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The Delayed Influence of a Teacher Education Program

A Longitudinal Study of Elementary Mathematics Teachers' Belief Development

Amanda G. Sawyer

Abstract

Some researchers found that preservice mathematics teacher education programs have little effect on elementary mathematics teacher beliefs. A preservice teacher education program can influence beliefs for some mathematics teachers. However, the question still remains as to what happens to these beliefs after the first 2 years of teaching. In this study, I interviewed and observed three elementary mathematics teachers 10 years after their preservice education program ended to investigate their current beliefs about the nature of mathematics, teaching mathematics, and learning mathematics and how these beliefs compared to those they held during their second year of teaching. The data show that teacher education programs might not initially influence individuals, but later, the individuals could become aware of their beliefs and take practices taught in teacher education programs into consideration. The data also support the finding that teacher education programs can have a lasting impact on these participants.

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Introduction

Policy makers in the United States are dealing with teacher shortages in every state as of the 2017–2018 school year (Strauss, 2017). To fill the teaching gap, changes to state policies were implemented that focused mainly on teacher education programs or alternative licensing programs (Aragon, 2016), despite that 51% of teachers reported leaving the workforce because of their workload and 53% reported leaving because of working conditions (Castro et al., 2018). Some states, such as Oklahoma, Utah, and Arizona, have gone as far as to allow teachers to be hired without any formal training (Strauss, 2017). This leaves the public asking if teacher education programs matter.

Some researchers have found that teacher education programs have little effect on preservice teachers' teaching practices and beliefs (e.g., Raymond, 1997; Scott, 2005). The researchers observed little to no change in the preservice teachers' beliefs at the end of their teacher education programs (Raymond, 1997) and found that after they enter the teaching profession, many teachers regress back to beliefs constructed prior to their teaching education programs (Scott, 2005; Swars et al., 2009). Spangler et al. (2012) found that a teacher education program could have some influence after 2 years of teaching, yet like the other authors mentioned, they have not studied their teaching program's longitudinal effects.

I studied three of the participants from Spangler et al.'s (2012) investigation to document their beliefs and elicit their perspectives of their teacher education program after 10 years of teaching. The purpose of this study was to determine these teachers' current beliefs about the nature of mathematics, teaching mathematics, and learning mathematics, as well as to discover the impact of their teacher education program over time. These teachers graduated from the same teacher education program with the same mathematics methods instructor and had participated in a prior study where only one participant identified her teacher education program as being influential after her second year of teaching. Through four interviews, two surveys, and three classroom observations, I examined the following questions:

1. What are these elementary teachers' beliefs about the nature of mathematics, teaching mathematics, and learning mathematics after 10 years of teaching?

2. How have these elementary teachers' beliefs about the nature of mathematics, teaching mathematics, and learning mathematics changed since their second year of teaching?

3. What was the impact of their teacher education program?

This study provides a bridge between the preservice teacher education research on belief change and the research focused on belief change conducted with in-service teachers. I gained insight into how the teachers formed their initial beliefs and teaching practices and what happened to their beliefs after their teacher education program.

Literature Review and Theoretical Framework

Pajares (1992) argued that "beliefs are the best indicators of the decisions individuals make throughout their lives" (p. 307). Thus beliefs are important constructs to study, and many researchers have investigated belief structures, systems of beliefs, and categorizations for identifying beliefs (e.g., Ernest, 1989; Leder et al., 2002). In this chapter, I begin by defining beliefs, belief structures, and belief systems. Then, I describe how I categorize beliefs in my study and provide an overview of the literature on belief change and factors influencing beliefs. I conclude this chapter by describing related literature that was found to be significant through my investigation.

Because there are a variety of definitions for *beliefs*, I used the definition offered by Philipp (2007): "psychologically held understandings, premises, or propositions about the world that are thought to be true" (p. 259). I selected this definition because it identifies beliefs' psychological aspects. Beliefs are individually constructed, yet they are influenced by social and cultural factors. By this definition, beliefs are individual psychological understandings about the social world that individuals find valid.

Beliefs Framework

I adopted Ernest's (1989) framework to classify beliefs. Ernest viewed teaching mathematics as dependent on three key elements: (a) the teacher's mental contents or schemas, particularly the system of beliefs concerning mathematics and its teaching and learning; (b) the social context of the teaching situation, particularly the constraints and opportunities it provides; and (c) the teacher's level of thought processes and reflection (p. 249).

As explained in Table 1, Ernest (1989) defined three views of the nature of mathematics (the instrumentalist view, the platonist view, and the problem-solving view) and three beliefs about mathematics teachers' roles (instructor, explainer, and facilitator). Ernest also identified two different teacher views on student learning: passive recipients of knowledge and active constructors of knowledge.

Ernest (1989) explained, "These three philosophies of mathematics, as psychological systems of belief, can be conjectured to form a hierarchy" (p. 250). In this hierarchy, the instrumentalist view is the lowest level, and the problem-solving view is the highest level. The beliefs about the nature of mathematics were found to have significant correlation to teachers' beliefs about mathematics teaching and learning. Ernest suggested that individuals who held a platonist view of mathematics would be more likely to enact an explainer's role in the classroom and to view learners as passive recipients of knowledge.

Teachers' Beliefs and Teaching Practices

Researchers have searched for ways to influence teachers' beliefs to change teaching practices (Ambrose, 2004; Philipp, 2007; Stuart & Thurlow, 2000; Swars et al., 2007). Special mathematics methods classes have been studied to determine belief change (Philipp, 2007; Stuart & Thurlow, 2000), as have specified field activities and failed teaching experiments (Ambrose, 2004). From these studies, the authors have come to some conclusions about how to affect teachers' beliefs.

First, teachers need to reflect on their current beliefs for a change to occur (Cooney et al., 1998; Kagan, 1992; Stuart & Thurlow, 2000). Different activities have been investigated to help foster personal reflection and initial belief change for teachers (Kagan, 1992; Stuart & Thurlow, 2000). However Vacc and Bright (1999) explained, "It is not clear whether pre-service teachers' education programs can structurally accommodate these needed 'reflection events'" (p. 107).

Second, beliefs are relatively stable and take significant time to be influenced (Cooney et al., 1998; Swars et al., 2007). Swars and her team of researchers determined that change was most significant after the second methods course, and preservice teachers either kept those beliefs or regressed to earlier beliefs during their student teaching (Swars et al., 2009). Ambrose (2004) speculated that past beliefs might not be changed through teacher education programs, but rather individuals added new beliefs to their belief structure. Scott (2005) found that preservice elementary

Table I Adopted Ernest (1989) Belief Classifications

Beliefs	Belief classifications
Mathematics	<i>Instrumentalist:</i> Mathematics is a set of unrelated facts and rules used to solve problems.
	Platonist: Mathematics is a unified body of knowledge discovered by man.
	<i>Problem solving</i> : Mathematics is a continually changing body of knowledge created by man.
Teacher's role	<i>Instructor</i> : Teachers facilitate the mastery of skills through corrected actions.
	<i>Explainer</i> : Teachers facilitate conceptual understanding by modeling unified knowledge.
	<i>Facilitator</i> : Teachers facilitate problem solving though selected questioning.
Learning	<i>Passive reception of knowledge</i> : Students replicate specified skills and behaviors through modeling.
	<i>Active construction of knowledge:</i> Students work with the mathematics to explore and construct understanding of different topics.

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teachers at the beginning of their programs had similar experiences with traditional teaching practices in school as their graduating counterparts, but the graduating students had a greater likelihood of wanting to learn about and build on children's mathematical experiences. When practice and theory clashed, preservice teachers tended to be influenced by sources offering practical advice, for example, practicing teachers (Scott, 2005). Forgasz and Leder (2008) explained,

In many of the reports which contain positive accounts of functional changes in the prospective teachers' beliefs it was nevertheless concluded that the extent to which these changes would eventually be translated into practice in classrooms could only be a matter of speculation. (p. 179)

Third, the research identified, after preservice teachers became practicing teachers, that they encountered many obstacles. For example, teachers' mathematical knowledge (Halai, 1998), students' classroom behavior (Steele, 2001), preconceived notions about student needs (Sztajn, 2003), and teachers' everyday duties (Quinn & Wilson, 1997) were found to impact teaching practices for in-service teachers. Overall, the researchers found that primary teachers who did experience belief change were subject to reflection on their own teaching practices, which did promote teachers to enact their beliefs in their classrooms (Clarke, 1997; Senger, 1998).

Fourth, teacher educators generally strive to help preservice teachers experience mathematics learning in ways consistent with the reform movement in the hope of influencing their beliefs about mathematics teaching and learning. However, beliefs are not easily affected by teacher education programs (Hiebert et al., 2002; Hiebert et al., 2003; Raymond, 1997). Hiebert et al. (2003) described their goals for teacher education as preparing preservice teachers to learn to teach for mathematical proficiency. They explained, "Even if the current knowledge base identified the complete set of skills and dispositions for effective teachers, it is unlikely that prospective teachers could acquire these competencies in a relatively brief preparation program" (p. 205). McDiarmid (1989) constructed a course to have students reconsider their beliefs, but he stated,

Despite abundant evidence that prospective teachers do reconsider their initial beliefs and orientations, that they begin to understand the folkways of teaching they have learned are not merely unreflective but, in some respects, downright damaging, I am skeptical about the effects of the course. (p. 20)

Hiebert et al. (2002) explained one reason for this disconnect between practices reinforced in teacher education programs and practices implemented in schools as coming from a lack of communication between educational researchers and school practitioners. Because of this disconnect and lack of change in traditional beliefs, many researchers found that teacher education programs had a minimal impact on preservice teachers' beliefs (e.g., Hiebert et al., 2002; Raymond, 1997).

Methodology

To determine the impact of a teacher education program, I conducted a longitudinal case study on three participants, looking into their beliefs after 10 years of teaching. Literature has stated that individuals are not aware of their beliefs, so researchers must interpret participants' understandings using multiple strategies to ensure accurate representations of their views (Cooney et al., 1998; Pajares, 1992; Rokeach, 1968). Using methodological triangulation of the three interviews, one focus group interview, two surveys, and three classroom observations, I investigated the individuals' beliefs and factors influencing those beliefs.

Context

I sampled my participants from a previous study on teachers' beliefs titled "Learning to Teach Elementary Mathematics." In the previous investigation, two cohorts of elementary education students (15 total) were followed through 2 years of their teacher education program and into their first 2 years of teaching. Across the initial 4 years of the study, the teachers participated in interviews and class-room observations and completed the Integrating Mathematics and Pedagogy (IMAP) belief survey (Philipp et al., 2007). From the "Learning to Teach Elementary Mathematics" study, Spangler et al. (2012) identified that pedagogical coursework and field experiences changed some preservice teachers' beliefs. I determined if this change lasted over time and what factors contributed to maintaining or changing these beliefs.

I became a researcher on the initial investigation after the data collection process was completed. Because my interaction with the data did not include meeting the participants, I maintained separation from the original study. It must be noted that the original study was not discussed with the participants during any part of this investigation, and if the participants made a connection with the original study, they did not state or identify it to the researcher. This investigation began in their 10th year of teaching, and the previous investigation ended at the end of their 2nd year of teaching; therefore no data were collected between their 2nd and 10th years of teaching.

Participants

I chose three of the 15 participants with pseudonyms Laura, Jayne, and Jennifer from the original study to investigate how their beliefs had changed since the end of the initial study. As shown in Table 2, I selected Laura, Jayne, and Jennifer because they displayed three different patterns of belief development during the prior study (Spangler et al., 2012).

The three participants were White females in their early 30s who had taught for at least 10 years, were teaching in schools in the same southeastern state at the time of this study, and all had the same mathematics methods instructor, with pseudonym Dr. Mathis. Both Laura and Jayne were currently teaching first grade in two different school districts, both of which were identified as Title I schools. Jennifer was currently teaching fifth grade at a non–Title I school. None of their schools could be considered rural (i.e., serving fewer than 25 students per square mile; Georgia Department of Education, 2021).

Data Collection

I collected data on each individual by conducting (a) three face-to-face interviews, (b) one focus group meeting with all participants, (c) three 1-hour classroom observations, (d) the IMAP belief survey (Philipp et al., 2007), and (e) the Known Factors Affecting Mathematical Belief Change (KFABC) survey. As seen in Table 3, after each item was collected, I analyzed the materials to help inform the next data collection stage to produce the most accurate beliefs for all participants.

Data collection proceeded as follows. First, I conducted an initial hour-long classroom observation of each teacher and the first interview. The purpose of the first interview was to elicit the teacher's current beliefs. In the first interview, I asked nine different questions, each related to the individual's mathematical beliefs. I used two of Cooney et al.'s (1998) similes to help my participants articulate their beliefs. Each teacher was asked to complete the following sentences: "Teaching mathematics is like _____" and "Learning mathematics is like _____"." Per Cooney et al., I was interested in the participant's explanation of each simile rather than the particular simile she picked. The last seven questions were based on questions from my pilot study investigating the beliefs of an experienced elementary teacher. (Please see Sawyer, 2017, for more details on the construction of these questions.) During the classroom observations, I focused on the richness of the mathematics, the role

Table 2

Participants' Beliefs From Spangler et al. (2012)

Participant			
Stage	Laura	Jayne	Jennifer
Belief about the nature of mathematics Initial Second year	instrumentalist platonist	platonist problem solving	instrumentalist instrumentalist
Belief about mathematics learning			
Initial	passive	active	passive
Second year	active	active	active
Belief about mathematics teacher's role Initial Second year	instructor facilitator	explainer facilitator	explainer explainer

each participant took as a mathematics teacher, and the role the students took in the classroom (Hill et al., 2008). I identified the practices the participants enacted in their classrooms to determine their aligned beliefs.

Next, I asked each participant to complete the IMAP survey. The IMAP (Philipp et al., 2007) is a web-based survey including video clips, open-response questions, and written teaching episodes. The survey allows for branching based on a participant's early responses to capture more fine-grained beliefs than would be possible by administering an identical instrument to all participants. The IMAP survey contains 16 items and takes approximately 1 hour to complete. I scored participants' responses using the rubric the survey developers provided to describe the participants' beliefs about the nature of mathematics, teaching mathematics, and learning mathematics (Philipp et al., 2007).

Then, I asked each participant to complete the KFABC survey. I developed the KFABC survey to collect data about the participants' backgrounds through a pilot study unrelated to the participants' first investigation. Research in mathematics education identified some influences affecting belief development (Raymond, 1997; Richardson, 1996), and I coupled these findings with a pilot study I conducted to create the survey. For example, the participants' teaching experiences were discussed using the following questions from the survey:

1. How have your students influenced the way you view how people learn mathematics?

2. How did your school administration affect the way you teach?

Table 3 Research Se	quence
Date	Sequence of research
August	 Conducted initial interview for each participant Conducted initial classroom observation for each participant Had each participant take the IMAP Belief Survey Had each participant take the KFABC Survey Analyzed KFABC Survey and constructed second interview protocol from responses
September	 Conducted second interview for each participant Conducted second classroom observation for each participant Analyzed second interview and construct focus group interview protocol from responses
October	 Conducted focus group interview Conducted third classroom observation for each participant
November	 Analyzed all the data and constructed interpretation of each individual's beliefs and factors affecting beliefs
December	 Emailed each participant interpretation of their data Conducted the final interview based on participant responses to interpretations

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To ensure validity and reliability, I tested the instrument with 12 former teachers and had the group help to analyze the data to determine if the instrument collected the data it was designed to collect. The KFABC survey is a web-based, 20-item open-response questionnaire that takes approximately 1 hour to complete.

From the responses on the KFABC survey, the second interview was created to focus on the teachers' explanations of their influences. I also used their responses to the KFABC survey during the focus group interview. I asked the participants about similarities I found, and all three participants identified in their surveys that their teacher education program was an influence. I probed how they interpreted these experiences during the focus group. I was also interested in how they responded to each other's comments about factors affecting their change over time to see how they would respond to their shared experiences.

Finally, I conducted the third interview as a member check. By giving my participants my initial interpretation of the data, I confirmed the validity of my findings. After the participants read the stimulus text, they confirmed my interpretation of their beliefs or suggested modifications, either through a telephone interview or by email.

Data Analysis

All interviews were fully transcribed for analysis, and all surveys and field notes from classroom observations were typed for analysis. These documents were loaded into HyperRESEARCH (Hesse-Biber, 1993) for ease of coding (see Figure 1). Then, I used two analysis techniques. I followed the categoriza-

Figure I Example of HyperRESEARCH Coding

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tion method from Spangler et al. (2012) to stay consistent across studies. The participants' beliefs about the nature of mathematics, learning mathematics, and teaching mathematics were categorized using Ernest's (1989) framework. I coded data related to beliefs about the nature of mathematics, teaching mathematics, and learning mathematics for each participant. Leatham (2006) argued that beliefs are constructed in a sensible system for each teacher even if they do not appear sensible to an outsider, so when I found a contradiction between a person's beliefs and practices, I continued investigating to better understand the participant's perspective until I understood its fit with Ernest's categories, as seen in Table 4.

Once I determined possible beliefs, I wrote a 5- to 10-page summary of each participant's beliefs. I asked each participant to read the summary and indicate whether it reflected her views. The participants generally said that the summaries were an accurate reflection of their views, and I only changed one participant's word choices from her interviews at her request.

Limitations

The study was limited by the number of participants and the kinds of data collected on the participants. First, out of the 15 initial participants in the original study of beliefs, only three teachers were selected for this study. The three teachers were contacted by Dr. Mathis initially to determine if they would be interested in participating, so they were aware of their former methods instructor's involvement in the investigation, which may have influenced their responses to my data collection.

Table 4 Coding Method

5		
Code	Characteristic of code	Example
Belief about mathematics teacher's role	When a participant describes how a mathematics teacher should teach concepts to their students	I would sit at that podium, and I would speak the truth of math. Then they were like a little congregation.
Belief about nature of mathematics	When a participant describes mathematics as a subject	Math is like a book. There is a way to do it right and you can go to the book to find out how.
Belief about learning mathematics	When a participant describes the way individuals should learn	Practice. Students only learn through doing the math through practice.

Findings

Laura

Since graduating from college, Laura had taught in the same district in two different schools. She taught at an elementary school for four years in fourth grade, and she taught at a primary school for six years, five of which were in first grade and one of which was in kindergarten. During that period, she earned her master's degree in early childhood education, got married, and had two children. Spangler et al. (2012) found that Laura experienced the largest belief change between her preservice teacher education program and her second year of teaching. Laura's beliefs were more stable across the next eight years, although she did experience some slight changes. Laura identified her teacher education program as having had the greatest influence on her beliefs over time, and she attributed her continued belief change through her 10 years of teaching to the program and her methods instructor. Table 5 provides an overview of Laura's beliefs over the span of both studies.

Laura's Beliefs About the Nature of Mathematics

At the time Laura was starting her teacher education program, Spangler et al. (2012) characterized her as having an instrumentalist view of mathematics, which is a set of rules to be memorized, because of her previous school experiences. After 2 years of teaching, Laura displayed a platonist view of mathematics, explaining that from her mathematics content courses, she learned how mathematics was developed, and from working with students, she realized that mathematics was more than rules—yet

Table 5 Laura's Beliefs Over Time	
Stage	Belief
Belief about the nature of mathematics Initial 2nd year 10th year	instrumentalist platonist problem solving
Belief about teaching mathematics Initial 2nd year 10th year	instructor facilitator facilitator
Belief about learning mathematics Initial 2nd year 10th year	passive active active

she still believed that mathematics has a set structure (Spangler et al., 2012).

After 10 years of teaching, Laura held a problem-solving view of mathematics because she saw it not as a fixed body of knowledge but as a continually expanding field of inquiry. She expressed her view of mathematics by stating, "Conceptually, there is probably not an end, and with all the new developments in technology, there are a lot of places we never thought math was going to go." Laura saw how new mathematics was continually created through technological development, and she wanted her students to be able to explore mathematics in their own unique ways so they, too, could construct their own mathematical understandings.

Laura's Beliefs About Teaching Mathematics

Spangler et al. (2012) categorized Laura's initial beliefs about teaching mathematics as matching Ernest's instructor category because she emphasized the need to receive "correct answers" from her students. She explained that Dr. Mathis "opened her eyes" to a new way of understanding mathematics, thus changing her beliefs into her second year of teaching to what Spangler et al. (2012) classified as having a facilitator orientation about the teaching of mathematics.

After 10 years of teaching, Laura still believed in being a facilitator in the classroom. With time came self-reflection and awareness, and she was able to describe her past teaching practices, from which I inferred her beliefs about teaching. She stated,

At the beginning of my career, I was much more focused on standing up in the front and telling you the truth and you practicing it. And now I try to be more like OK let me introduce it, let you go out and try it, and then we will come back and see what you think about it.

Her teaching practice changed because her beliefs about teaching changed. Initially, she believed in telling students the concepts. After some reflection, she believed that for students to understand the concepts, they needed to experience "struggle time," allowing them to grapple with the mathematics first to help them understand it in a conceptual manner. She stated, "The first time that they get ahold of a concept I like to just kind of throw it out there and let them try it a little bit and see what happens." I observed "struggle time" during a whole-class lesson in which students had time to discuss their findings and ideas with their classmates to make sense of their problem. Thus Laura still believed in facilitating students' learning after 10 years of teaching.

Laura's Beliefs About Learning Mathematics

In the "Learning to Teach Elementary Mathematics" study, Laura initially showed little evidence of believing students should be actively engaged with mathematical ideas. When asked about her mathematical classroom in her second year of teaching, she explained, "I want a lot of hands-on, a lot of experimenting kind of things, and let them figure [out the problem] because that makes it so much more meaningful." She believed that students needed to actively construct knowledge.

After 10 years of teaching, Laura still believed that learning mathematics was an active process. When asked if she would describe learning as like watching a movie, she stated, "That is so passive. It would definitely be one of the more active verbs not just watching something." She believed mathematics instruction should be hands-on because "I think that is what they need at this age." In her class, she consistently provided manipulatives for students to touch and use to construct their understandings. For example, the Rekenrek was a manipulative used to help construct number sense by helping them solve the addition and subtraction problems on cards they were given.

Laura believed in students actively participating in learning mathematics and rejected the use of worksheets without any manipulatives. Parents would even ask her why their students did not come home with worksheets of mathematics problems. She explained, "I'm like, 'It is really much better, I promise.' It [the mathematics task in class] actually has pattern blocks instead of a picture of a pattern block." Therefore, after 10 years of teaching, Laura believed that students learn mathematics through active construction of knowledge.

Jayne

Jayne entered the teacher education program displaying a platonist view of the nature of mathematics, explainer views of teaching mathematics, and an active view of learning mathematics (Spangler et al., 2012). From her preservice experience into her second year of teaching, Jayne progressed in her beliefs, despite that little improvement was available to be made. She experienced little to no change in her beliefs over the next 8 years of experience because she reached the highest level in Ernest's classification after her second year of teaching, as shown in Table 6.

Since leaving her teacher education program, Jayne had taught first grade in the same school for 10 years, experiencing change in both her community and students. She was married, had two children, and earned a master's degree in early childhood reading and literacy. These life-changing events reinforced her previous beliefs about the nature of mathematics, teaching mathematics, and learning mathematics.

Jayne's Beliefs About the Nature of Mathematics

Spangler et al. (2012) initially categorized Jayne as holding a platonist belief about the nature of mathematics. Jayne viewed mathematics as an interconnected jigsaw puzzle: "You see bits and pieces of things, and you know they're all going to fit together, but it takes time for you to learn them first." In her second year of teaching, she emphasized that there were "different ways to solve problems," and

she assessed her students' understanding through multiple methods. After 2 years of teaching, Jayne demonstrated a problem-solving view of mathematics.

After 10 years of teaching, Jayne still held a problem solver's view of mathematics. She even again described mathematics as a jigsaw puzzle, but she no longer believed everything must fit together the same way. Jayne explained that "math is like a puzzle" with multiple pathways that students can construct to understand a problem. She explained, "I feel like the end result can come out to be the same, but the way we approach it or think about it would be different." She believed teachers should not stifle the creative process and saw this creativity in her students by observing the "way my children think differently."

Jayne designed activities in her class centering on problem solving. In an observation during her 10th year of teaching, Jayne taught a problem-solving lesson by asking her students to work in pairs to manipulate and construct a cube and a rectangular prism using marshmallows and toothpicks without explaining the process first. This is one example illustrating Jayne's problem solver's view of mathematics after teaching for 10 years.

Jayne's Beliefs About Teaching Mathematics

Jayne initially believed that it was the mathematics teacher's role to explain concepts to her students, and she continued to reinforce this belief in facilitating learning (Spangler et al., 2012). For example, she was observed in each of her classes asking questions to elicit higher-order thinking. Therefore, after her second year of teaching, Jayne adopted a facilitator role as a teacher (Spangler et al., 2012).

After 10 years of teaching, Jayne viewed teaching mathematics as a combination of modeling and facilitating, and the transition between the two corresponded

Jayne's Beliefs Over Time	
Stage	Belief
Belief about the nature of mathematics Initial 2nd year 10th year	platonist problem solving problem solving
Belief about teaching mathematics Initial 2nd year 10th year	explainer facilitator explainer/facilitator
Belief about learning mathematics Initial 2nd year 10th year	active active active

Table 6 Jayne's Beliefs Over Time

to when her students were developmentally ready. Jayne stated, "I don't think I can expect my kids to do something if I haven't modeled the appropriate strategy or given them the appropriate tools to solve the problem." The modeling process, which corresponded to the needs of her students, was necessary for both the mathematics and the instructions of the activities. She explained, "I feel like as the year progresses, I kind of start to cut the string, less of me modeling, and more of you reading and trying something first." She demonstrated her belief in facilitating learning during the lesson using the toothpick and marshmallow constructions because she did not model how to construct the figure. In another observation, she modeled how to measure items first as part of her lesson on using nonstandard units. After 10 years of teaching, Jayne believed teachers needed to explain and facilitate learning for their students.

Jayne's Beliefs About Learning Mathematics

Jayne's beliefs about student learning stayed constant throughout her mathematics education program and into her second year of teaching. She believed students needed to actively construct knowledge of mathematics to understand the concepts (Spangler et al., 2012). Jayne consistently held the same belief, after 10 years of teaching, that students learn mathematics through active participation. She had her students using tools to understand the mathematical concepts in each lesson I observed. For example, in her last observation, the students constructed a unit of measurement from their own foot to measure objects around the room. Jayne stated, "I just feel like a picture in a book where they have to measure the beads as opposed to actually measuring it themselves, there is no comparison. I just don't feel like it is meaningful to my kids." By having the students construct a unit of measurement and then practice measuring items, she believed her students were able to develop understanding of measurement.

Jennifer

Compared to Laura and Jayne, Jennifer's belief change was significantly different. Jennifer experienced little influence on her initial beliefs from her preservice teaching experience. After she started teaching, Jennifer began to reflect on her beliefs and define what she believed to be best practices for her students, as shown in Table 7.

Over the 10 years, Jennifer experienced various events shaping her views on the nature of mathematics, teaching mathematics, and learning mathematics. She lived in three different states and taught elementary school in second through fifth grades. She was married, had a child, earned a master's degree in curriculum and instruction, earned a specialist designation in educational leadership, experienced job transfers, was divorced, and moved back to her home state.

Jennifer's Beliefs About the Nature of Mathematics

Jennifer initially held an instrumentalist view of mathematics and believed it to be a set of rules that she must transmit to her students (Spangler et al., 2012). Jennifer's view of mathematics did not change after she started teaching. In her second year of teaching, she explained how she taught items by checking them off a list, showing that her belief that mathematics consists of rules still remained. Spangler et al. reported that Jennifer stayed consistent with her instrumentalist views throughout the initial 4-year study.

After 10 years of teaching, Jennifer experienced a drastic change in her beliefs, resulting in her holding a problem solver view of mathematics. She saw mathematics as continually changing by her students being able to solve problems in "unique and creative ways." She described her belief by defining what she did not believe, showing her reformed views. She explained, "Cooking with a recipe? Definitely not, because that is like the whole algorithm. You do it this way: step 1, step 2, step 3, step 4. That is not the way that I think of math." She enacted this belief by implementing problem-solving activities in the classes I observed. From her reflection, she explained how she considered that her methods courses "forced us to really think about math—not just accept the algorithms." After teaching for several years, she understood why the methods courses challenged her beliefs about the nature of mathematics and "forced" her to view mathematics not as a set of rules but rather as a problem-solving process.

For example, in the first lesson I observed in her 10th year of teaching, Jennifer had her students find the most cost-efficient way of sodding a field, as shown

Jennifer's Beliefs Over Time	
Stage	Belief
Belief about the nature of mathematics Initial 2nd year 10th year	instrumentalist instrumentalist problem solving
Belief about teaching mathematics Initial 2nd year 10th year	explainer explainer facilitator
Belief about learning mathematics Initial 2nd year 10th year	passive active active

Table 7 Jennifer's Beliefs Over Time

in Figure 2. The students were given this task to explore the mathematics, and Jennifer introduced the activity by saying, "I didn't give you how I would solve the problem because you might think of it differently than how I would like to do it." Consequently, the students were able to take risks and use their problem-solving skills to solve the problem.

Jennifer's Beliefs About Teaching Mathematics

Jennifer held an explainer view of teaching during her junior year of college, and her beliefs stayed constant through her teacher education program and second year of teaching (Spangler et al., 2012). However, after 10 years of teaching, Jennifer experienced a change of beliefs about teaching mathematics. Jennifer initially admitted that when she first started teaching, she would ask other teachers what she needed to do for her students, but she later came to realize the importance of the information she had been taught in her program. The methods instructor modeled teaching techniques that "inspired" Jennifer to change how she viewed mathematics teaching, consequently influencing her teaching practice. Jennifer explained, "It made me realize if I just stand up there and explain it, there are a few kids that will learn that way, but the majority of the students will not learn in that way."

Jennifer acknowledged her change and identified that she no longer believed in being the bearer of knowledge. She displayed this belief in the lessons I observed. For example, during the first observation, she wanted her students to come

Figure 2	
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Jennifer's Task From Observation 1

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Note. From Georgia Department of Education, CCGPS Frameworks Student Edition (2013), https://www.georgiastandards.org/Georgia-Standards/Frameworks/5th-Math-Unit-1.pdf, p. 76.

up with their own ways of solving the problem, so she purposefully did not show them any examples of how to solve the task. She explained, "I am giving them the instruction, but they are the ones that are actually doing the hard work." She viewed teachers as coaches, and she believed that students should do the mathematics. She jokingly stated that her undergraduate education did have a significant impact on her beliefs, but "it just took me 10 years to get there." After 10 years of teaching, Jennifer viewed the mathematics teacher's role as a facilitator of mathematics.

Jennifer's Beliefs About Learning Mathematics

Jennifer's initial view of students' learning was consistent with Ernest's classification of students being passive recipients of knowledge. She based this view on her own learning during her K–12 schooling experience. Her teachers treated her as a passive recipient of knowledge; thus she believed that was how you teach. Through her teacher education courses, Jennifer understood that students need to know why things happen in mathematics. Jennifer equated active learning with using manipulatives, which also aided in her desire to make math fun. By her second year of teaching, Spangler et al. (2012) viewed Jennifer as moving toward viewing students as active learners who need interaction to learn.

After 10 years of teaching, Jennifer came to see manipulatives as a way for students to develop and demonstrate conceptual understanding, and she defined active learning as engaging with conceptual mathematical ideas. Jennifer acknowledged that this belief about active learning initially stemmed from her interest in making mathematics fun. However, now, she explained, "I want math to be fun, and I want the kids to enjoy class, but my goal is not to entertain them. My goal is to help them learn." She reached this goal of learning by building "a solid foundation, which in my mind is number sense." Jennifer believed students needed to engage with the manipulatives to build a conceptual understanding of the mathematical concepts rather than memorizing patterns or formulas. After 10 years of teaching, I categorized Jennifer as still believing in actively engaging students in mathematics.

Mathematics Teacher Education Program Influencing Beliefs

When asked to rank the events that were most influential to their belief development, the participants ranked their mathematics teacher education program as first, explaining that it had the largest impact on their beliefs about teaching and learning mathematics.

Influencing Beliefs on Nature of Mathematics

Once the participants took their content and methods courses in their math-

ematics teacher education program, they identified the instruction as causing them to "challenge" their previous views on mathematics. As Laura explained,

When I was in her [Dr. Mathis's] class, I recognized that I was thinking a different way about the algorithm, and I do remember constantly thinking in her class, "Yeah right. I am probably never going to use manipulatives in the fourth grade because no one uses base 10 blocks in fourth grade." But, I did buy them and did use them because my kids were not learning the way Saxon said magically should be happening.

These courses allowed them a place to reflect about their past understandings of mathematics fostered by their past teachers and textbooks. Through the reflection and content of the methods courses, the participants changed how they viewed the subject.

Influencing Beliefs on Teaching Mathematics

The three teachers said their mathematics methods courses introduced them to "how teaching was supposed to be done," influencing their beliefs about teaching mathematics. Jayne additionally stated that Dr. Mathis's courses were the only classes from which she could still use the materials today. She explained, "I always looked forward to getting out of my other classes to make it to her class. It was like one of the few classes where I felt as if it was not a waste of my time. It was like 'Now I am going to learn something." Dr. Mathis challenged their ideas of teaching and made them think about what they were doing in a new ways, which influenced their beliefs about teaching mathematics.

Influencing Beliefs on Learning Mathematics

The mathematics teacher education program also influenced their beliefs about learning mathematics. The method courses reinforced their beliefs about the active construction of knowledge by demonstrating how to use manipulatives to teach mathematics conceptually. Their methods instructor was identified as helping to reinforce how to build students' conceptual understanding of mathematical topics. After they became full-time teachers, the participants implemented many of the activities they learned in those courses.

Discussion

The participants' teacher education program had a long-term influence on their beliefs. The preservice teachers experienced two mathematics methods courses and three content courses, and the lessons learned from these experiences were lasting.

Delayed Influence of Teacher Education Program

Some researchers observed little to no change in preservice teachers' beliefs

at the end of their teacher education programs (e.g., Raymond, 1997; Scott, 2005) and found that after they entered the teaching profession, many teachers regressed back to beliefs constructed prior to their teacher education programs (Scott, 2005; Swars et al., 2009). However, this study suggests that teacher education programs can influence teachers' beliefs years later, even if their programs did not initially influence them.

Raymond (1997) speculated that teacher education programs might not initially have a large influence on beliefs. My participant who initially did not find her teacher education program as influential demonstrated that teachers might need the gift of time to allow them to reflect and become aware of their beliefs. Each preservice teacher filtered her teacher education program through her personal perceptions, influencing how she translated events, which was consistent with Grant et al.'s (1998) findings. Thus some teachers might need more experiences with students to highlight how their beliefs might be contradictory to the realities of teaching. My participants' constructions of beliefs suggest that more time might be needed to fully observe a belief change in teachers.

Long-Term Influence of Teacher Education Program

Across their 10 years of teaching, participants either kept the beliefs they constructed during their teacher education program or progressed in their beliefs because of information learned during the program. The participants noted that they still implemented activities and lessons advocated in their mathematics methods classes throughout their 10 years of teaching. They explained that the beliefs they constructed during those courses still held true. While preservice teachers might gripe about the mathematics content courses because they require preservice teachers to explain their thinking on and understanding of the foundations of mathematical concepts, they found the courses to be an extremely valuable experience, helping them have a long-term belief change about the nature of mathematics.

Implications

Teacher education's influence over time can have specific implications for the mathematics education community as well as policy makers. Because teaching allowed the participants to reflect on their past experiences in light of what they learned in their teacher education program, I suggest that beginning teachers need to be given support throughout their induction years to reflect on what they have learned. Beginning teachers need assistance through their first years of teaching to build confidence in their abilities and to reflect on their beliefs. Therefore mathematics teacher educators should continue to focus on teachers' belief changes after the teachers complete their teacher education programs.

Policy makers also need to pay attention to this long-term influence. Much debate has centered on the effectiveness of teacher education programs and

whether alternate certification programs, such as Teach for America, could be more effective (Ballou & Podgursky, 2000; Darling-Hammond et al., 2005). The debate has gone as far as policy makers restructuring teacher certification systems to deemphasize education training and to make student teaching and education coursework optional in some states (Aragon, 2016). However, few longitudinal studies of teacher education programs or alternative certification programs are available to provide evidence for policy makers on how influential these programs can be for teachers. This study provides evidence that teacher education programs can have a long-term impact on their graduates. Additional studies of this nature, of both standard and alternative certification programs, are warranted before decisions are made about eliminating particular aspects of teacher education programs. The question remains whether the longitudinal effects seen from my participants' teacher education programs could be replicated with other certification programs. Also, the long-term effectiveness of alternative certification programs could be difficult to determine due to the high turnover rate (Benner, 2000). Therefore policy makers should become aware of the longitudinal effects of these programs and the influences they have on future teachers before making far-reaching decisions on teacher education programs.

Conclusion

From the data, I concluded that teacher education programs could have a delayed influence; thus researchers should look past the first 2 years of a teacher's career to determine if their teacher education program is influential. This raises new questions about past research studies on teacher education programs. Would researchers be able to find their programs to be influential in changing beliefs if they had looked at their participants across their careers? In conclusion, this research reinforces the argument that teacher education programs matter and that we need these programs to influence future educators.

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