the side opposite the larger angle becomes longer than the side opposite the smaller angle, and thus in the triangle PSR, $\overline{PQ} + \overline{QS}$ must be greater than $\overline{PR}$. Since $\overline{QR} = \overline{QS}$, then $\overline{PR} < \overline{PQ} + \overline{QR}$. Similarly, $\overline{PQ} + \overline{PR}$ and $\overline{QR} + \overline{PR}$ are greater than $\overline{QR}$ and $\overline{PQ}$, respectively. Thus, $\overline{PR}$ is greater than both $\overline{QR} - \overline{PQ}$ and $\overline{PQ} - \overline{QR}$, that is, $|\overline{PQ} - \overline{QR}| < \overline{PR}$.

2.2. An overview of the main features of the theory of didactical situations in mathematics

This study is set within the perspective of Guy Brousseau's Theory of Didactical Situations in mathematics. The methodological principle of TDS assumes the application of the target mathematical knowledge in a situation that preserves meaning, i.e., the mathematical knowledge at stake appears in a sense as an optimal solution to the given problem (Mangiante-Orsola, Perrin-Glorian & Strømskag, 2018). During the 1990s, Brousseau emphasized the importance of the notion of the milieu in the theory and insisted that a particular piece of mathematical knowledge intended to be taught can be modeled by didactical situations in which interactions between students, the teacher, and the milieu occur (Miyakawa & Winsløw, 2009). This milieu provides feedback on students' own actions and it comprises everything that affects learning in the situation such as students' prior knowledge, other students, materials or symbolic objects, and classroom arrangement. In order to deal with the given problem, the student needs to take action on the milieu, make hypotheses, validate or reject them, develop strategies as if trying to win a game, and evaluate the feedback from the milieu. Here, the term 'game' is used as a metaphor for the actions of teachers and students. That is to say, the students interact with a problem and 'win the game' if they solve it, and (ii) the teacher takes the initiative and makes the regulations and evaluations about the students' game in situations (Bahn & Winsløw, 2019).

According to the Theory of Didactical Situations, learning is seen as the adaptation of students' knowledge to the milieu, but unlike the assumptions in Piagetian theory, the milieu is to be created and managed by a teacher to encourage learning. The function of a milieu depends on the situation and its characteristics. A situation may indeed be either didactical where the teacher interacts with the student and the subsystem of the milieu, or a-didactical where the teacher does not interact. A-didactical situations are provided especially in the context of the game. The rules of the game and the game-winning strategies of the opposing one or team planned in interaction with the milieu create these a-didactical situations that give feedback to the others at each stage. Unlike didactical situations in which the teacher's purpose of teaching and teaching content are evident, a-didactical situations refer to situations in which the teacher conceals her/his purpose and goals from the students even for a certain period and allow students to engage in actions independently of the teacher's intervention and guidance (Hersant & Perrin-Glorian, 2005).

A-didactical situations work ideally when the teacher steps back to let students make their own discoveries to create significant learning experiences. However, the teacher is still in charge of ensuring that the students take responsibility for solving the problem, which is called the devolution process. In other words, a-didactical situations are not simply autonomous work of students on a mathematical problem, rather they are devolved by the teacher in a didactical situation referred to as a situation of devolution. Effective realization of the responsibility transfer is possible with a good understanding of the mutual expectations between the teacher and the students in a given situation, which is called a didactic contract (Brousseau, Sarrazy & Novotna, 2020). This contract regulates the interaction of students and teachers in the situation as well as establishes their respective responsibilities concerning the mathematical knowledge at stake. The didactic contract is not necessarily stable throughout the teaching and learning process, but it can evolve and change with the progress in knowledge and expectations.

A-didactical situations basically occur in three different ways: (i) situations of action, (ii) situations of formulation; (iii) situations of validation (Brousseau, 1997). The first a-didactical situation to be experienced after delegating the responsibility to students concerning new knowledge in didactical situations is the situation of action in which the milieu is completely devoid of intention to instruct. With the help of the objective milieu, whose role is to provide feedback on students' actions, the students try to understand the problem. During the process of action, the feedback from the milieu should be taken into account by the learners and strategies should be guided accordingly. Positive feedback approves students' actions while negative feedback requires students to revise the actions taken. The next a-didactical situation is the situation of formulation that allows students to construct hypotheses explaining how to solve the problem. Since the possible strategies that emerge in the process of action are not in the status of a mathematical proposition, that is, they are not expressed as rules, the situation of formulation is a process in which students begin to formulate and develop hypotheses or claims about the problem, usually based on the shared experience in the process of action. The a-didactical situation that comes upon completion of the formulation is the situation of validation in which the explicit verification or refutation of formulations produced by the students is attempted. In the process of validation, the teacher serves as a chair in theoretical debates and only intervenes to organize the debates, attracts students' attention to potential inconsistencies, and motivates them to incorporate more precise mathematical principles to justify their hypotheses (Brousseau, 1997).

These three a-didactical situations are followed by another didactical situation called the situation of institutionalization in which the teacher encourages students to relate target mathematical knowledge gained in the setting set up by the teacher to existing and established knowledge so that it can be used in other settings. Thus, contextualized knowledge acquires the status of cultural knowledge to be decontextualized by having the characteristics of law rather than just an answer to a mathematical problem (Brousseau, 1997). Therefore, the model of a didactical situation includes a-didactical situations and a didactic contract. The teacher plays an important role during the devolution and institutionalization which are two important phases for regulating the didactic contract. While devolution ensures the conditions for adaptation, institutionalization ensures the conditions for acculturation (Artigue, Haspekian & Corblin-Lenfant, 2014).

3. Methodology

3.1. Research design

Qualitative research, in an exploratory vein, is usually preferred for illuminating, elucidating, or interpreting an event or phenomenon in its natural context to generate an in-depth and multi-faceted understanding of a complex issue under investigation (Creswell & Creswell, 2017). This research study lends itself well to the use of a qualitative exploratory case study (Yin, 2014), where the case of interest is the learning of the triangle inequality principle in a unique and dynamic context within the theory of didactical situations, which entailed uncovering and reporting the complex interactions occurring between three important elements: students, teacher, and milieu.

3.2. Participants

The study was conducted with a total of sixteen (eight males, eight females) eighth-grade students from one public school located in a suburban area of Zonguldak. The participants of the study were volunteered based on both convenience and criterion sampling techniques of the purposive (or purposeful) sampling method in qualitative research (Patton, 2014). The fact that students did not have any specific knowledge, skills, or expertise about the target learning outcome of the eighth-grade mathematics curriculum phrased as "relating the sum or difference of the lengths of two sides of a triangle with the length of the third side" (Ministry of National Education, 2018, p. 74) was used as a criterion for selecting participants. On the other hand, considering the fifth, sixth, and seventh-grade mathematics curriculum outcomes, it was supposed that the necessary prerequisite knowledge was available for participants to achieve this specific learning outcome. Besides, since one of the authors of this study assumed the role of teacher-researcher, researchers found it more convenient to recruit participants from the school where this author worked as a mathematics teacher in order to easily overcome the problems that might arise during the research process.

The students participating in this study worked in groups of four for two lessons. Research shows that any student in groups of three or four can take a more active role in the group by being involved in the whole process, working better, and being productive (Davidson, 1990). Students working in collaboration are capable of solving more explorative tasks than students working on their own (Roschelle & Teasley, 1995). Group work presents important opportunities for students to exhibit their different knowledge, skills, and abilities in learning settings and to realize the social interaction necessary for learning (Webel, 2013). Therefore, the group work for this study is more appropriate, as it provides an environment in which students construct their mathematical knowledge. In other words, the best way for students to reflect and share their thoughts on learning mathematics is in a social setting where group work is practiced. This reflection and sharing offer important opportunities for the students to rearrange the previous concepts they have and the new concepts they have created in their minds (Slavin, 1991). Moreover, students in the group can better help each other if they are mixed in terms of their abilities. One of the effective ways of forming a mixed group is to be formed by teachers who know the cognitive and affective characteristics of their students well (Baki, 2018). In this sense, participants of this study were assigned into four groups of four students each by their teacher (who is also a researcher), named Group A (two males, two females), Group B (two males, two females), Group C (two males, two females), and Group D (two males, two females). The participants in each group were coded according to the group name to which they belonged. For example, students of Group A were coded as Student A1, Student A2, Student A3, and Student A4. Particular attention was also paid to ensuring that the groups were as homogeneous as possible among themselves in terms of both students' average mathematics achievement from the 5th to 7th grades and gender distributions. The average mathematics achievement scores for each group range from 70.33 to 71.47 out of 100 (M = 70.92, SD = 0.49).

3.3. Data collection

The data gathered in this study included participant observation of two lessons lasting 40 minutes each during the implementation of the didactical situation designed to have students engage in an open-ended exploratory task called 'Fragrant Strawberries'. Students are expected to discover the triangle inequality with this task, which is about the strawberry-growing process of two sisters. The task featuring two sheep was prepared in a way that would attract the attention of the students, arouse their curiosity, and contain more than one solution strategy. The researcher kept field notes of direct observations of interactions between students and the milieu organized by the teacher while the participants worked through the task aimed at meeting the intended learning outcome of the study. The observation form was also drawn up to assist the observation process while collecting data on how key behaviors were performed for each phase of the didactical situation. To facilitate the analysis of the data, the group discussions during each class period were also videotaped with the participants' consent. Students' written work from each lesson was also collected. Before the data collection process, the task was also piloted with two groups of four students not involved in the study. The expert opinions were also taken into consideration. Necessary amendments were made based on the feedback taken to improve clarity and to ensure that the task was suitable for the didactical milieu. The following is the presentation of the task to be implemented in this study from the perspective of TDS.

Devolution phase. At the devolution stage, the teacher not only makes students accept responsibility for an a-didactical learning situation but also accepts the consequences of transferring this responsibility. The teacher assumes that students will be able to solve the problem on their own, based on the information given, prior knowledge or experience, and the reasoning during the discussion of the problem. First, the teacher expresses the problem situation as follows:

Two sisters visiting their grandfather want to grow strawberries in the garden, but they cannot decide exactly where to grow strawberries. Realizing that grandchildren cannot make a decision, their grandfather shares with them the suggestion that strawberries may be planted in the triangle ABC formed as a result of meeting two sheep that are tied with a rope to two fixed bamboo poles A and B which are at a fixed distance from each other, as shown in Figure 1. What should be the lengths of the pairs of ropes used to tie sheep so that they can form a triangular region when they meet? If you want to establish a rule for the lengths of the pairs of ropes used to tie sheep, how do you express it in general?

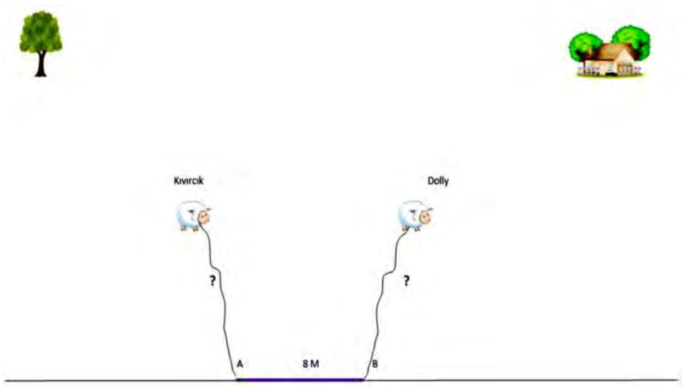


Figure 1. *Visual representation of the task: 'fragrant strawberries'*

After explaining the problem situation, the teacher talks about the details of the problem situation. First, she states that ropes to which two sheep are tethered are sufficiently stretched when the sheep meet at point C. She also mentions that the vertices of the triangle ABC formed will be point C and two fixed points A and B where the bamboo poles are located. Since points A and B are fixed, the distance between these two points is also fixed. She also adds that any length used in meters in the drawing will be taken as centimeters. All these details are explained with a simple model brought into the classroom to embody possible triangles that sheep can form. Students are also allowed to ask questions to better understand the problem situation, and they are encouraged to express it in their own sentences.

Action phase. After the devolution phase, the action phase begins. The teacher sets up a milieu for students to engage with before completely withdrawing from the scene. This is assumed to become an a-didactical situation in which students interact with the presented problem using their own reasoning without teacher intervention and assistance. As a result of interactions with the milieu, students can realize that each sheep draws a semicircle and that the semicircles must intersect to form the desired triangular region. Students who do not notice the semicircular movements of sheep can draw different lengths with a ruler to form a triangle. Students are also likely to seek help or advice from the teacher when all reasoning is being made at this stage, but the teacher must act in a way that does not break the didactic contract.

Formulation phase. The main goal of the formulation phase is to build a language to formulate students' strategies and agree on some common meaning. Personal observations gained in the action phase are to be communicated for others to share the experience and make up their minds about the strategies developed. Students need to exchange and compare observations among themselves and are expected to make some hypotheses about the solution to the problem situation with the help of specific strategies expressed in the group. Individual strategies through verbal expressions during the action stage can be expressed mathematically at the formulation stage. For example, in the action phase, a statement such as *'The sum of the lengths of the ropes will be greater than 8 cm'* can be expressed as $a + b > 8$, or a statement such as *'The sum of the lengths of the ropes will be less than 8 cm'* can be expressed as $a + b < 8$.

Validation phase. The validation phase is about the establishment of theorems. Students in the validation phase are expected to defend the hypotheses put forward by their groups and continue to interact with the milieu to analyze the hypotheses from the opposite groups. Since the validity of the hypotheses presented by one group is tested by other groups, it is supposed that there will be intense scientific debates between the groups. Furthermore, since both parties (the teacher and students) agree to comply with the didactic contract underlying all teacher-student interactions in the classroom, at the validation stage the teacher acts as a neutral chair in classroom debates, intervenes only to structure these debates, draw students' attention to possible inconsistencies, and encourage them to bring more convincing rationale for the strategies proposed.

Institutionalization phase. In the phase of institutionalization, the teacher helps students to link their new experience with the existing and established knowledge. She plays the role of a representative of the mathematical community of inquiry to find how this knowledge can be used for a new purpose or in a new context to solve other problems. At this stage, the teacher first reminds the students that the lengths of the ropes are the lengths of two sides of the triangle formed. She relates students' valid strategies for constructing various triangles to 'the triangle inequality theorem', and reinforces the institutionalization of new knowledge through whole-class discussion and justification. The new knowledge will also be reinforced in different problems where students use it to derive knowledge that is decontextualized.

3.4. Data analysis

Data analysis of the study was done through a deductive approach. Deductive content analysis involves analyzing data according to an available framework. It begins with existing studies and theories, and its purpose is to validate or expand knowledge within a theory (Braun & Clarke, 2006). The conceptual framework used in this study, based on Brousseau's theory of didactical situations, guided the interpretation and analysis of the data. All video recordings were transcribed verbatim and merged with the field notes taken during the observations. Video transcripts and field notes were analyzed using deductive qualitative analysis. Pre-set coding schemes were formulated considering the phases of the theory of didactical situations, i.e., the phases of devolution, action, formulation, validation, and institutionalization. After the pre-set coding schemes were assigned to the transcripts and related field notes, the video recordings were played back several times to ensure the validity of the data. Field notes were checked by cross-analyzing with the transcripts to ensure consistency. Students' written work was also analyzed in a similar way to supplement the video recordings and field notes during the observation period. Pre-set coding schemes were then revised with new incoming data. Participants were also encouraged to review the data through member checks to minimize any possible bias and to develop trustworthiness. To facilitate inter-coder reliability, one mathematics education researcher with more than fifteen years of experience in analyzing qualitative data was invited to serve as an external rater. Thus, during the final stage of analysis, the authors and this mathematics education researcher double-checked the pre-set coding schemes to ensure interrater reliability (Miles & Huberman, 1994). The raters resolved all disagreements and revised the coding schemes until a full agreement was reached to establish the credibility of the data.

4. Results

The devolution phase started with the teacher reading the problem situation aloud and continued with explaining the details of the problem situation to the students. This phase lasted about 15 minutes. A model prepared by the teacher (Figure 2), which was thought to provide a better understanding of the problem situation, was demonstrated to the class and the problem situation was embodied.

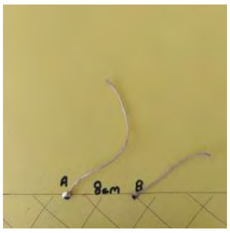


Figure 2. *A model that embodies the movements of sheep*

The teacher visualized the movements of the sheep with the help of the ropes on the model and gave some specific examples of rope lengths that form the triangular region (10 cm by 12 cm) and those that do not (3 cm by 4 cm). While explaining the problem situation with these examples, the dialogue between some students and the teacher was as follows:

> ***Teacher:*** Can we form a triangular region when we take the lengths of the ropes as 3 cm by 4 cm?
> ***Student A3:*** The ropes are not long enough to touch each other end to end.
> ***Student B2:*** (Nodding her head) Yes.
> ***Teacher:*** What if we take the lengths as 4 cm by 7 cm?
> ***Student C1:*** Then it becomes a triangle, Miss.

Meanwhile, one of the students intervened and asked:

> ***Student B3:*** Well, but quite a lot of lengths can be given as examples. Are we going to write them all? We can't write!

Although Student (B3) understood that she needed to find some pairs of rope lengths to form triangles, she did not understand that generalization should have been made using these length values. In response, the teacher said:

> ***Teacher:*** I don't want you to write them all. I would like you to express your pairs of rope lengths as a general rule. So…
> ***Student B1:*** (Turning to B3) For example, like... (She is thinking) Hmm...since we can't write all of them, we show the even numbers as '2n', just like that.
> ***Teacher:*** You can think of it this way.
> ***Student B3:*** Okay, now I got it.
> ***Student A1:*** I got it, too.

In this way, the teacher facilitated the understanding of the problem situation and tried to answer all questions from students in a way that would not break the didactic contract. It was then asked if any student was willing to explain the problem situation in their own words, and it was seen that all students in the class were willing to do so. After the devolution phase was over, the next phase was started.

The action phase began by handing out an activity sheet, compass, ruler, and blank paper to each student. For 15 minutes, the students tried to guess the pairs of rope lengths that could form triangles by trial and error. Then, they worked on hypotheses that would generalize the way of drawing triangles based on the values they found. The hypotheses that emerged during the action phase were as in Table 1.

Table 1. *Hypotheses by the students in the action phase*

Hypotheses	Truthfulness of Hypotheses	Students
Hypothesis 1: If the rope lengths are equal to each other, a triangle can be formed. (Isosceles Triangle)	False	A1, A2, B4, D3, D4, C1, C4
Hypothesis 2: If the rope lengths are 8 cm each, a triangle can be formed. (Equilateral Triangle)	False	A1, A2, D3, C1
Hypothesis 3: If both rope lengths are odd, a triangle cannot be formed. (Except for Isosceles or Equilateral Triangles)	False	D3
Hypothesis 4: If the rope lengths are close to each other, a triangle can be formed.	False	A3, A4, D4
Hypothesis 5: If the rope lengths form an acute-angled triangle, a triangle can be formed.	False	A2
Hypothesis 6: A triangle can be formed with rope lengths that are not relatively prime.	False	B2
Hypothesis 7: If one of the rope lengths is less than 5,5 cm and the other is less than 6 cm, a triangle cannot be formed.	False	C2
Hypothesis 8: If one of the rope lengths is greater than 7,9 cm and the other is less than 6 cm, a triangle cannot be formed.	False	C3
Hypothesis 9: If the sum of the rope lengths is greater than 8 cm, a triangle can be formed.	Partially True	B1, B3, D1, D2

It was seen that students often put forward Hypothesis 1, Hypothesis 2, Hypothesis 4, and Hypothesis 9. Of all the hypotheses, only one was found to be partially true and the others were false. Since most of the students tended to draw acute-angled triangles while proposing hypotheses, they were unable to combine some side lengths to form triangles. In the obtuse-angled triangles drawn, it was determined that angle C is generally constructed as an obtuse angle, while angles A and B are constructed as acute angles. In addition, students mostly preferred to draw isosceles or equilateral triangles. For example, it was observed that seven students expressed Hypothesis 1, which was one of the false hypotheses. Students who put forward this hypothesis stated that if the lengths of the ropes were not taken equal, the straight edges would not join together to form a closed plane shape bounded by three-line segments. As shown in Figure 3, one of the students expressed it as follows:

> …as a result of my trials, I realized that if the two sides are equal, I can form a triangle by joining them from the upper endpoints, but when I choose different lengths such as 12 cm and 10 cm, it does not work. (Student A1)

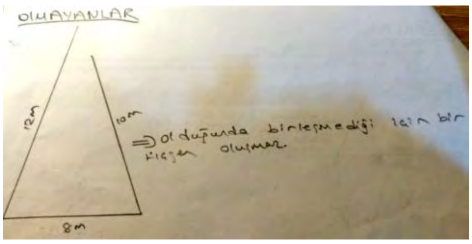


Figure 3. *Drawing by Student A1 in the action phase*

Hypothesis 9, which was classified as partially true, was suggested by four participants. For instance, Student D1, one of the participants who proposed this hypothesis, realized that the movements of the sheep were semicircles and made the drawing in Figure 4 using the side lengths of 3 cm by 4 cm. When the student noticed that the circles did not intersect, she noted on the activity sheet that *"the shape remains open"*.

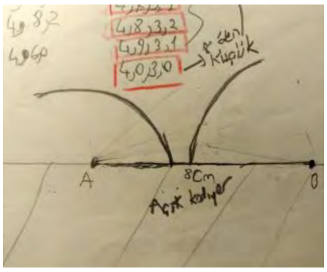


Figure 4. *Drawing by Student D1 in the action phase*

Next, she wrote down the pairs of lengths as shown in Figure 5, and grouped their sums greater than 8 cm, equal to 8 cm, and less than 8 cm. She realized that the sum of the rope lengths (edges) must be greater than 8 cm to form a triangle. She then made a note on the activity sheet that *"To form a triangle, the length of one side must be less than the sum of the lengths of the other two sides"*. Since she only related the sum of the lengths of two sides of the triangle to the length of the third side, she partially established the relationship between the side lengths of a triangle.

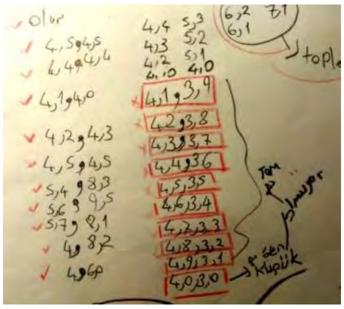


Figure 5. *Sample pairs of lengths checked by Student D1 in the action phase*

In the formulation phase lasting about 20 minutes, students in their groups exchanged and compared ideas among themselves. They discussed the hypotheses they produced during the action phase and tried to find out whether they were true or false. They co-developed the hypotheses based on the feedback received from the milieu.

After in-group discussions and interactions with the milieu, hypotheses put forward by each group at the formulation phase were as in Table 2.

Table 2. *Hypotheses by the students in the formulation phase*

Groups	Hypotheses		
A	If the side lengths of [AC] and [BC] are equal and greater than or equal to 5 cm, a triangle can be formed.	If the side lengths are 8 cm each, a triangle can be formed.	
B	If the sum of the side lengths of [AC] and [BC] is greater than the side length of [AB], a triangle can be formed.		
C	If the side lengths of [AC] and [BC] are equal to each other, a triangle cannot be formed.	If the side lengths are 8 cm each, a triangle cannot be formed.	If there is a difference of 1 cm between the side lengths of [AC] and [BC], a triangle can be formed.

D	If the sum of the side lengths of [AC] and [BC] is greater than the side length of [AB], a triangle can be formed.		

For example, the exchange of ideas in Group D started with the dialogue between Student D1 and Student D3 to formulate their observations and agree on some common meanings.

Student D1: I think everyone should share what they have found.
Student D3: First, I thought of special triangles. I formed the triangle by trying isosceles and equilateral triangles. Then, I tried different lengths. Those that do not form triangles are always odd numbers.
Student D1: How so? (Surprised) Can't a triangle be formed with 5 cm and 11 cm? I think it's a triangle.
Student D4: I think the lengths should be close to each other to form a triangle.
Student D1: How close should they be chosen?
Student D4: (Shuffles the papers) Look if I take 3 cm by 7 cm, I cannot connect them.
Student D3: In fact, now it seems to me that they do not join up when they are relatively prime.
Student D1: It has nothing to do with being relatively prime or odd numbers. Just listen to me and you'll understand. I think the sum of the rope lengths should be more than 8 cm (showing Figure 6). For example, I took 4,5 cm by 4,5 cm, and it worked.

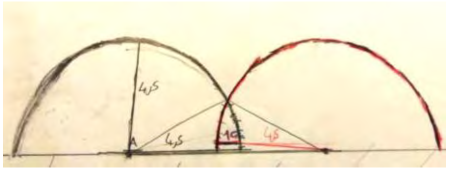


Figure 6. *Drawing by Student D1 for the lengths of 4,5 cm by 4,5 cm in the formulation phase*

Student D2: Exactly! That's when it becomes a triangle. It doesn't work if the total length is less than 8 cm.

In-group discussions revealed that student D3 often focused on acute-angled triangles when drawing triangles.

Student D2: (Turning to Student D3) For example, in this example (showing Figure 7) you did it wrong. The lengths you draw can be connected.

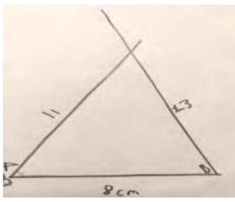


Figure 7. *Drawing by Student D3 for the lengths of 11 cm by 13 cm in the formulation phase*

Student D4: Yes, that's what I thought too, but I think it would be because these lengths are close to each other.
Student D2: (Turning to Student D4) Lengths of 2 cm and 3 cm are also close to each other, even closer than the pair of 11 cm and 13 cm. (Showing Figure 8) Let's see if it works.

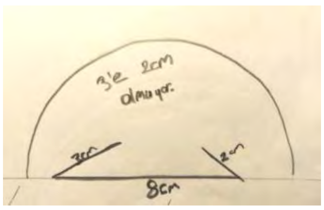


Figure 8. *Drawing by Student D2 for the lengths of 2 cm by 3 cm in the formulation phase*

Student D4: (Thinking) That's right, they are not connected end to end. Then it's not about being close to each other.

However, Student D3 continued to maintain that the lengths were either relatively prime or odd. Therefore, Student D1 asked the teacher for help. The teacher responded in a way that did not break the didactic contract.

Teacher: You must all agree with the hypotheses that you put forward as a group. You can convince each other more easily if you give specific examples while explaining your ideas.

Then, Student D1 made the drawing in Figure 9. She explained to Student D3 that triangles with side lengths of 5 cm and 11 cm could be drawn with the help of a compass.

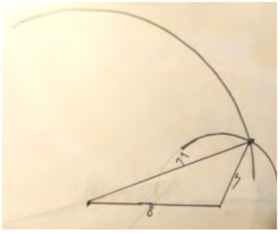
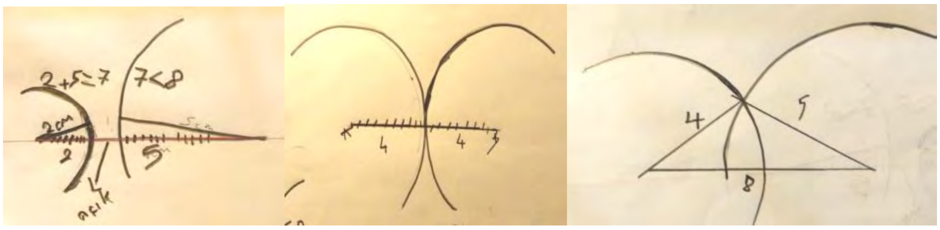


Figure 9. *Drawing by Student D1 for the lengths of 5 cm by 11 cm in the formulation phase*

Student D3: Of course, that's why I couldn't connect them…I never used the compass while drawing.
Student D1: (Turning to Student D3) Exactly.
Student D2: The compass makes a lot of sense… we first draw the circles to find the intersection point.
Student D4: I got it now.

Student D1 and Student D2 argued that circles did not intersect if the sum of the side lengths was less than 8 cm. In this case, it was observed that the triangle could not be formed. In cases where the sum of the side lengths was equal to and greater than 8 cm, the drawings in Figure 10 were made for each case.



(a) (b) (c)

Figure 10. *Drawings by Group D for the sum of the side lengths (a) less than, (b) equal to, and (c) greater than 8 cm in the formulation phase*

Based on these drawings and in-group discussions, it was seen that the claims of Student D1 and Student D2 were accepted by other members of Group D, and they came up with the following hypothesis: *"To form a triangle, the sum of the lengths of two sides must be greater than the length of the third side. That is, the length of any side must be less than the sum of the lengths of the other two sides."*

Next, all the hypotheses put forward by the groups during the formulation phase were written on the classroom whiteboard. Thus, the validation phase started as Group A began to explain its hypotheses to the other groups. This phase, in which each group defended its own hypothesis, lasted about 20 minutes.

Student A2: We were only able to construct an equilateral triangle with each side length of 8 cm, or isosceles triangles with equal side lengths of 5 cm or greater.
Student D1: You said that the lengths of the ropes should be equal, but in the drawings we made (showing Figure 11), we were able to form a triangle even if we took them differently.

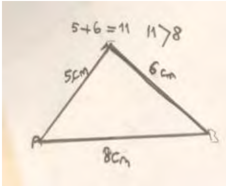


Figure 11. *Drawing by Group D for the lengths of 5 cm by 6 cm in the formulation phase*

> ***Student B1:*** (Turning to Group A) The drawing we made also refutes your hypothesis (showing Figure 12).

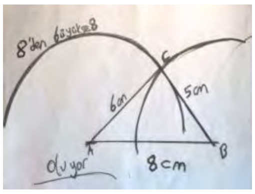


Figure 12. *Drawing by Group B with the lengths of 5 cm by 6 cm in the formulation phase*

After Group B and Group D explained how to use the compass to form a triangle, other groups stated that their hypotheses might not be true. Then, Hypothesis 9 considering only the sum of the side lengths to draw a triangle was explained by both Group B and Group D together to the whole class. For this, various drawings were made with the sum of the side lengths less than and greater than 8. In this way, it was seen that all groups were convinced that triangles could be drawn if the sum of two side lengths was greater than 8 cm. For example,

> ***Student C3:*** …then our hypothesis would be disproved. We said we can't draw an equilateral triangle, but 8 plus 8 is 16. That's bigger than 8.
> ***Student A3:*** The hypothesis of not being an isosceles triangle was also disproved. You said that 10 cm by 10 cm or 6 cm by 6 cm cannot be taken, but they can be taken. The sum of 10 and 10 is greater than 8. Add 6 to 6, again greater than 8.
> ***Student B3:*** Yeah, it's all been refuted.

Since there was no conversation among the students yet associating the difference between two side lengths with the third side length, depending on the didactic contract the teacher tended to deepen the discussions by asking students to think of different examples.

> ***Teacher:*** …then, to form a triangle, you claimed that the sum of the side lengths must be greater than 8 cm. What else could the side lengths be?
> ***Student B1:*** It is also drawn with 3 cm by 12 cm.
> ***Student D1:*** I don't think it can be drawn.
> ***Student C1:*** But we just showed it. When you add it up, it becomes 15. Isn't it large enough?
> ***Student B3:*** It was sufficient for the sum to be greater than 8. Why not this? Seems a little silly to me.

It was observed that all students, except Student D1, thought that a triangle could be drawn with side lengths of 3 cm by 12 cm. Each student tried to draw the triangle on their own. The trials, which were done as individual work, soon turned into group work. The drawing of Group B using the side lengths of 3 cm by 12 cm is given in Figure 13.

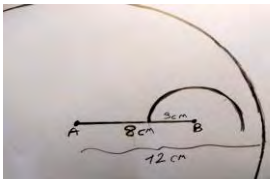


Figure 13. *Drawing by Group B with the lengths of 3 cm by 12 cm in the second formulation phase*

During in-group discussions, Group B and Group D hypothesized why a triangle could not be drawn with side lengths of 3 cm by 12 cm, while Group A and Group C did not put forward any hypothesis (see Table 3). At the end of in-group discussions, each group presented its hypotheses to the class for consideration and critique.

Table 3. *Additional hypotheses by the students in the second formulation phase*

Groups	Hypotheses	Additional Hypotheses
B	If the sum of the side lengths of [AC] and [BC] is greater than the side length of [AB], a triangle can be formed.	The difference between the side lengths of [AC] and [BC] should be less than 8 cm.
D	If the sum of the side lengths of [AC] and [BC] is greater than the side length of [AB], a triangle can be formed.	The difference between the side lengths of [AC] and [BC] should be less than or equal to 8 cm.

Group B and Group D both agreed that the difference between the lengths of two sides should be less than the length of the third side to draw a triangle, but they disagreed on whether the difference of the lengths could be equal to 8 cm or not. Since there was no consensus on this issue, discussions among all groups again ensued on what it might be. All groups together tried to find out whether a triangle was formed by taking two side lengths with a difference of 8 cm. They made the drawing in Figure 14 with two side lengths of 7 cm and 15 cm. Then, they noticed that the circles with a radius of 7 cm and 15 cm were tangent to each other.

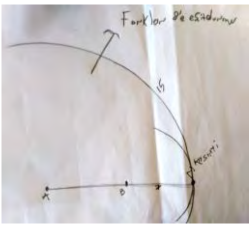


Figure 14. *Drawing by all groups with the lengths of 7 cm by 15 cm in the second validation phase*

Accordingly, with the contribution of all groups, it was shown that the sum of the side lengths should be greater than 8 cm while the difference between them should be less than 8 cm to form a triangle by the end of the validation phase.

After the validation phase, the institutionalization phase started. This phase lasted approximately 10 minutes. The knowledge developed from the verified hypotheses was attempted to be decontextualized by the teacher to give it the status of reusable mathematical knowledge for students in a new context.

> ***Teacher:*** We have seen that we cannot take the side lengths randomly to form a triangle. (Drawing Figure 15) Let the unknown side lengths of the triangle be a and b, respectively. You have shown that in order to draw a triangle a+b must be greater than 8 cm and a-b must be less than 8 cm.

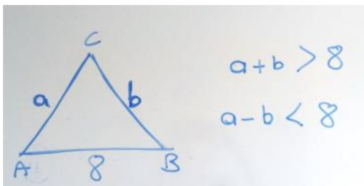


Figure 15. *Drawing by the teacher with the lengths of a by b in the institutionalization phase*

The teacher continued to assist students to recognize the knowledge gained in the previous phases and transform it into knowledge used to solve other problems.

> ***Teacher:*** The length of the side [AB] does not always have to be 8 cm. It can take different values. (Drawing Figure 16) Let the length of the side [AB] be c. We can write the general inequality statement so that all triangles can be drawn. We call it the triangle inequality. That is, for any triangle, the sum of the lengths of two sides must be greater than the length of the third side, and the difference between their lengths must be less than the length of the third side. In addition, if the longer side is subtracted from the

shorter side, the difference between the side lengths will be negative. So, we write it in absolute value. Let's not forget that either.

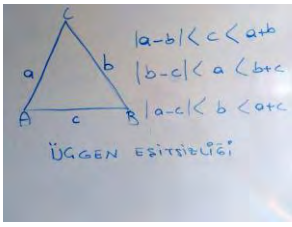


Figure 16. *Drawing by the teacher for the institutionalization of the triangle inequality*

In this way, for the knowledge to be institutionalized it was seen that the teacher tried to provide students with scholarly knowledge to make contextualized knowledge gain the status of classroom knowledge so that it could be used in other settings. By presenting additional problems about the triangle inequality, students were also allowed to internalize the knowledge they had just discovered, and then the lesson was over.

5. Discussion

The findings of the study revealed that student behaviors expected to take place within the framework of Brousseau's didactical situations in mathematics occurred at every stage as stated in the theory. In this sense, the devolution stage was the stage in which the responsibility of the problem solving passed from the teacher to the students, and the action stage was the stage in which the first strategies and hypotheses for the solution of the given problem were put forward. The formulation stage manifested itself as a stage in which individual hypotheses formed in the action stage turned into group hypotheses along with the interactions among students in each group. The validation stage was also completed as a process in which all proposed hypotheses were checked by the whole class. After the hypotheses established in the action phase were checked in the formulation and validation phases, respectively, appropriate solutions were produced for the problem, and contextualized knowledge obtained in the institutionalization phase was decontextualized by the teacher. It is also noteworthy that the validation phase was the stage in which intense and considerable discussions were held between the groups. It turned out to be a stage in which more than one way to the solution was adopted, different strategies were created, incomplete or incorrect hypotheses put forward by the groups were completed or refuted, and the groups that solved the problem guided the groups that could not solve it. Therefore, the validation phase emerged as a dynamic force in which all aspects of the problem-solving process were discussed and handled by the students.

As students progressed from the action phase to the validation phase, the strategies and methods used to solve the problem also evolved. While the problem-solving strategies used in the action phase were the basic ones such as trial and error and simplification of the problem, high-level strategies such as reasoning and proof were preferred in the formulation phase and validation phase. Students who could not identify the right strategies at the action stage realized the right strategies at the formulation stage, and groups who could not identify the right strategies at the formulation stage realized the right strategies at the validation stage. All hypotheses that were partially put forward at the action stage were associated with each other at the formulation stage and syntheses were made for the solution of the problem. As a result of the evaluations made by the group members, the hypotheses were revised according to the suggestions made during the formulation phase. Hypotheses that were thought to be flawed were transformed into new hypotheses by making use of the other hypotheses presented. They were eventually either rejected or modified to formulate better ones with the whole class discussion. Student interaction at the whole class level during the validation phase helped transform the partial hypotheses produced in the action phase into general hypotheses accepted by all students. These findings are in line with earlier findings arguing that the mathematical ideas leading up to the discovery of new knowledge are gradually formulated and justified as stages progress (Brousseau, Brousseau & Warfield, 2001, 2013; Brousseau & Gibel, 2005). Moreover, classroom discourse and social interaction also encourage the recognition of connections between ideas and the reorganization of knowledge (Lampert, 1986) and can have a powerful influence on academic motivation and achievement (Light & Littleton, 1999; Wentzel, 1999). Therefore, it is thought that the interaction amongst the learners in the classroom, which is quite common in a-didactical learning situations, plays an effective role in students' discovery of the mathematical knowledge at stake.

Even though most of the students stated that they understood the problem situation during the devolution phase, the transfer of responsibility to students could not be made immediately as a result of the questions they asked about the issue. The devolution phase progressed through a cycle of questions from students and feedback given to them. Hence, the devolution phase had a cyclic structure in itself. This cycle seemed to be effective for understanding the problem situation in the class. Once students encountered the problem, there was silence for a short time and they tried to understand it. This can be explained by Piaget's cognitive disequilibrium. When confronted with a problem situation in which learners identify inconsistencies or incompatibilities in their existing understanding, a state of cognitive equilibrium moves towards cognitive disequilibrium. The desire to maintain cognitive equilibrium drives the learners to reorganize their existing cognitive structure. In order to regain cognitive equilibrium, they attempt to understand the problem to get rid of the disturbing situation. This process also facilitates the construction of new knowledge (Ormrod, Anderman & Anderman, 2019). Therefore, in this study, students who realized that solving the problem requires understanding the problem frequently asked the teacher questions to better comprehend the problem situation. They tried to express it in their own words. Some students also tried to answer each other's questions and commented on their own. Although the devolution of an a-didactical learning situation is the act by which the teacher makes the student accept the responsibility for a problem after making the necessary arrangements and explanations without breaking the didactic contract, some students took an active role in the transfer of responsibility to students by contributing to the understanding of the problem situation.

Moreover, it is worthwhile noting that ideas build on previous learning played an important role in understanding the problem. The inability to use prior knowledge in problem-solving caused students to be passive at some phases. Students who could not use their prior knowledge properly to solve the problem could not establish appropriate strategies and hypotheses at the end of the action phase. This led to the acceptance of false hypotheses at the formulation stage and in turn prepared the environment for students to present false hypotheses at the validation stage. However, students must learn mathematics by connecting new ideas to prior knowledge and actively building new knowledge from prior knowledge and experience (NCTM, 2000). Therefore, students need to integrate their prior knowledge into the problem-solving process by seeing the connections among mathematical ideas in order for the a-didactical learning phases to be effective enough. On the other hand, inadequate or inaccurate prior knowledge can also serve as a source of confusion. For example, students generally preferred acute-angled triangles, isosceles or equilateral triangles, or triangles with side lengths close to each other while making trial drawings to form their hypotheses. The triangle images that existed in their minds may have led them to draw such triangles (Gal & Linchevski, 2010; Hershkowitz, 1989; Kaur, 2015), but this may also have adversely affected the accuracy of the hypotheses put forward throughout the phases.

Although the phases of didactical situations follow a certain order in a hierarchical structure, it has been observed that each of them is closely related to the other and there can be transitions from any phase to the others when necessary. For instance, the validation phase hosted a new a-didactical process in case the target mathematical knowledge was not sufficiently appropriated. Thus, while performing the validation phase, a-didactical learning situations were repeated by returning to the action phase, formulation phase, and validation phase. In such a case, the role of the teacher is especially important. As illustrated by Laborde and Perrin-Glorian (2005) and Sensevy, Schubauer-Leoni, Mercier, Ligozat, and Perrot (2005), the regulation of an interplay between the didactic contract and the milieu in a-didactical situations is significant to ensure that students progress smoothly in the process of solving a mathematical problem. Therefore, the teacher of this study meticulously followed the process at each phase and carefully decided where and how to intervene in order not to break the didactic contract. She fulfilled the mutual obligations of the teacher and students in the a-didactical learning situations and did not indicate the target knowledge that she intended the students to learn.

It is also remarkable that the concrete material used to model the problem situation, which was brought to the classroom during the devolution phase, and the visualizations made in this way drew students' attention to the problem situation. This model offered students concrete representations of abstract ideas and flexible movement among them to solve the problem. Thus, modeling mathematics with a concrete object facilitated the understanding of the problem situation in the devolution phase and allowed this phase to progress efficiently. As stated by NCTM (2014), using concrete models helps students develop and test their ideas and they can articulate clear mathematical arguments for why geometric relationships are true. Hence, the concretization of mathematics is an accepted approach in terms of supporting students' learning processes as it is an effective way of embodying abstract concepts and realizing meaningful and worthwhile learning outcomes (Gravemeijer, 2007; Groshong, 2016). Therefore, it is believed that using the concrete model of the problem at the devolution phase enabled students to go into details of the problem and better understand it by making inferences about the characteristics of the problem situation.

6. Conclusion

Overall, the evidence from the analysis of didactical situations aimed at students learning the triangle inequality principle has suggested that designing a learning environment based on the theory of didactical situations to explore the mathematical knowledge at stake is beneficial for giving the students a sense of discovery. In this way, students demonstrated a range of strategies to make sense of and solve the problem by reflecting on their thinking during the problem-solving process. They worked out different methods of evaluation in solving the mathematical task and defended them in a way that invited others to implement their chosen strategies. Devising strategies for solving the problem mathematically is expected to provide them to gain ways of mathematical thinking and reasoning, habits of persistence and curiosity, and confidence in unusual situations that benefit them well beyond the classroom. Mathematical thinking and reasoning with deductive arguments, including making and generalizing strategies are essential because they serve as a foundation for developing new insights and promoting sound mathematical understanding (NCTM, 2021). By getting students to talk about their informal strategies, we can also help them become aware of and develop their implicit informal knowledge by building on their thinking and the thinking of others. It encourages students to engage in classroom interaction and provides learning opportunities for them to explain and justify their reasoning based on what they know to expand their thinking skills (NCTM, 2014). In this regard, the findings of this study support that employing a-didactical situations for teaching geometry can promote such practices. It not only facilitates the purposeful exchange of ideas through whole-class discussion that ensures a common understanding of mathematical ideas but also allows students to create their own learning adventures. Hence, it paves the way for learning to learn. Moreover, even if special emphasis is placed on the definition of the milieu, it is not easy to fully understand what kind of feedback this milieu offers or how mathematical knowledge can gradually emerge from interactions in the social setting of the situation. It is also important to understand the dynamic nature of the didactic contract in terms of the roles that both the teacher and students undertake. Further research in these areas may shed more light on the details of the theory of didactical situations in mathematics.

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