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# AN INVESTIGATION OF PROSPECTIVE ELEMENTARY MATHEMATICS TEACHERS UNDERSTANDING OF THE FORMAL DEFINITION OF THE LIMIT OF A SEQUENCE

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**Abstract:** The purpose of this study was to investigate the understanding of prospective elementary mathematics teachers of the formal definition of the limit of a sequence. The participants were 32 prospective mathematics teachers (8 male and 24 female) who were in their third year of study in a four-year mathematics teacher preparation programme Istanbul University-Cerrahpasa in Türkiye. The data were collected via a test that was developed by the authors. The findings of the study demonstrate that the prospective elementary mathematics teachers had difficulty in defining the concept of limit of a sequence and in determining the graphical representations of the variables and the inequalities in the formal definition. Moreover, they also had difficulty comprehending the relationship between the variables of $\varepsilon$ and $n_0$, and hence, when a numerical value was assigned to one of these variables, they were not able to find an appropriate value for the other. It was concluded that the prospective mathematics teachers generally tended to memorize the formal definition of the limit of sequences but lacked conceptual understanding of its meaning. Therefore, they were not able to reason about the variables it contained.

**Key words:** limit, sequence, $\varepsilon$-N definition, calculus instruction, prospective mathematics teachers

## 1. Introduction

The concept of limits is an essential aspect of calculus and constitutes a substantial portion of the mathematics taught at the undergraduate level. According to Tall (1992), this concept serves as a gateway for the transition to advanced mathematical thinking. In this regard, some of the primary concepts covered in calculus are special types of limits, although they are not always referred to by this name. For instance, derivatives are the limit of a difference quotient, and the definite integral is the limit of a Riemann sum; both of these concepts are fundamental to calculus, thus underscoring Tall's view.

Generally, in the introductory calculus courses offered at the undergraduate level, the concept of limits is addressed in two different contexts: namely, the investigation of the limits of single-variable real valued functions and the investigation of the limits of real number sequences. There is a huge body of research that concentrates on the different aspects of teaching the concept of limits in the former context. Some of these studies have focused on producing conceptual frameworks to describe the learning process (Cottrill et al., 1996; Przenioslo, 2004; Roh, 2007; Williams, 2001). Some studies have also aimed to document the difficulties that students encounter in learning the subject (Bezuidenhout, 2001; Cornu, 2002; Sierpińska, 1987; Tall & Vinner, 1981); while others have comprised investigations of students' understanding of the formal definition of limits (Baki & Çekmez, 2012; Doruk et al., 2018; Kabael et al., 2015; Oktaviyanthi et al., 2018; Swinyard & Larsen, 2012). The studies that focused on the formal definition of the concept have reported various difficulties and misunderstandings on the part of students. For instance, Baki and Çekmez (2012) revealed that prospective mathematics teachers had problems understanding and representing the variables and inequalities within the formal definition in the Cartesian coordinate system. Their findings also revealed that students tended to memorize the definition without comprehending the logical structure. Similarly, Tall and Vinner (1981) reported that students could not interpret the roles of the quantifiers "$\forall$" and "$\exists$" in the formal definition, and hence were not successful in proving the existence of a limit. In this regard, Kabael et al. (2015) suggested that the main sources of difficulty for students were the inability to use quantifiers included in the formal

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definition in a meaningful way, as well as their inefficacy in associating the dynamic nature of the process with the inequalities in the formal definition and their corresponding intervals. In line with Kabael et al., Doruk et al. (2018) reported that most of the participants in their study did not demonstrate a mature understanding of the formal definition of limits and hence could not interpret the symbols $\varepsilon$ and $\delta$ or explain they relate to each other.

The symbolic form of the formal definition of the limit of a convergent sequence consists of a three-level quantification: "$L \in \mathbb{R}.\ \lim_{n\to\infty} a_n = L \Leftrightarrow\ \forall\varepsilon > 0\ \exists n_0 \in N$ s.t. $\forall\ n > n_0\ |a_n - L| < \varepsilon$". The verbal counterpart of this definition can be stated as follows: "The limit of the sequence $(a_n)$ is $L$ if and only if for each $\varepsilon$-neighbourhood of $L$ there is a natural number such that the terms of the sequence whose index is greater than that natural number falls into that neighbourhood". Accordingly, for a sequence to have a limit, it must satisfy the proposition (referred to as *formal proposition* throughout this article) on the right-hand side of the *if* conjunction. Considering the many aspects embedded in the symbolic form that must be comprehended to fully grasp the meaning of the definition, the majority of students may not be expected to comprehend all of its complexities at once. These aspects include an apprehension of what is signified by the absolute-value inequality with regard to the terms of the sequence and the limit value; a recognition of the role of the index $n_0$ and its dependence on the parameter $\varepsilon$; and a comprehension of the roles of the universal/existential quantifiers and how they relate to each other.

Despite an abundance of studies that have focused on the limit concept in the context of functions, few have dealt with this concept in the context of sequences (Larsen et al., 2017). Among these, only a limited number have focused on the formal definition. However, Roh (2010) discussed the formal definition of the limit concept in the context of sequences in a study that explored undergraduate students' understanding. This group had been instructed on the limit of a sequence without addressing the $\varepsilon$-$n_0$ definition regarding the logical structure within the formal definition by focusing on the relationship between $\varepsilon$ and $n_0$. To this aim, Roh prepared $\varepsilon$-strips made of translucent paper in rectangular shapes to embody the $\varepsilon$-neighbourhood of the limit value of a sequence. Two definitions, which were structured around the use of the $\varepsilon$-strips by a fictitious calculus student, were presented to the participants. Afterward, Roh conducted interviews in which the students were asked to apply these definitions to algebraically represented sequences to determine whether they were valid to define the limit of an arbitrary sequence. The findings of the study showed that the process of counting, in which the number of terms inside and outside of an ε-strip were determined, was a key factor in understanding the logical relationship between $\varepsilon$ and $n_0$ in the formal definition, and some of the students could not properly employ this ability in determining the limit of a sequence. The interviews with the students also revealed that in order for students to grasp the logical structure in the formal definition, the following three essential components must be conceptualized: *i)* Finding $n_0$ corresponding to $\varepsilon$ before completely diminishing $\varepsilon$ to 0, *ii)* recognizing that the expression 'any $\varepsilon$' suggests the arbitrariness of error bounds, and *iii)* understanding that the arbitrariness of $\varepsilon$ suggests that $\varepsilon$ tends to 0.

In an earlier study, Przenioslo (2005) found similar results to those of Roh (2010), as well as uncovering additional difficulties in reasoning about the limit of a sequence using $\varepsilon$-strips. One false conviction that students held in that study was that if the terms of a sequence started falling into a strip of an arbitrary width centred at the limit value, then successive terms could not fall outside of it. Another was that a single strip in which the terms of the sequence approached the limit value was a sufficient condition for the existence of the limit. In addition, it was reported that many students tended to believe that the approaching of all terms had to be monotonic. Moreover, an interesting observation of Przenioslo was that a majority of her students believed that a sequence could have a limit at a natural number. Likewise, Cory and Garofalo (2011) found that some of the misconceptions reported in the literature about the limit of functions also occur in the context of sequences as a result of failure to successfully interpret the formal definition; for instance, their participants expressed the belief that a sequence cannot reach its limit value and that a constant sequence is not convergent.

It is widely acknowledged that the various representations of mathematical concepts (e.g., algebraic, graphic and verbal) comprise complementary information that helps students construct a deeper understanding (Ainsworth, 1999). As to the verbal component, for instance, Friedlander and Tabach (2001) contend that it provides a natural environment for comprehending the context of a problem and has the potential to make learning more meaningful. Moreover, Ainsworth et al. (1997) assert that a specific representation may serve to support and assist in the interpretation of a more complicated representation of a mathematical concept. Having considered the complexity and density of the algebraic representation of the formal definition, this assertion is particularly true in the case of the limit of a sequence. Hence, comprehension of the verbal and graphical representations of the definition seems necessary for students to develop a thorough understanding of the formal definition. Accordingly, the authors of this manuscript share the view of Cory and Garofalo (2011) that a good understanding of the limit of a sequence requires the facility to analyse various aspects of the formal definition by using words and diagrams. Therefore, it is necessary to investigate students' ability to interpret the formal definition via different representations. However, as outlined in the previous paragraph, the research on the formal definition has primarily focused on students' performance in terms of the relation between the variables of $\varepsilon$ and $n_0$. Thus, the current study aims to contribute to the gap in the literature by investigating students' ability to connect the different representations of the formal definition. With this objective in mind, the following research question was addressed: What understandings do students develop regarding the different representations of the formal definition of the limit of a sequence?

## 2. Method

This section presents information regarding the study method, the participants, the data collection tool and the data collection process.

### 2. 1. Sample

The participants in this study included 32 prospective elementary mathematics teachers (24 female and 8 male) who were enrolled in the elementary mathematics teacher education programme at Istanbul University-Cerrahpasa in Türkiye. The participants were in their third year of study, and their participation took place 3 weeks after they had been instructed about the limit of real number sequences in the context of a calculus course. The instructional format in the course consisted mainly of direct lecturing. The students were also encouraged to take part in class discussions and to share their ideas with the class; however, the implementation was not supported with the use of specific software designed for mathematics education. The teaching of the topic included the presentation of the formal definition of the limit of a sequence, along with problems that asked students to prove the limit of a sequence using the formal definition. The course was taught by a lecturer who was not among the authors. The lecturer had six years of experience in teaching the subject at the undergraduate level.

Prior to the implementation of the test used as a data collection tool, the students were informed about its purpose and asked for their voluntary consent to participate. In addition, they were informed that their identities would not been shared, nor would their work be used for any other purpose. The implementation took place in a classroom, and the students were given 40 minutes to complete the test.

### 2. 2. Instrument

To examine students' understanding of the formal definition of the limit of a sequence, a test consisting of two sections was developed by the authors (see Appendix 1). The qestions in the test did not include limit forms in which the terms of the sequence diverge to infinity. The first section consisted of the first three questions, and the remaining questions formed the second section. The two sections were handed out on two separate sheets, and after each student had finished the first section, they received the second section. The reason for delivering the questions on two separate sheets was to eliminate any possible effect of the algebraic expressions in the fourth and subsequent questions on the students' answers to the preceding questions. The objectives evaluated by each question on the test are provided in Table 1.

**Table 1.** *The Objectives that the Questions Targeted in the Test*

| Q. | Focused learning outcomes |
|---|---|
| 1 | Being able to define the limit of a sequence. |

| | |
|---|---|
| 2 | Being able to determine the graphical representations of the variables in the formal definition. |
| 3 | Being able to recognize the algebraic statement that represents the behaviour of a convergent sequence presented graphically. |
| 4 | Being able to recognize the formal proposition of the limit of a convergent sequence presented algebraically. |
| 5 | Being able to determine an appropriate value of $n_0$ for a given value of $\varepsilon$ and to interpret its uniqueness. |
| 6 | Being able to recognize the verbal and algebraic representations of the relationship between $\varepsilon$ and $n_0$. |
| 7 | Being able to identify the algebraic representation and the limit value of a convergent sequence which is presented via formal proposition. |
| 8 | Being able to determine appropriate values for $\varepsilon$ and $n_0$ when the terms of a convergent sequence are presented graphically. |

After the first version of the questions was prepared by the authors, it was presented to the lecturer to have him confirm that the knowledge required for answering the questions was appropriate for the students' level. The test questions were also reviewed by two scholars in the field to verify the content validity. Next, the test was piloted with a separate cohort of students to assess the wording of the questions and to determine the time required to finish the test. After the implementation of the final version, the authors performed a quick inspection of the students' responses to decide on the categories to be produced for each question on the test. Following the first round, the authors performed a second examination individually to classify students' responses according to the categories that they had produced. Afterward, the authors met to check and resolve any conflicts among the classifications and finalized the categorizations of the students' answers. For reporting purposes, the students' responses presented in this manuscript were translated from Turkish to English while preserving the essence of the meaning.

## 3. Results

### 3. 1. Defining the Limit of a Sequence

The aim of the first question on the test was to determine students' knowledge about the definition of the limit of a sequence. With this aim, the students were asked to state the formal definition of the limit of a sequence, or if they could not give the formal definition, to explain it in their own words. Their responses to the question were categorized as correct, incomplete, incorrect or blank. In addition, the responses that were categorized as true, incomplete or incorrect were classified as either formal or informal. The frequencies and percentages of the response types are presented in Table 2.

**Table 2.** *Frequencies and Percentages of the Categories of Students' Responses to the First Question*

| | **Correct** | | **Incomplete** | | **Incorrect** | | **Blank** |
|---|---|---|---|---|---|---|---|
| | ***Formal*** | ***Informal*** | ***Formal*** | ***Informal*** | ***Formal*** | ***Informal*** | |
| *n* | 8 | 2 | 4 | 8 | 2 | 3 | 5 |
| % | 25 | 6 | 13 | 25 | 6 | 9 | 16 |

As Table 2 illustrates, thirty-one percent of the students gave the correct formal or informal definition for the limit of a sequence. The students whose answers were categorized as *formal correct* provided the correct formal definition of the limit of a sequence. Those whose responses were categorized as *informal correct* provided verbal statements that carried the same meaning of the formal definition. For example, one of the students whose response was classified in this category stated that "*the limit of a sequence represents the value at which its terms gradually converge. That is, for any given open neighbourhood of L, if a finite number of terms reside outside of that neighbourhood, then L is the limit value*."

On the other hand, it was found that 38% of the students supplied incomplete definitions. Among these, the responses that in part included inequalities and/or quantifiers in the definition were classified as *formal incomplete*, whereas responses given as a verbal expression in the same manner were classified as *informal incomplete*. For example, the response of a student classified as *formal incomplete* was as follows: "$\forall\varepsilon > 0 \quad |a_n - a| < \varepsilon$. *If this holds true, then the limit of* $(a_n)$ *is a, and* $-\varepsilon < a_n - a < \varepsilon$

*means that the limit of the sequence is $a$*". On the other hand, one student, whose response was classified as *informal incomplete*, responded with the following: "*The limit of a sequence: the value that the terms of the sequence take on as the index goes to infinity*".

As Table 2 also indicates, sixteen percent of the students provided incorrect statements for the definition. The students whose responses were classified as *formal incorrect* stated the direction of the inequality regarding the index of the terms incorrectly. On the other hand, those responses classified as *informal incorrect* included statements regarding the limit of a convergent series, rather than the limit of a sequence.

### 3. 2. Determining the Graphical Representation of Variables in the Formal Definition

The purpose of the second question on the test was to determine whether students were able to discern the graphical representation of the variables in the formal definition in the Cartesian coordinate system. To this aim, the students were presented with a pictorial scenario of a limit of a convergent sequence and asked to match the variables $\varepsilon$, $n_0$, and $L$ in the formal definition with the symbols included in the question. Their responses were categorized as correct, incorrect or blank. The frequencies and percentages of the response types are presented in Table 3.

**Table 3.** *Frequencies and Percentages of the Categories of Students' Responses to the Second Question*

| | Correct | Incorrect | Blank |
|---|---|---|---|
| $n$ | 22 | 4 | 6 |
| % | 69 | 12 | 19 |

As indicated in Table 3, sixty-nine percent of the students were able to determine the graphical representations of the variables in the formal definition. The students who provided an incorrect answer to the question either matched "p" with "$\varepsilon$" and "s" with "$L$" or "p" with "$n_0$" and "q" with "$L$".

### 3. 3. Recognizing the Algebraic Statement that Represents the Behaviour of a Convergent Sequence Presented Graphically

The purpose of the third question on the test was to determine whether students were able to recognize the proposition that renders the behaviour of a convergent sequence represented graphically in Cartesian coordinate system. The question was a multiple-choice type; therefore, the students' answers were categorized as correct, incorrect or blank. The frequencies and percentages of the responses are presented in Table 4.

**Table 4.** *Frequencies and Percentages of the Categories of Students' Responses to the Third Question*

| | Correct | Incorrect | Blank |
|---|---|---|---|
| $n$ | 20 | 11 | 1 |
| % | 63 | 34 | 3 |

As shown in Table 4, approximately one third of the students failed to recognize the correct proposition that represents the behaviour of the sequence. An examination of their responses revealed that this failure could be attributed to three causes. These were, in order of prevalence: (1) failure to choose the correct value for the index; (2) failure to discern the correct direction of the inequality sign; and (3) confusing the variables of $n_0$ and $L$ with one another.

### 3. 4. Recognizing the Formal Proposition of the Limit of a Convergent Sequence Presented Algebraically

The aim of the fourth question on the test was to determine students' ability to recognize the formal proposition that states the limit value of a convergent sequence represented algebraically. The students' responses were categorized as correct, incorrect or false. The frequencies and percentages of the responses to the question are presented in Table 5.

**Table 5.** *Frequencies and Percentages of the Categories of Students' Responses to the Fourth Question*

| | Correct | Incorrect | Blank |
|---|---|---|---|
| *n* | 27 | 4 | 1 |
| % | 84 | 13 | 3 |

As indicated in Table 5, a majority of the students were able to recognize the correct proposition that represents the limit value of the sequence. All the students whose answers were categorized as incorrect concluded that the correct proposition was given in option c. This implies that these students confused the roles of $n_0$ and $\varepsilon$ with each other.

### 3. 5. Finding an Appropriate Value of $n_0$ Depending on Given $\varepsilon$ and Interpreting Its Uniqueness

The purpose of the fifth question in the test was to determine whether students could work out an appropriate value of $n_0$ for a specific value of $\varepsilon$ and interpret the uniqueness of that value. The question included an algebraically represented convergent sequence, along with its limit value. The frequencies and percentages of the students' responses are presented in Table 6.

**Table 6.** *Frequencies and Percentages of the Categories of Students' Responses to the Fifth Question*

| | Correct | Incorrect | Blank | Uniqueness of the value of n_0 | | |
|---|---|---|---|---|---|---|
| | | | | *Unique* | *Not unique* | *No comment* |
| *n* | 7 | 11 | 14 | 3 | 3 | 26 |
| % | 22 | 34 | 44 | 9 | 9 | 82 |

As Table 6 demonstrates, only about one fifth of the participants were able to find an appropriate natural number for the index ($n_0$). Moreover, among these, only three were able to correctly interpret the uniqueness of the value that they determined. It was observed that the students who found an appropriate natural number for the index, but failed to interpret its uniqueness, had confused the uniqueness of the index and the uniqueness of the limit value with each other. For instance, one student commented as follows: "*It must be unique, because for a sequence to be convergent, its limit value must be unique*". The students whose responses were categorized as incorrect either made computational errors or could not decompose the absolute value inequality to determine the boundary points for the index.

### 3. 6. Recognizing the Verbal and Algebraic Representations of the Relationship between $\varepsilon$ and $n_0$

The aim of the sixth question on the test was to assess students' ability to recognize the algebraic and verbal representations of the relationship between $\varepsilon$ and $n_0$. Accordingly, the students were given an algebraically represented convergent sequence, together with a question that asked for the minimum value that the index can take on for a specific value of $\varepsilon$. The students were asked to choose which of the statements among the given options had the same meaning as the question in the problem statement. This question differed from the others in that two of the given four options were correct; therefore, to provide a correct answer, options "a" and "c" would both need to be chosen in their responses. The responses in which the students chose both options were categorized as correct, whereas the responses in which a participant chose only one of the correct options were categorized as partially correct. The numbers of responses that fell in each category are given in Table 7.

**Table 7.** *Frequencies and Percentages of the Categories of Students' Responses to the Sixth Question*

| | Correct | Partially correct | | Incorrect | Blank |
|---|---|---|---|---|---|
| | | a | c | | |
| *n* | 7 | - | 18 | 2 | 5 |
| % | 22 | - | 56 | 6 | 16 |

As indicated in Table 7, nearly one fifth of the students were able to recognize both the algebraic and the verbal representation of the question in the problem statement. However, roughly half of the students chose only option "c" as the correct answer.

### 3. 7. Identifying the Rule and Limit Value of a Convergent Sequence from Its Formal Proposition

The seventh question on the test asked students to identify the rule and the limit value of a convergent sequence represented by the formal proposition. The students' answers to the question were categorized as correct, incorrect or blank. The frequencies and percentages of the responses are presented in Table 8.

**Table 8.** *Frequencies and Percentages of the Categories of Students' Responses to the Seventh Question*

| | Correct | Incorrect | Blank |
|---|---|---|---|
| *n* | 25 | 5 | 2 |
| % | 78 | 16 | 6 |

As the numbers in Table 8 suggest, the majority of the students were able to identify the rule of the convergent sequence and the limit value based on the information provided by the formal proposition.

### 3. 8. Determining Appropriate Values for $\varepsilon$ and $n_0$ When the Terms of a Convergent Sequence Presented Graphically

The final problem on the test included a pictorial representation of the behaviour of a convergent sequence in the Cartesian coordinate system. The terms of the sequence were presented with points such that the abscissa and ordinate of a point that represented a particular term of the sequence were equal to the index of the term and the term itself, respectively. The picture also included four open intervals centred at the limit value, depicted with dashed line segments. The problem consisted of four sub-questions. In general, the main aim of the sub-questions was to assess students' ability to interpret the relationship between the parameter ε and the index $n_0$ based on graphical data.

In the first sub-question, the students were required to find two appropriate natural numbers for the index for a given constant value of $\varepsilon$ based on the graphical data. The students' responses were categorized as correct, partially correct, incorrect or blank. The partially correct category was composed of responses in which students provided only one appropriate natural number for the index. The frequencies and percentages of students' responses are presented in Table 9.

**Table 9.** *Frequencies and Percentages of the Categories of Students' Responses to the Question 8-a*

| | Correct | Partially correct | Incorrect | Blank |
|---|---|---|---|---|
| *n* | 5 | 6 | 9 | 12 |
| % | 16 | 19 | 28 | 37 |

As Table 9 demonstrates, a small portion of the students were able to find two appropriate values for the index for which the terms of the sequence satisfied the inequality given in the problem statement. On the other hand, more than half of the students either provided inappropriate values or did not attempt to answer the question. Among the students whose answers were categorized as false, four gave one of the values (e.g., $L$+0.1) marked on the y-axis. The others took into consideration only the upper bound, which was $L$+0.2, of the inequality $|a_n - L| < \varepsilon$; and they gave either one or two of the values of 4 and 5 for the index, because the fifth and successive terms were less than $L$+0.2.

The second sub-question was similar to the first. However, in this question, students were asked to find the minimum natural number that the index could take on so that the inequality would hold true. The students' answers were categorized as correct, incorrect or false. The frequencies and percentages of students' responses are presented in Table 10.

**Table 10.** *Frequencies and Percentages of the Categories of Students' Responses to the Question 8-b*

| | Correct | Incorrect | Blank |
|---|---|---|---|
| *n* | 2 | 21 | 9 |
| % | 6 | 66 | 28 |

As the figures in Table 10 reveal, a very small portion of the students were able to determine the least value that the index could take on. Upon examination of the answers categorized as incorrect, it was found that 8 students claimed that 2 was the least appropriate value that the index could take on. As shown in the graph, the second term was the minimum term of the sequence. It seems this fact had a decisive impact on the students' responses. The remaining students in this group claimed that the answer was 9. Again, the ninth term was the minimum term of the sequence within the interval (*L-0.3, L+0.3*). Similar to previous students, these, too, seemed to have interpreted the term "*minimum*" in the problem statement as qualifying not the value of the index but of the terms.

The aim of the last sub-question was to assess the students' ability to determine appropriate values of ε based on graphical data for a given index value. Accordingly, the students were asked to find two proper values of $\varepsilon$ that satisfied the proposition given in the problem statement for the given index value. The students' responses were categorized as correct, partially correct, incorrect or blank. The category of partially correct was composed of responses in which students provided only one proper value for the parameter. The frequencies and percentages of students' responses are presented in Table 11.

**Table 11.** *Frequencies and Percentages of the Categories of Students' Responses to the Question 8-c*

| | **Correct** | **Partially correct** | **Incorrect** | **Blank** |
|---|---|---|---|---|
| *n* | 10 | 4 | 6 | 12 |
| % | 31 | 12 | 19 | 38 |

As the numbers in Table 11 suggest, more than half of the students either provided an incorrect answer or did not attempt the question. Among the students whose answers were categorized as incorrect, two provided negative values for the parameter $\varepsilon$, and the remaining gave values greater than 0.5, which was the greatest one in the graph. It seems that those students who claimed that $\varepsilon$ must be greater than 0.5 did not take into account the information "$n_0 = 5$" given in the problem statement, and, consequently, they thought that the interval to be constructed must include all the terms of the sequence depicted in the graph.

## 3. Conclusion and Discussion

This study sought to examine students' understanding of the different representations of the formal definition of the limit of a sequence. The findings revealed that students had varying levels of understanding regarding different representations of the concept. First, the majority of the students were unable to state the formal definition. This finding is in line with the study of Cory and Garofalo (2011), in which they stated that students had difficulty in recalling the formal definition. However, with respect to students' performance on the second question, there was a sharp contrast; that is, some of the students who could not provide the algebraic statement of the formal definition in the first question successfully determined the graphical representations of the variables in the formal definition in the Cartesian coordinate plane. From one point of view, Pinto and Tall (2002) demonstrated that in the process of constructing meaning for quantified statements, a student may build from his/her imagery and give meaning to a formal definition based on a generic image that backs his/her statements. Similarly, Mamona-Downs (2001) contended that when students first encounter the limit concept, they simultaneously strive to make sense of the symbolism and struggle to link resulting meaning to informal images. This reasoning may account for the contrast. In other words, instead of memorizing the formal algebraic statement of the definition, students seemed to have developed generic mental images in their minds to resort to in problem situations.

From another point of view, the studies in the literature have often found that students have difficulty writing formal mathematical expressions, and that it takes considerable time to achieve this skill (Epp, 2016; Selden & Selden, 1995). In this regard, Mamona-Downs (2001) also commented that very few students possess the natural facility to grasp such statements with ease in a short time, whereas the vast majority of students require a great deal of time to comprehend the meaning. The findings in this study support this assertion. The difference between students' success on the first and fourth questions indicates that most of those who could not provide the algebraic statement for the formal definition were able to recognize the formal proposition that represents the given information. A similar success pattern also occurred in the students' answers to the first and seventh questions. For the seventh question, many of the students were able to identify the algebraic representation and the limit value of a convergent sequence that was presented via the formal proposition. With all this in mind, it can be concluded that most of the students could recognize the algebraic and graphical representation of the formal definition, but they could not produce the formal definition in written form.

One of the key skills that a student must attain to comprehend the logical structure within the formal definition is an understanding of the relationship between the index ($n_0$) and the parameter ($\varepsilon$). The students' responses to the questions that addressed this relationship revealed that they had difficulty in understanding and interpreting the logical structure. With respect to their performance on the sixth question, it can be concluded that the majority recognized the resulting inequalities when they were asked to find the minimum value that the index could take on for a given value of the parameter. However, the same students did not recognize the verbal representation of the same question; in other words, they were unable to correctly explain the meaning of the question on the basis of the distance of the terms from the limit value. This finding is in line with the observation of Selden and Selden (1995), in which they reported that students failed to link an informal expression with its structured formal counterpart in mathematical contexts. Furthermore, although the majority of the students recognized the formal proposition within the formal definition, their performance on the fifth question indicates that they could not find an appropriate value of the index for a given value of parameter by setting up the relevant inequalities and solving them. Moreover, they did not successfully interpret the uniqueness of the index. Taking all of this into account, it can be concluded that most of the students who participated in the study had not established the logical structure concerning the relationship between the index and the parameter. In particular, the poor performance of the students on interpreting the uniqueness of the index indicates that they did not fully comprehend the role of the existential and universal quantifiers in the definition, and as a consequence, they could not establish a firm understanding as to the dependence of the index on the parameter. This issue is echoed in several the studies in the literature (Cory & Garofalo, 2011; Kidron & Zehavi, 2002; Pinto & Tall, 2002).

The difficulties that the students experienced in the algebraic context in linking the parameter and the index were also observed in the graphical context. For instance, in the final question on the test, the students were asked to reason about the dependence of the index on the parameter based on the graphical data of a convergent sequence represented as a function from the set of natural numbers to the set of real numbers. Similar to their performance on the preceding questions, the vast majority of the students could neither interpret the uniqueness of the index nor find the value that the index can take on based on the graphical data. This pattern of performance indicates that the students' failure in this regard was not a consequence of lack of ability in manipulating and solving algebraic inequalities, but a lack of conceptual understanding regarding the relationship between the index and the parameter. As a result, they resorted to irrelevant information in providing answers to the questions. To sum up, the majority of the students failed to develop a sound understanding regarding the formal definition in the graphical context. However, Mamona-Downs (2001) argues that graphical representations of sequences help students to recognize that a sequence is a function and thus, they provide a valuable medium for examining the relationship between the index and the parameter. Based on this, we recommend that future research focus on designing teaching that exploits the potential of the graphical context to provide students with the opportunity to recognize the logical structure within the formal definition.

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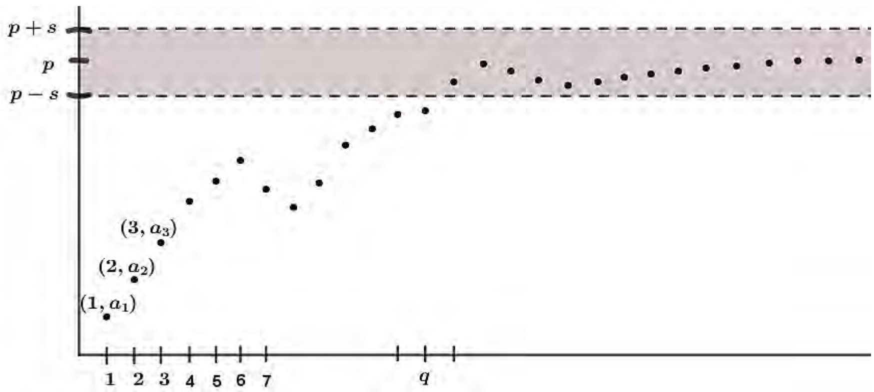
## Appendix 1

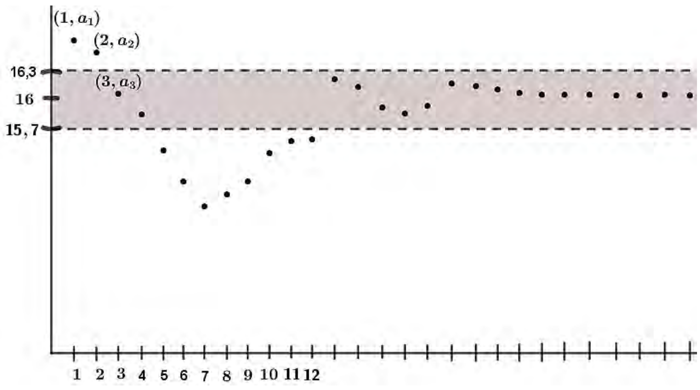
**1)** State the formal definition of the limit of a sequence. If you do not remember the definition, try to describe it in your own words.

**2)** $(a_n)$ is a sequence and $\lim_{n\to\infty} a_n = L$. The graph below depicts the terms of sequence $(a_n)$, which is defined via the function $f: N \to R$, $f(n) = a_n$. Match the variables $\varepsilon, L$ and $n_0$ in the formal definition with the given symbols.




| | |
|---|---|
| $\varepsilon$ | $p$ |
| $L$ | $q$ |
| $n_0$ | $s$ |

**3)** The graph to the right depicts the terms of a sequence $(a_n)$, which is defined via the function $f: N \to R$, $f(n) = a_n$. Which one of the algebraic expressions listed below represents the behaviour of the terms of the sequence?




a) $n > 2 \Rightarrow |a_n - 16| < 0{,}3$

b) $n > 16 \Rightarrow |a_n - 12| < 0{,}3$

c) $n > 12 \Rightarrow |a_n - 16| < 0{,}3$

d) $n < 12 \Rightarrow |a_n - 16| > 0{,}3$

**4)** $Let\ (a_n) = \frac{3n^2+5}{n^2}$ and $\lim_{n\to\infty} a_n = 3$ is known. Which one of the algebraic expressions listed below states that the limit of the given sequence is 3?

a) $\forall\varepsilon > 0\ \exists n_0 \in N\ such\ that\ \ if\ n > n_0\ then\ |a_n - 3| < \varepsilon$

b) $\forall\varepsilon > 0\ \ \exists n_0 \in N\ such\ that\ \ if\ \ n_0 > n\ then\ |a_n - 3| < \varepsilon$

c) $\forall n_0 \in N\ \ \exists\varepsilon > 0\ such\ that\ \ if\ n > n_0\ then\ |a_n - 3| < \varepsilon$

d) $\forall n_0 \in N\ \ \exists\varepsilon > 0\ such\ that\ \ if\ \ n_0 > n\ then\ |a_n - 3| < \varepsilon$

**5)** $\lim_{n\to\infty} \frac{2n-3}{n} = 2$ is known. Find an appropriate value of $n_0$ that satisfies the condition in the formal definition for $\varepsilon = 0{,}1$ . Is there only one appropriate value of $n_0$ for the given $\varepsilon$? Please justify your answer.

**6)** $\lim_{n\to\infty} \frac{2n-3}{n} = 2$ is given. What is the minimum value that $n_0$ can take on if $\varepsilon = 0{,}1$?

Which question or questions listed below has/have the same meaning as the above question?

a) What is the position of the first term such that the distances of the following terms to 2 are less than 0.1?

b) What is the position of the first term such that the distances of the following terms to 2 are greater than 0.1?

c) What is the minimum value of $n_0$ such that if $n > n_0$ than $\left|\frac{2n-3}{n} - 2\right| < 0{,}1$?

d) What is the minimum value of $n_0$ such that if $n > n_0$ than $\left|\frac{2n-3}{n} - 0{,}1\right| < 2$?

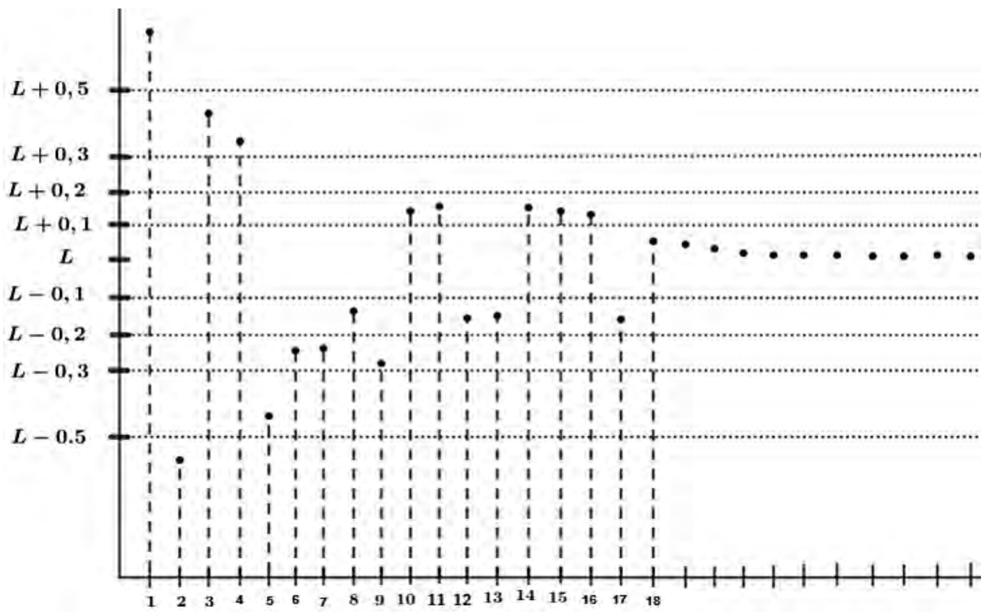
**7)** "$\forall\varepsilon > 0\ \ \exists n_0 \epsilon N\ such\ that\ if\ n > n_0\ than\ \left|-\frac{6n+3}{n-1} + 6\right| < \varepsilon$" Which one of the statements below corresponds to the given expression?

a) $(a_n) = -\frac{6n+3}{n-1}$ and the limit of $(a_n)$ is 6.

b) $(a_n) = -\frac{6n+3}{n-1}$ and the limit of $(a_n)$ is -6.

c) $(a_n) = \frac{6n+3}{n-1}$ and the limit of $(a_n)$ is -6.

**8)** The below graph depicts the terms of a sequence $(a_n)$, which is defined via the function $f\colon N \to R$, $f(n) = a_n$. Based on the data provided in the graph, answer the following questions.



a) Let $\varepsilon = 0{,}2$ . Find two appropriate values of $n_0$ such that the proposition "$n > n_0 \Rightarrow |a_n - L| < \varepsilon$" holds true. $n_0 =$…………

b) Let $\varepsilon = 0{,}3$ . What is the minimum value that $n_0$ can take on such that the proposition "$n > n_0 \Rightarrow |a_n - L| < \varepsilon$" holds true? $n_0 =$…………..

c) Let $n_0 = 5$ . Find two appropriate values of Ɛ such that the proposition "$n > n_0 \Rightarrow |a_n - L| < \varepsilon$" holds true.

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