Preschoolers' ways of experiencing numbers

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In this paper we direct attention to 5–6-year-olds' learning of arithmetic skills through a thorough analysis of changes in the children's ways of encountering and experiencing numbers. The foundation for our approach is phenomenographic, in that our object of analysis is differences in children's ways of completing an arithmetic task, which are considered to be expressions of their ways of experiencing numbers and what is possible to do with numbers. A qualitative analysis of 103 children's ways of encountering the task gives an outcome space of varying ways of experiencing numbers. This is further analyzed through the lens of variation theory of learning, explaining why differences occur and how observed changes over a prolonged period of time can shed light on how children learn the meaning of numbers, allowing them to solve arithmetic problems. The results show how observed changes are liberating new and powerful problem-solving strategies. Emanating from empirical research, the results of our study contribute to the theoretical understanding of young children's learning of arithmetic skills, taking the starting point in the child's lived experiences rather than cognitive processes. This approach to interpreting learning, we suggest, has pedagogical implications concerning what is fundamental to teach children for their further development in mathematics.

Keywords: arithmetic, numbers, phenomenography, preschoolers, variation theory

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1 Introduction

Research on early mathematics education from the last four decades provides multiple observations of children solving arithmetic problems, offering a comprehensive picture of the strategies children use and the common trajectory of arithmetic skills development (see e.g., Baroody, 1987; Baroody & Purpura, 2017; Fuson, 1992). This body of knowledge has influenced many curricula and guidelines for teaching mathematics in the early years (Cross et al., 2009; Sarama & Clements, 2009). What has still not been revealed, however, is what children explicitly learn when developing a more advanced understanding of numbers that becomes useful in their arithmetic problem-solving. This calls for taking an educational perspective in interpreting children's arithmetic skills. In this paper, we aim to contribute to filling this knowledge gap by offering an alternative approach to describing children's learning, starting from how numbers appear to them (in a phenomenological sense; see Marton & Neuman, 1990). This, we suggest, has implications regarding what is fundamental to teach for children's mathematical development. More specifically, we aim to answer the following research questions:

1. How can preschool children's ways of experiencing numbers in an arithmetic task be described?
2. What distinguishes changes in their ways of experiencing numbers over time?

These questions are answered through empirical research involving 5–6-year-old preschool children solving an arithmetic task characterized as a "missing addend problem". Through the lens of phenomenography and the variation theory of learning (Marton, 2015), we analyze the children's encounter with numbers and how their ways of experiencing numbers differ and change over time.

1.1 Learning arithmetic skills in the early years

There are many documentations of children's strategies in arithmetic problem-solving and the trajectory of these strategies. Baroody (1987) is a researcher who is often referenced, describing the development starting with counting skills closely connected to physical countables (enumeration) and consecutively integrating other skills of significance for arithmetic problem-solving (such as the cardinality principle and the succession of numbers on the number line). In line with this way of describing development, when more skills are mastered, this allows the child to make use of mental representations such as the number line, without having to construct the number sequence every time by counting from one, alleviating the cognitive load. Crucial in the development of arithmetic skills is presumably the child's ability to construct units and an understanding of numbers as compositions of units in a part-whole relationship (Baroody, 2016). Nevertheless, there are empirical observations of children who do not develop these mental representations and continue constructing numbers by counting single units in all tasks they encounter. In other words, they do not invent efficient arithmetic strategies in which number relations (based on the composition of units) can be applied (Neuman, 1987; Ellemor-Collins & Wright, 2009).

The research based on cognitive science primarily emphasizes counting as the foundation for arithmetic development (Baroody, 1987; Fuson, 1992). However, an overemphasis on counting strategies may delay children's development of more advanced mathematical skills, according to Cheng (2012, p. 30), because "preschool children who receive continuous encouragement when using counting strategies are reluctant to try the new more advanced decomposition strategy. ..., these children prefer to use such seemly easier and effortless counting strategy". It has been shown that low-achieving students, even when they are older, often rely on counting strategies, which suggests that counting skills on their own will not develop arithmetic understanding and efficiency in problem-solving (Ahlberg, 1997; Christensen & Copper, 1992; Neuman, 2013). Determining the quantity of a missing part or difference when changes in quantities occur that cannot be enumerated (because missing parts are unknown, not "visible"), particularly in larger number ranges, demands a sense of numbers as constituting a part-whole relationship (Baroody, 2016; Peters et al., 2012).

The need to experience numbers' part-whole relations was also confirmed in a recent study of preschool children's arithmetic skills, concluding that children knowing the cardinality of numbers (e.g., the last number word said when counting items one-after-the-other means the whole set of counted items) and ordinality (numbers have internal relations to one another: adding one makes the next number in the counting sequence) is not sufficient for solving even simple arithmetic tasks. It is only when children realize that numbers (in addition to their features of cardinality and ordinality) can be seen as a triad of related numbers that they are able to solve arithmetic tasks or compare sets without concrete countables available (Björklund, Marton et al., 2021). Thus, learning to solve arithmetic problems constitutes the development of a complex of skills that might not be explained by solely constructing mental representations. As we suggest in this paper, the child's perspective on numbers, and how numbers appear to them, may be a necessary addition to our knowledge of how children learn basic arithmetic skills.

There have been attempts to describe how numbers are understood by children, not least by Piaget (1976/1929), who described intellectual development as qualitative changes in perceptions. His seminal work (Piaget, 1952) concerns how children structure their experiences into knowledge. The structuring process, in Piaget's view, results in similarities and differences that constitute psychologically real entities. Such a psychological formal structure is assumed to be applicable to different concepts. However, Piaget's thesis has been criticized for not explaining why a child is then able to express an advanced conception of some phenomenon presented in one situation but fails to do so in another, even though the encountered task or concept seems to be similar (Smedslund, 1977). This way of describing children's development of number knowledge places the focus on the child rather than the child's experienced world. Starkey and Gelman (1982) present an alternative view to the Piagetian one, proposing that an early understanding of arithmetic is related to particular principles, and that the understanding of these principles proceeds through increasingly complex levels. This view involves the child perceiving not any phenomenon in general but rather numerical phenomena specifically. That is, attention to certain principles is necessary in order for numbers to be understood (see Gelman & Gallistel, 1978). What the research mentioned above focuses on is how knowledge is constructed and transformed in children's development, from less effective toward more effective and valid knowledge.

The interest in understanding children's qualitatively different ways of understanding numbers can be found in several contemporary studies, while taking different theoretical perspectives to interpret what numbers mean to children. For example, Lavie and Sfard (2019) describe the development of children's reasoning with quantities over a prolonged period of time and conclude that number words are indeed commonly used but bear different meaning in children's problem-solving. That is, number words are not necessarily used for enumerating but often instead for estimating and comparing in a sense of "more or less". Already in the 1980s, Neuman (1987) presented a study of children's number knowledge with a strong emphasis on how numbers' meaning appears to them. Based on interviews with school-beginners, she aimed to theorize children's ways of creating concepts of numbers and described this as a trajectory starting from the prenumerical, moving through the early numerical, and ending in numerical concepts. *Prenumerical concepts* are expressed in children's intuitive or learned gestalts of quantities, known as subitizing or recognizing patterns of, for example, two pairs making four, but if separated (spatially) the two pairs would be conceptualized as different. *Early numerical concepts* include several ways of attending to numbers, such as a primitive way of seeing number words as relating to quantities but lacking numerical meaning or making use of number words for "fair sharing", meaning that partitioning is focal to the child in an intuitive sense but the exact quantity (number) is irrelevant; thus, any number word is possible as an answer. Furthermore, some children show an understanding of numbers as "names", which means the number words are seen as names of objects: When adding 4+5 the child answers 5, as the fifth object is added, resulting in the last said counting word being the answer. According to Neuman, this indicates that the number concept is purely ordinal in character and that 5 does not include 4 or any smaller number but is rather a label given to a specific object. Some of Neuman's observations revealed that the size of a quantity is primary to a discrete number of items, a category called "estimates". Number words, then, rather mean "much" or "a little", and there are no computational strategies related to this number concept. Neuman states that these early numerical conceptions are integrated toward an indissociable meaning of cardinality and ordinality, thus leading to numerical concepts. *The numerical conceptions* Neuman found in her studies were "structuring" and "counting", which allow the child to determine the answer to "how many" questions as an exact number of units. Counting is subordinate to structuring, however, as it is important to recognize and also be able to create patterns to represent part-whole relations. If the child's conception of number is restricted to the counting category, according to Neuman's studies this will lead to mathematics difficulties because numbers are then measured only in their smallest single units, which leads to difficulties in keeping track and the cardinal and ordinal meanings of numbers appearing as parallel, leaving the part-whole relation undiscerned. Here, Neuman highlights the theoretical basis of phenomenography, in which learning is regarded as changes in conceptions. For example, incorrect answers to simple arithmetic problems do not imply an absence of learning but can indeed reflect qualitatively different ways of understanding numbers. For instance, regarding "names" and "estimates", which are error-prone conceptions, according to Neuman these are both important parts of children's creation of number concepts that will eventually develop into more advanced number concepts. Similarly, children may very well be able to complete an addition task correctly, but their strategies reveal different conceptions of numbers, of which one (structuring) is a path to development and the other (counting) is not.

What stands out in the research on early numerical learning and development is the (methodological) need to interpret children's actions as expressions of awareness. Ahlberg (1997) clarifies this as two levels of descriptions: Strategies or ways of handling numbers are what can be captured in an observation, but what we need to make interpretations of is what a child is focally aware of in a problem-solving situation and how the child structures this information. How the latter is conceptualized, however, depends on the researcher's theoretical perspective, which is why we sometimes find contradictory explanations of how children learn arithmetic skills (see Björklund, Marton et al., 2021). Ahlberg conducted a study similar to

Neuman's, taking the same theoretical approach and finding similar categories. However, she takes the interpretation one step further, describing children's ways of handling numbers and relating them to their ways of experiencing the meaning of numbers. She concludes (1997, p. 35) that "... using these different ways of handling number children's awareness is directed towards various aspects of them. These different aspects of numbers presented in the children's awareness constitute their understanding and consequently they understand the meaning of numbers in qualitatively different ways". According to Ahlberg (1997), the different ways of understanding numbers are, as: i) number words, ii) extents, iii) position in a sequence, and iv) composite units. These different ways of understanding numbers are explained by what is foregrounded in the child's awareness. In this sense, learning arithmetic skills entails experiencing and simultaneously perceiving these as different aspects of number. However, Ahlberg does not elaborate on how this is executed as a learning process that also includes the mathematical aspects (such as cardinality and ordinality). Even though Neuman and Ahlberg made great efforts to theorize children's understanding of numbers based on the different ways in which children experience numbers, they did not fully come up with a theoretically driven conclusion regarding *how* children come to change their way of experiencing numbers (and thus develop their arithmetic skills).

Regardless of whether one takes a cognitive or phenomenographic approach, children's handling of numbers (their strategy use) is not in a one-to-one relation with a certain way of understanding, even though some clues can be revealed from their actions. In sum, while there is no lack of observations of children using numbers with different meanings, our aim is to contribute theoretically underpinned explanations as to *why differences occur* and how children learn arithmetic skills.

2 Theoretical framework

The theoretical lens we apply in our study is phenomenography and variation theory. Phenomenographic research investigates different ways in which the same phenomenon can be experienced by a group of people (e.g., Marton, 1981). Its goal is to find and systematize forms of thought by which people interpret phenomena in their surrounding world. This directs attention to an experiential perspective that highlights individuals and their ways of experiencing (or seeing, perceiving) phenomena they encounter. Phenomenography is a research orientation with the aim to describe, and what it describes is conceptions. "Conceptions" tell what the phenomenon looks like to the individual (in our case, how numbers appear to the child), and have two intertwined features: the global meaning of the conceptualized phenomenon and a structural feature, which constitute the specific combination of aspects that are discerned and focused on. Thus, a conception (or a certain way of experiencing a phenomenon) is both a holistic experience of a phenomenon and at the same time constitutes a complex of discerned aspects of the same phenomenon (Marton & Pong, 2005). If some aspect that was previously undiscerned is suddenly discerned, this alters the global meaning to the person. Thus, in phenomenographic research, descriptions of conceptions are based on explorative forms of data generation and interpretative character of data analysis, resulting in qualitatively different categories (Svensson, 1997). This means that the results of a phenomenographic investigation comprise a group of persons' knowledge; not in terms of what is considered objectively right or wrong but in terms of the meaning a phenomenon in the surrounding world has for these persons. In recent phenomenographic studies, this focus on describing conceptions is labelled *ways of experiencing* phenomena (Marton, 2015).

The phenomenographic approach has significance for describing and investigating learning, taking its starting point in the meaning that appears to the learner. The phenomenographic research approach has been used for many years to describe students' ways of experiencing different phenomena as a point of departure for understanding why participating in the same teaching situation can result in different learning outcomes (Marton & Booth, 1997). However, it is not enough to learn that children convey different ways of experiencing; in educational studies, it is significant to also know *why* these differences occur. In the phenomenographic approach this is not explained in terms of cognitive deficits, for example, but as being due to differences in how the learning object appears to the children. Even so, in order to explain learning and how to advance the ways the learning object appears to a child, one needs to distinguish what constitutes the different ways of experiencing the learning object.

The main question in variation theory of learning (Marton, 2015) is what constitutes the learning of a specific content. A fundamental idea, based in phenomenography, is that learning entails changes in ways of experiencing a certain content, which is why a central question in the theory involves what the learner needs to "see" that will make this change. Ways of experiencing content constitute the learner's differentiation of aspects of that content (cf., Gibson & Gibson, 1955). The fundamental principle in variation theory is that the combination of the necessary aspects for handling numbers in an arithmetic task, arrived at by a particular child, defines his/her way of experiencing numbers. When a new (or rather, not previously attended to) aspect is discerned, this liberates a new way of experiencing numbers and thus what the child can do with numbers. In line with this way of reasoning, children's strategy use in arithmetic problem-solving thereby involves expressions of certain ways of experiencing numbers, which in turn is a function of discerned aspects of numbers.

3 Methods

To deepen our knowledge of children's learning of arithmetic skills, we studied how numbers are experienced by preschool children and what aspects of numbers appear to them that inform their use of arithmetic strategies. To gain these insights, we conducted interviews with 103 preschool children in their final year of preschool[1]. The interviews were conducted by researchers experienced in educational studies and interviewing children, and were held individually at the children's preschools. Tasks were given orally, and the children were encouraged through follow-up questions to explain how they had come up with their answer. They were also encouraged to use their fingers if they wanted to, but no other manipulatives or tools were offered other than what was part of the task. Nevertheless, some children made use of objects found in the room to support their reasoning.

All the children's legal guardians had given their informed consent for the children to participate in the study. The interviews were video-recorded to allow detailed analyses of the children's actions and utterances. If permission to video-record had not been given, detailed field notes were taken by an assisting researcher. The children participated in the task-based interview on two occasions (8-month interval). The children's mean age was 5 years 3 months at Interview I and 5 years 11 months at Interview II. The participants, from three suburbs outside a large Swedish city, all spoke fluent Swedish and were of mixed socioeconomic backgrounds.

[1] Preschool is a voluntary pedagogical practice in Sweden for children 0-5 years of age, with a high attendance rate (95% of 5-year-olds the year of the study and 85% of all children aged 1–5).

3.1 Data

In this paper we use responses to one task in the interview as our object of inquiry. The task was inspired by the "Guessing Game" task used in Neuman's (1987) study, in which the interviewer hides a number of buttons in two boxes and asks the child to guess how many there could be in each box. A similar number decomposition activity is the "hidden item task" in Tsamir et al.'s (2015) interview study with 5-6-year-olds, in which seven identical items were used, one set visible in the interviewer's hand and the rest hidden. The child was asked how many items were hidden in the closed hand. The task was repeated, altering the visible number of items.

Our version of the task, also given orally, includes seven identical glass marbles. The child is initially asked to count the marbles, which are lined up on the table. The interviewer then hides the marbles in her two hands and thereafter the child is asked how many marbles could be in each hand. In the second step, the interviewer opens one hand and lets the child see some of the marbles and asks the child to figure out how many are hidden in the closed hand. After each answer, the interviewer asks follow-up questions to encourage the child to reason about how s/he came up with the answer. The child is given the task three times, altering the partitioning of the seven marbles.

In the analysis we present here, we have selected only one part of the task – the interviewer shows four marbles in her opened hand and the child is to figure out how many are hidden (3) in the closed hand – and only the first round that the task is given. The task corresponds to common "missing addend" tasks in mathematics education, without relying on formal symbolic knowledge, and is thereby suitable for preschool children who have not yet attended formal arithmetic education.

Data for analysis consists of 189 observations of the 103 participating children (92 observations in Interview I and 97 in Interview II). Data was excluded if the child gave no response to the task.

3.2 Analysis

To answer our research question, we conducted two consecutive analyses. First, we did a qualitative analysis of the children's ways of experiencing numbers in the task in both interviews (189 observations in total). The unit of analysis was the observed instances of children's different ways of handling numbers, shown in both verbal utterances and gestures such as finger patterns. We followed the principles of variation theory (Marton, 2015); that is, the child acts in accordance with aspects that are discerned at a particular moment, which defines the child's way of experiencing numbers. For example, when shown four marbles in one hand, one child responds "If I add two it only makes six, so it has to be three" and another child "After four comes five, then six and seven, there are seven in the other hand". Considering the first child, we interpret the response as the child experiencing numbers' relation and thereby manages to handle the given part, the missing part and the whole as a cardinal set of composed units. The second child is interpreted to express a way of experiencing numbers as labels given to each item, why it is logical to that child that the last item is "seven", however not expressing a meaning of numbers as composed units and thereby not related to one another in a sense of cardinality. Different acts reflect different ways of experiencing the meaning of numbers. The results from such an analysis are the phenomenographic categories of meaning that appear to the children. This is reflected in our descriptive categories "numbers are experienced as…". These categories present an outcome space of a limited number of qualitatively different ways of experiencing numbers, and this variation is further explained in terms of discerned mathematical aspects. Thus, the analytical process is a constant interchange between interpretations of how numbers appear to the child and what aspects the child seems to discern, as expressed in words and gestures. The children's expressions are sometimes very subtle; the video recordings allowed for reiterate viewing. Each observation has been coded and categorized by two or more researchers, followed by collective discussions within the research group.

Initially, we coded each child's answer according to which numbers they gave as their answer. Thereafter, we categorized the answers into groups with similar answers and compared the children's ways of explaining their answers within each group. In some cases, children who answered with the same numbers were categorized differently as their different ways of experiencing numbers were identified based on their ways of explaining and reasoning about how they had come up with their answer.

What counts as the "same" conception can be expressed in linguistically different ways, and what can be seen as different conceptions can be expressed in similar language (see Neuman, 1987). Thus, interpreting children's conceptions or ways of experiencing numbers is a comprehensive process based on impressions from both verbal and gestural responses. For example: "After four comes three, maybe it's three? You start with five (raising index finger), then comes four (raising middle finger), and then comes three (raising ring finger)". The combination of verbal and gestural expressions by the child thereby reveals what she discerns (the phenomenon's structural features) and how numbers appear to her (the holistic meaning).

Six categories were found empirically (also reported in relation to other tasks in Björklund, Ekdahl et al., 2021 and Björklund & Runesson Kempe, 2019), and are to some extent similar to previous findings in studies with 6- and 7-year-olds (Ahlberg, 1997; Neuman, 1987). This means, the outcome space of the first analysis partly confirms earlier findings of children's ways of experiencing numbers and partly adds new ones, not described before. In another group of children, it may be possible to find yet additional ways of experiencing numbers (or lack what has been found in our, Ahlberg's or Neuman's studies). The large number of observations do however ensure that our study covers those ways of experiencing numbers that are common among children attending the last year of Swedish preschool.

Second, we selected 90 of the children for whom we had observations from both interviews in order to analyze the changes in their ways of experiencing numbers. This is presented in two ways: on a group level to give an overview of the trajectory of changes, and then on individual case level. The cases are analyzed on a micro-level to gain insights into what in particular constitutes their changed way of experiencing numbers in terms of discerned aspects of numbers. This micro-analysis contributes to our understanding of what the children actually learn to discern that changes their way of experiencing numbers.

4 Results

We present the results from our analysis in three sections: First, we describe the ways of experiencing numbers that appear in the empirical data. Second, we present changes in ways of experiencing numbers within the group of children, and third, we illustrate how changes are expressed empirically on an individual case level.

4.1 Ways of experiencing numbers

From all of the observations in both interviews, we find six categories of qualitative different ways of experiencing numbers that impact the children's strategies in completing the Guessing Game (see Table 1). Differences between categories appear in terms of discerned aspects of numbers, but there are also differences within each category in terms of how the discerned (and undiscerned) aspects are coming through in the children's acts and utterances.

Table 1. Ways of experiencing numbers expressed in the Guessing Game.

Numbers experienced as:
A. Numbers as Words
B. Numbers as Names
C. Numbers as Extent
D. Numbers as Countables
E. Numbers as Structure
F. Numbers as Known Facts

4.1.1 A: Numbers as Words

Number words are used without having any meaning of cardinality or ordinality. Children know that number words represent a certain category of words that are used in situations in which groups of items are handled. They use random number words either solely or in a random sequence or repeat a counting word from the given task. In the Guessing Game we observed this way of experiencing numbers among children who answered with random number words, such as Kevin: “Five, seven, thousand”. Even though the moment before the child counted, or at least recited, the counting sequence while pointing at the marbles one-to-one, there is no numerical relation foregrounded in the child’s utterance when asked how the marbles may be partitioned. In the task, the number of objects also exceeds the subitizing range, and as the child does not discern numbers’ cardinality or ordinality, counting to determine quantities is not an option – it is a procedure you use when asked “how many”, but the number words used do not have the meaning of a composed set.

4.1.2 B: Numbers as Names

When experiencing Numbers as Names, number words are ordered in a sequence and thereby have some relation to each other in terms of ordinality. In this sense, number words can describe “the n^{th}” object, as in an object following another object. However, there is no cardinal meaning involved, as in a consecutive word meaning “one more”. This has significant impact on how numbers are used and how a numerical task is encountered. Otto answers by first counting and pointing at the visible marbles – “One, two, three, six” – and then tapping on the knuckles and back of the interviewer’s closed hand: “One, two, three, four, six. Seven”. Otto’s actions indicate that numbers appear as single objects that are labelled with number words. He never answers with one word, as in labelling a collection, but always counts on the sequence starting from one. Another expression of experiencing Numbers as Names is observed when children answer with two consecutive number words: When one hand with four marbles is shown, Lydia confidently answers “Five there”, pointing at the closed hand. Giving two consecutive number words as an answer to how many objects there might be in the closed hand is a quite common response, even when the child confirms that there were seven marbles on the table from the start. The ordinal meaning appears in the foreground, for example by Sanna: “After four comes three, maybe it’s three? You start with five (raising index finger) then comes four (raising middle finger), and then comes three (raising ring finger)”. When experiencing Numbers as Names, the number words are closely connected to objects that are to be enumerated, which is why the words rarely exceed seven because the counting sequence and ordinality are foregrounded – the child labels objects starting from “one”. This sometimes leads to children answering “seven” when they see four marbles in the opened hand, even though they without difficulty enumerated the set of marbles to be seven when seeing them all on the table. When ordinality is foregrounded (and cardinality undiscerned) this makes sense to the child, as the marbles labelled one, two, three, and four are indeed visible in the opened hand and the marble known as “the seven” then has to be in the enclosed hand. When numbers are experienced as names, these cannot be added or subtracted from other names. Because the cardinal meaning is undiscerned, number words can not be seen as parts of a larger collection labelled with another word (or: the child is unable to see that four is part of the larger set, seven). Some children, like Malik, make attempts to operate with the names “After five comes four, after four comes three, it might be three there”, which indicates that the counting sequence supports him in maintaining attention on objects that are to be enumerated but are hidden in the interviewer’s closed hands (see Category D for advancements resembling of this way of operating with the counting sequence).

4.1.3 C: Numbers as Extent

When numbers are experienced as Extent, they have an approximate value that indicates that a cardinal meaning is discerned. The ordinality of numbers is not discerned, and the relation between numbers is limited to “more or less” in an un-distinct meaning, like Agnes: “Perhaps a little bit more than these (pointing to the opened hand with four marbles)”. In the Guessing Game, this way of experiencing Numbers as Extent is observed when children give answers characterized by some sense of plausible quantities related to the task. For example, Jamila says "I don't know how many there are, I have to look to know, but I think three". Some children give answers that are close but not correct. Characteristic of these instances is that the child does not give a reason for the answer or express his/her way of coming up with it; thus, there is no explicit relation between the numbers discerned that would enable the child to reason about why a certain number is a plausible answer. In cases in which the children motivate their answers they are described as guesses, which is likely because the lack of discerned ordinality hinders any proper operation with the numbers in the task. When children attempt to reason their way to an answer it is often directed at equality in their partitioning, such as Olivia suggesting "doubles": "Equally many as in the first one [opened hand]".

4.1.4 D: Numbers as Countables

In some children's ways of experiencing numbers, we see a strong influence of the ordinal aspect of number and some idea that numbers can relate to each other. The child discerns numbers constituting a set of items, thus having a cardinal meaning as well, but this set is experienced as added units of "ones". There is a clear difference to Category B, because here numbers are not connected to specific items but rather discerned as single units in themselves, which can be counted. Due to the dual meaning of numbers (cardinality and ordinality), it is possible to add and subtract by enumerating (and thus creating) sets in what is commonly known as the "counting all" strategy. William, for example, makes a finger pattern of four by raising and counting one finger at a time, then raising fingers on the other hand while counting all raised fingers from one, ending up with seven raised fingers together (four on one hand and three on the other), and then counting the last three raised fingers on the other hand. That is, he operates with the known numbers by representing them on his fingers but experiences them as added single units and has to create the numbers starting from one. It then becomes difficult to relate numbers to each other; they have to be operated on directly, and re-created, to be perceived. Another expression of this way of experiencing numbers is shown by Liam when answering: "Maybe there are five (pointing at the closed hand). Because there can be four, five, six. And seven, eight." The last utterance indicates that the numbers constitute countable (single) units: He counts on the counting sequence, and then counts or perceives how many number words were said.

4.1.5 E: Numbers as Structure

Experiencing Numbers as Structure is based on the child's discernment of numbers constituting composite sets or units, which may simultaneously be related to other units in a part-part-whole relation. In the Guessing Game we observe that children sometimes use finger patterns to represent numbers, particularly to structure numbers' parts and whole. This leads to their operating on the relation between parts and/or the whole simultaneously and finding the missing number (hidden marbles) through arithmetic reasoning (this differs from Category D, Numbers as Countables, in which numbers are single units constituting a set and adding two sets means that a new set is created from the single units of the two earlier ones). There are several actions that this way of experiencing numbers opens up for. One is shown by Sara, who creates some of the units by counting, "counting on", taking as a starting point the given number (4), keeping the whole (7) in mind, and adding on (3) by raising one finger at a time until the finger pattern seven is visible. Also, without fingers as an aid for structuring numbers, we can see the same way of experiencing numbers in children's reasoning toward their answer, such as Alex: "If I add two it only makes six, so it has to be three." The difference to Category D here is that the child simultaneously keeps the parts and the whole in the foreground, thus relating and reasoning about the four being part of the larger seven, like Mary: "One, two, three, four (raising one finger for each number word, then simultaneously showing two more fingers on the other hand, folding down the first four raised fingers and raising the fifth finger, now holding the two fingers and the single finger close together) three!" Seeing numbers in this way can also be observed, for example, when children start by showing a finger pattern on one whole hand and the thumb and index finger on the other, then moving the thumb on the whole hand to make a gap between the rest of the (four) fingers and thus creating a unit of the thumb and the two on the other hand, showing four, three, and seven at the same time, in this case not created by counting but rather by recognizing the units that the fingers represent.

4.1.6 F: Numbers as Known Facts

Experiencing Numbers as Known Facts means that children instantly recognize numbers as a part-whole relation; that is, numbers can be partitioned in different ways, and smaller numbers are parts of larger ones. This is shown when children give an instant (correct) answer. Most children also explain their answer in terms of retrieved facts, like Christa: "Because three and four make seven". This category differs from Category E, Numbers as Structure, in that the children do not compose and decompose numbers, for example on their fingers or verbally reason their way to an answer, but simply "see" the number relations.

4.2 Changes in ways of experiencing numbers on group level

In the following, we describe how children's ways of experiencing numbers (see the six categories described above) change over time on a group level. Table 2 gives an overview of how many observations were found within each category in Interviews I and II, only including children who responded to the task in both interviews (n=90).

A comparison between the two interviews shows that the changes are mainly positive. Categories A-C, which involve ways of experiencing numbers that do not impose any operations based on numerical features except for guessing and intuitive estimations of the size of the amount, dominate the first interview (84.4%) but have decreased to 24.5% in the second one. In both interviews, Category D is rare. Categories E and F, which express an awareness of number relations and open up for children to operate with numbers as part-whole relations, are also quite rare in the first interview but in fact dominate in the second one (74.5%). This means that, over the course of one preschool year, the children in general have changed from prenumerical to numerical ways of experiencing numbers and are consequently able to solve the Guessing Game using arithmetic strategies when they finish their last preschool year.

Table 2. Children's ways of experiencing numbers in Interviews I and II (N=90).
Note: Percentages do not add up to 100% due to rounding error.

	Interview I		**Interview II**	
Category	Frequency	Percent	Frequency	Percent
A. Words	4	4.4	0	0
B. Names	28	31.1	8	8.9
C. Extent	44	48.9	14	15.6
D. Countables	2	2.2	1	1.1
E. Structure	2	2.2	26	28.9
F. Known Facts	10	11.1	41	45.6
Total	90	100.0	90	100.0

In Table 3 we see how the changes in ways of experiencing numbers are distributed. There seems to be a hierarchy in the distribution, with all but five observations moving in the direction A toward E. The five exceptions involve four observations of children expressing their experience of Numbers as Extent (C) in Interview I and Numbers as Names (B) in Interview II, and one child expressing Numbers as Known Facts (F) in Interview I and Numbers as Structure (E) in Interview II. Ten children remain in the same category (4 in B and 6 in C). Nine children had already expressed Numbers as Known Facts in Interview I.

Table 3 further shows that there is a difference in how ways of experiencing numbers develop toward Categories E and F; that is, an awareness of numbers' part-whole relations that leads to opening up for children to complete the arithmetic task. Experiencing Numbers as Words (A) or Names (B) is found to change into experiencing Numbers as Structure or Known Facts among 21 of the children (22.2%), while children who experience Numbers as Extent (C) or Countables (D, however rarely observed) more often (38.8%) develop into the advanced ways of experiencing numbers (E and F).

Table 3. Transition between the categories, Interview I to II, N=90.

		Interview I						
		A. Words	B. Names	C. Extent	D. Count.	E. Structure	F. Known Facts	*Total Int. II*
Interview II	A. Words	-	-	-	-	-	-	-
	B. Names	-	4 (4.4%)	4 (4.4%)	-	-	-	8 (8.9%)
	C. Extent	2 (2.2%)	6 (6.7%)	6 (6.7%)	-	-	-	14 (15.6%)
	D. Countables	-	-	1 (1.1%)	-	-	-	1 (1.1%)
	E. Structure	2 (2.2%)	9 (10.0%)	12 (13.3%)	2 (2.2%)	-	1 (1.1%)	26 (28.9%)
	F. Known Facts	-	9 (10.0%)	21 (23.3%)	-	2 (2.2%)	9 (10.0%)	41 (45.6%)
	Total Int. I	4 (4.4%)	28 (31.1%)	44 (48.9%)	2 (2.2%)	2 (2.2%)	10 (11.1%)	90 (100.0%)

Note: Percentages do not add up to 100% due to rounding error.

4.3 Changes in ways of experiencing numbers on individual case level

Table 3 shows that the categories Numbers as Names (B) and Numbers as Extent (C) were the most common ways of experiencing numbers in the first interview. In the second interview, most children in these two categories were categorized as experiencing Numbers as Structure or as Known Facts. In this part, we illustrate this change by analyzing four children's ways of experiencing numbers on the two interview occasions and particularly the changes that have occurred.

4.3.1 From Numbers as Names to Numbers as Structure

The change from experiencing Numbers as Names to Numbers as Structure is significant to the child's learning of arithmetic skills, because of the foregrounded cardinality and number relations that appear in the child's awareness in the latter category. It seems critical that the child discerns how number words label not the concrete objects but rather a set that can be composed of any objects. This change in how numbers appear to the child opens up for relating sets to each other for a comparison of quantities, but also how sets (and thus numbers) relate in a part-whole fashion.

The example of Mary will illustrate the specificity of the change from experiencing Numbers as Names in the first interview to experiencing Numbers as Structure in the second one:

Excerpt 1: Mary, Interview I

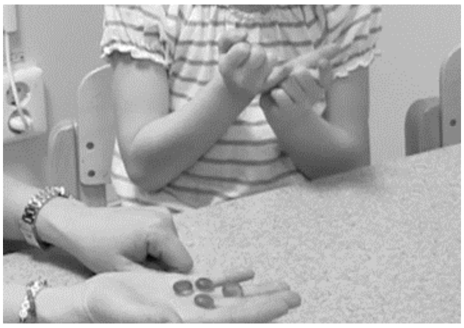
I: (shows four marbles in her opened hand) If there are four there, how many are there in this [closed] hand?
Mary: Seven.

In Interview I, when Mary is shown four marbles in one hand and asked how many are in the other, she answers that there are "seven" in the other hand. This answer is typical of the children who experience Numbers as Names. We interpret her answer as an illustration of her discerning the ordinal but not yet the cardinal aspect of number – the marbles are labelled with number words, making the answer "seven" perfectly logical, as the seventh marble is indeed hidden in the interviewer's hand. The lack of discerned cardinality meaning comes through in that she did count the marbles one-by-one before starting the game, but her "seven" does not constitute a set of seven items (if so, she would realize that there cannot be a set of seven marbles hidden when she sees four in the opened hand).

In the second interview, Mary approaches the game in a quite different way, using her fingers to structure numbers in a relationship of a whole and its included parts:

Excerpt 2: Mary, Interview II

Mary: Four. (Instantly identifies that there are four marbles in the opened hand.)
I: There were four there. If you now know there are four there, how many are there here then?
Mary: Okay. (Raises four fingers one at a time on one hand (Picture a) and then adds three fingers, showing one whole hand and two fingers. Thereafter, she holds up only the three added fingers to the interviewer (Picture b).) Three.


(a)

(b)

Mary handles the task in a way that shows her experiencing numbers in a more comprehensive way than before, now discerning more aspects of numbers, which allows her to handle numbers differently. Numbers are no longer names labelling objects, as she represents the marbles, even the hidden ones, on her fingers: Four, seven, and three thereby have cardinal meaning for her (see Picture a, in which she makes a pattern of four fingers) and not only ordinal (the fourth object). Thus, the number words do not address the marbles per se, but rather the representatives (fingers) that she is able to structure in order to determine the number of the hidden set of marbles. She does this by first structuring the numbers on her fingers, by which she discerns the relationship between the numbers, seeing four and the hidden part three in the total of seven (see Picture b). The difference between the first and the second interview is that in the second one Mary shows that she has now discerned not only ordinality but also cardinality, as well as the part-whole relation of numbers, and is able to keep these aspects foregrounded at the same time in order to complete the arithmetic task. She is also able to see units within units, for instance when she sees that one finger on her right hand and the two fingers on her left hand make a new unit of three (3/1/2), an additional indication of her experiencing Numbers as Structure.

Another example of a similar change in ways of experiencing numbers, but expressed somewhat differently, is done by Clara. In the first interview, she answers that there are “five” in the closed hand. This is interpreted as her likely experiencing Numbers as Names, as answering with a consecutive number indicates that ordinality is in the foreground of her awareness:

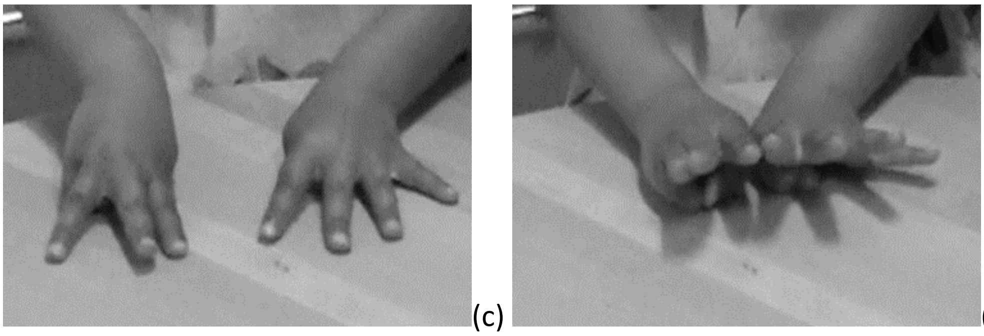
Excerpt 3: Clara, Interview I

I: (Opens her hand with four marbles) How many is this?
Clara: (Counts the marbles, pointing at them one-by-one) One, two, three, four.
I: Four. How many do you think there are in that hand, if there are four there?
Clara: Five, I think.
I: (Opens her other hand, showing three marbles).
Clara: Three!

A possible interpretation of how the numbers appear to Clara is that “five” represents a partition of an imagined number line up to seven, whereby five serves as a limit between “the four“and “the five”, in an ordinal sense. To some extent she is able to discern that a set of marbles comprises “three” in number, but this is isolated from any awareness of sets related to other sets. Thus, she can answer the “how many” question by either counting one-by-one or subitizing small sets but does not yet discern any number relations. This results in her experiencing numbers as isolated units and necessary to set to the concrete objects, or as in Excerpt 3 above, the counting sequence as an order of number words. This indicates, however, that numbers can be represented in an orderly fashion, which is indeed an important aspect to discern, but is not sufficient for forming a way of experiencing numbers that enables arithmetic reasoning. This, on the other hand, is something we can see evidence of in Interview II:

Excerpt 4: Clara, Interview II

Clara: Four there, and three there.
I: Now I’m curious. Why do you think there are four there and three there?
Clara: Because four plus three is seven (Models three and four on her fingers (Picture c) and puts the fingers together to show seven (Picture d)).
I: Okay, let’s check. There were four there. How many are there there then?
Clara: Three.



(c) (d)

In the second interview Clara instantly answers that there are three in the closed hand and says “Because four plus three is seven”. She structures the numbers on her fingers to show the interviewer how three and four together literally make seven, even though her instant answer indicates that she experiences the number relation as Known Number Facts.

The change in her ways of experiencing numbers is shown in what Clara is able to do with numbers. In the second interview she is able to show why three is the missing part by structuring seven on her fingers, showing the parts and the whole simultaneously. She sees the numbers involved as composite sets, and with this shows that she has discerned cardinality. She has also discerned ordinality, as she simultaneously relates numbers to each other in accordance with the counting sequence, adding smaller units to make the whole seven. Her way of moving the represented numbers (finger patterns) together is one way of structuring numbers that shows her awareness of the part-whole relation. That is, seeing how three and four are both parts of seven in a structural way is a powerful advancement from her earlier way of experiencing numbers.

4.3.2 From Numbers as Extent to Numbers as Structure

In the first interview, many children expressed their experiencing Numbers as Extent, which indicates that the ordinality aspect of numbers is undiscerned. These children do seem to have a sense of numbers’ manyness, but due to the absence of ordinal meaning they cannot organize numbers or sets according to quantity other than in an approximate sense. Consequently, they do not have any repertoire for operating with numbers, either to determine exact quantities, for instance in comparison, or to find a hidden or missing set. Nevertheless, they experience that numbers are related to “more or less”, which allows them to make guesses when asked “how many”.

Sofie is categorized as experiencing Numbers as Extent in the first interview:

Excerpt 5: Sofie, Interview I

I: What do you think?
Sofie: (Looks around in the room) As many as, the cookies.
I: As many as the cookies there. How many cookies are there then? (Sofie brings the cookies to the table and places them in a group).
Sofie: (Points at each cookie) One, two, three, four. Wait (Counts again) One, two, three, four, five.
I: Five, you think there are five.

In the first interview Sofie counts the marbles in the opened hand, thus having an idea of number words used in a procedure in which you point and say words in a consecutive order. However, when asked how many marbles there are in the other hand, she makes no attempt to account for the already visible ones, as related to the unknown set; instead, she looks around the room and at a bookshelf with toys near the table. We infer that she experiences some sense of cardinality, as she expresses quantity by saying “as many as the cookies”. As she does not discern any (numerical) relation between the set of marbles and the set of cookies, figuring out the quantity of a hidden set of items is not possible. The change in her way of experiencing numbers in the second interview is apparent, as she then clearly discerns exact numbers and relates them to each other in completing the Guessing Game:

Excerpt 6: Sofie, Interview II

I: How many are there in that one?
Sofie: Four (Shows four fingers on her right hand, then on her left hand raises the little finger, and immediately after this the thumb and index finger simultaneously). Three. (see Picture e)


(e)

Sofie immediately sees that there are four marbles in the opened hand, without counting. When asked how many marbles there are in the other hand, she shows a finger pattern of four and thereafter identifies the missing part as a set constituted of one finger on the right hand and two more fingers on the left hand (in the same way as Mary does, above). She sees the numbers as composite sets that have cardinal meaning. She is able to compose a unit of three, from one finger on the right hand and two fingers on the left, indicating her discerning of the relation between and within the numbers and thereby having developed her way of experiencing Numbers as Structure.

5 Discussion

In this paper we set out to describe preschool children's ways of experiencing numbers in an arithmetic task and what might distinguish changes over time. We have approached these questions by suggesting that the child's perspective on numbers and how numbers appear to them may be a necessary addition to our knowledge of how children learn basic arithmetic skills. Our qualitative analyses resulted in six different ways of experiencing numbers, distinguished by which aspects of numbers are discerned by the children. From a longitudinal perspective, we have shown how children's ways of experiencing numbers change and, more specifically, which aspects become critical to discern in order to develop arithmetic skills. Some categories presented in this paper are comparable to previous findings in studies with 6- and 7-year-olds (e.g., Neuman, 1987; Ahlberg, 1997). Particularly Ahlberg's theorizing ambition has influenced the current study, that different ways of handling numbers mean that children's awareness is directed at various aspects of numbers and that this constitutes their understanding of numbers. What Ahlberg did not determine in her research was *how* different ways of handling numbers are connected to ways of experiencing numbers and specifically discerned (or not discerned) aspects. Our study may contribute to fulfilling this ambition by specifically pointing out the difference in how children handle numbers depending on their discerning ordinality, cardinality or both of them and number relations simultaneously.

Earlier studies (Björklund, Marton et al., 2021) have shown that which aspects of numbers children discern is linked to their repertoire of arithmetic strategies. Some strategies, according to the large body of research in the field, are known to be error-prone, such as counting single units if it is the only strategy used by the student (e.g., Ellemor-Collins & Wright, 2009; Neuman, 1987). In our study, we rarely see any counting-based strategies among the preschoolers in either Interview I or II. This could be taken as an indication that it may not be necessary to introduce counting-based strategies in early arithmetic education, as children obviously do not need to experience Numbers as Countables at any particular point in time; they seem to be able to discern number relations and coordinate cardinality and ordinality meaning in numbers and thus learn to experience Numbers as Structure anyway. Or as Neuman (1987) would describe it, go from non-numerical to numerical conceptions without numbers appearing as countables. A consequence, then, would be that the children do not risk getting stuck in single-unit counting strategies, but instead appropriate numbers as constituting composite sets that can be de-composed and re-composed as means in solving arithmetic tasks (e.g., Cheng, 2012). Taking one's starting point in the child's lived experiences rather than cognitive processes, the key might thus be not to primarily attend to children's skills or abilities (such as frequency of using a certain strategy) but rather to focus on how numbers appear to them and support their discerning aspects that emphasize numerical units and relations. Our analysis of the variation in ways of experiencing numbers supports Neuman's suggestion that children's errors or success in completing arithmetic tasks may be induced by different ways of experiencing numbers; that is, experiencing Numbers as Words, Names, or Extent reflect very different ways of seeing numbers, while the result of completing a task may be the same number word. It is therefore necessary to highlight what appears as focal in the child's way of experiencing numbers, in order to fully understand what (aspects of numbers) are critical for children's ways of experiencing numbers to change into conceptions that allow for more powerful strategies to be used. For example, children who experience Numbers as Extent or Countables are in our study seen to more often develop more advanced ways of experiencing numbers as structure or known facts. This needs though to be the object of further inquiry, examining whether it indicates that experiencing Numbers as Extent is the path to more advanced ways of experiencing numbers or if it is merely an effect of a larger number of observations among this particular group of children. However, according to our observations we can draw the conclusion that during their last preschool year the majority of the children do learn to discern cardinality and ordinality as well as numbers' part-whole relations.

Observing children answering "how many"-questions, for example with random number words or irregular counting sequences, is not new; Fuson (e.g., 1992) and others have presented similar observations among preschoolers in several studies. What we wish to add to this field of research, however, is interpretations of what numbers mean to the children, how numbers appear to them. This would help us understand why children answer with random numbers when the moment before they were able to point and recite the counting sequence and repeat the last uttered counting word as an answer to “how many”. At the beginning of this paper, we claimed that there is a lack in the field of knowledge concerning what it explicitly is that children learn when they develop a more advanced meaning of numbers, which consequently leads to their using powerful strategies in arithmetic problem-solving. This is partly discussed in Björklund, Marton et al. (2021) in terms of children needing to learn to discern certain aspects of numbers. What the current paper contributes in addition to this is how the discernment of some (but not all) aspects of numbers constitutes a variation in ways of experiencing numbers. This study of ours is theoretically grounded in phenomenography and variation theory. This leads to an emphasis on the emergence of “conceptions”, or ways of experiencing some phenomenon. This means that we use the theoretical framework to describe what the “numbers” phenomenon looks like to the individual, determined by both the global and structural meanings of the conceptualized phenomenon. In line with this, we have intended to describe the variation in ways of experiencing numbers (that is, the global meaning appearing to the child) and how a certain way of experiencing numbers is constituted (that is, the structural meaning of the phenomenon of numbers). The combination is our theoretical contribution, which adds to what, for example, Neuman (1987) and Ahlberg (1997) described and theorized some decades ago.

The connection between discerned aspects of numbers and the way of experiencing numbers that is highlighted throughout the current paper is not only a theoretical contribution. We suggest that it is a key to early mathematics education, as it offers an explanation of children’s different ways of encountering arithmetic tasks and what they need support in discerning in order to develop their ways of experiencing numbers. In particular, it becomes evident that experiencing numbers as composed units that can be related, composed and de-composed is an essential aspect to discern in order to develop arithmetic skills, as shown in the empirical examples. Thus, what educational practices should facilitate is opportunities to explore and experience numbers as representing composed sets. What aspects children discern may be difficult to “see”, but how children experience numbers’ meaning might be the entrance point to understanding their knowledge and skills, as well as what support they need in learning to discern critical aspects.

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