Developing mathematical problem-solving skills in primary school by using visual representations on heuristics

Susanna Kaitera and Sari Harmoinen

Faculty of Education, University of Oulu, Finland

Developing students' skills in solving mathematical problems and supporting creative mathematical thinking have been important topics of Finnish National Core Curricula 2004 and 2014. To foster these skills, students should be provided with rich, meaningful problem-solving tasks already in primary school. Teachers have a crucial role in equipping students with a variety of tools for solving diverse mathematical problems. This can be challenging if the instruction is based solely on tasks presented in mathematics textbooks. The aim of this study was to map whether a teaching approach, which focuses on teaching general heuristics for mathematical problem-solving by providing visual tools called Problem-solving Keys, would improve students' performance in tasks and skills in justifying their reasoning. To map students' problem-solving skills and strategies, data from 25 fifth graders' pre-tests and post-tests with non-routine mathematical tasks were analysed. The results indicate that the teaching approach, which emphasized finding different approaches to solve mathematical problems had the potential for improving students' performance in a problem-solving test and skills, but also in explaining their thinking in tasks. The findings of this research suggest that teachers could support the development of problem-solving strategies by fostering classroom discussions and using for example a visual heuristics tool called Problem-solving Keys.

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Correspondence:
susanna.kaitera@oulu.fi

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1 Introduction

During the primary school years, students develop their understanding of concept of numbers and fluency in arithmetic skills (FNBE, 2016, p. 307). Learning mathematical procedures is important, but it is also crucial to equip students with strong problem-solving, reasoning, and thinking skills (e.g. Lester, 2003; Pehkonen et al., 2013) to give tools for functioning in a complex, unpredictable future. Mathematical problem-solving requires skills to apply variety of different solution strategies and models (Leppäaho, 2018, p. 374). It is not uncommon that while students may excel on routine exercises (those that they have already seen and practiced), they fail to solve problems that differ from those they have previously encountered (OECD, 2014).

Traditional teaching approaches often focus on learning mathematical facts and procedures. Teachers could take advantage on creating learning environments, which engage students in investigating problems and seeking solutions in an active manner. (Pehkonen et al., 2013, 13.) Näveri et al. (2011, p. 169) point out that if teachers rely on using routine tasks in mathematics lessons, also the learning of students stays on the routine level. Mathematical thinking skills can be developed via problem-solving (e.g. Schoenfeld, 1985; Lester, 2003, Leppäaho, 2018), and on the other hand, problem-based teaching methods can be used to foster deeper understanding.

The importance of developing mathematical reasoning and problem-solving skills is also recognised in international assessments, such as PISA and TIMSS. In PISA the problem-solving competence is defined as “an individuals’ capacity to engage in cognitive processing to understand and resolve problem situations where a method of solution is not immediately obvious” (OECD, 2014, p. 30).

As Leppäaho (2018, p. 368) points out, mathematical problem-solving is learned only by practising it repeatedly. Mathematics can actually be taught *through* problem-solving (see for example Schoenfeld, 1985; Hiebert, 2003; Lester, 2013). This teaching method enables students themselves to engage with meaningful, rich problem tasks and instead of superficial procedure-learning, develop understanding of mathematical concepts and methods. Students should have possibilities to explore a variety of different and unfamiliar problems, even though they would not yet master certain methods or algorithms (Goldenberg et al., 2003, p. 28).

Developing students’ mathematical thinking and problem-solving skills have been flagged as important goals in Finnish National Core Curricula for basic education (FNBE 2004; FNBE 2016). Students should be guided not only to solving problems, but also finding and modifying them (FNBE, 2004, p. 158). According to the mathematics curriculum in Finland, instruction should “support the development of the pupils’ skills in presenting their mathematical thinking and solutions to others in different ways and with the help of different tools” (FNBE, 2016, p. 307).

Expressing mathematical ideas and justifying thinking can be challenging for primary-school aged students, but as Finnish Curriculum (FNBE, 2016, p. 306) underlines, it would be important to learn to communicate ideas and collaborate with peers. Collaborative problem-solving situations, identifying and discussing ideas and participating in explanation-building discourse can help learners in developing their thinking skills (Scardamalia & Bereiter, 2014, p. 3). Collaborative problem-solving situations are excellent opportunities to explore also complex problems, because different examples and explanations by group members enable better understanding (Sears & Reagin, 2013).

In this research, fifth grade students were introduced general heuristics, which were understood to serve as stepping stones in solving non-routine mathematical problems. At the beginning of the school year, it appeared that many students seemed to struggle in mathematical tasks and especially in explaining their problem-solving processes in written form. Students were introduced to concrete tools called Problem-solving Keys, which were modified from Strategy Keys based on work by Herold-Blasius (2021). The aim was to provide students with a visual reminder of heuristics for mathematical problem-solving tasks. Similar heuristics were outlined also in the Singaporean Mathematics Syllabus 2013 and used as a reference when classifying and modifying the Keys for teaching purposes in Finland (Kaitera, 2021).

The research aimed to map fifth graders' skills and strategies before and after the intervention, which was designed to offer wide variety of mathematical problems and techniques to solve them. The interest was in finding out if the problem-oriented teaching approach influenced on how students solved mathematical problems, which required proportional reasoning.

This research aimed to answer the following questions:

1. What kind of influence did teaching approach, which focused on mathematical problem-solving, have on students' general performance in proportional reasoning tasks and abilities to explain thinking?
2. What kind of differences appeared in students' use of erroneous and correct problem-solving strategies between pre- and post-tests?

The study outlines possibilities to develop mathematics teaching towards a direction, in which students become more active participants in learning process and develop their mathematical problem-solving skills. Another aspect was to answer the 21st-century demands for analysing the teaching practises and creating knowledge as a practicing teacher (see for example Niemi & Nevgi, 2014). The study includes features of a teaching experiment and in this report is referred to as such.

2 Theoretical framework

2.1 Mathematical problem-solving: focus on heuristics

In many countries, mathematics curricula emphasize the importance of exploring versatile problem-solving activities. These have been a part of mathematics classrooms for a long time, but there is still confusion on what it means in practice. Teachers often understand it as solving word problems (e.g. Lester, 2003; Näveri et al., 2011), or even solving simple, routine arithmetic tasks presented in mathematics textbooks (Näveri et al., 2011). In this study, students were provided with non-routine tasks, which require skills to devise and implement a plan (Polya, 1945/1973) and combine previously learned solution strategies in a novel way (Lester, 2013; Leppäaho, 2018).

An ability to solve mathematical problems in different contexts is an important skill, which can, and should be taught at schools. To be able to invent and test strategies, students need to have basic skills and understanding of problem-solving processes. As Leppäaho (2018, 374–375) points out, in addition to mathematical skills (e.g. how students can use different strategies), for example motivational aspects and reading and writing skills play important roles in an individual’s capacity in mathematical problem-solving situations.

Mathematical problem-solving techniques are often called heuristics (Polya, 1945/1973; Schoenfeld, 1985; Goldenberg et al., 2003). Heuristics can be described as non-rigorous, general suggestions for strategies, which can be helpful when solving different types of problems. Learning these techniques and becoming familiar with different problem-solving methods helps students to tackle mathematical problems also in unfamiliar contexts.

Heuristics were linked to everyday teaching by Polya in his book “How to solve it” (1945). Polya outlined a simple four-step problem-solving process, and the following phases are often referred to when defining heuristics:

1. Understanding the problem: what is being asked? What is known, what is unknown?
2. Creating a plan for solving the problem, considering whether the type of the problem is already familiar, choosing the most appropriate heuristic.
3. Solving the problem by carrying out the plan and assessing whether the steps are correct.

4. Looking back and checking if the answer makes sense. (Polya, 1973, 5–6.)

Important first steps of understanding a problem and choosing methods for solving the task are often forgotten when describing elements linked to mathematical problem-solving (Näveri et al., 2011, p. 169). School mathematics often emphasizes teaching certain algorithms to fit certain types of problems instead of providing a wider variety of general tools for problem-solving (Näveri et al., 2011; Leppäaho, 2018). Another important aspect linked to problem-solving can be derived from Polya's views: he outlined the function of the last phase as not only reviewing the process but also discussing it (1973, p. 6).

Heuristics are not the same as algorithms: they rarely prompt a solution, while carrying out an algorithm, which is suitable for a certain type of mathematical problem, leads to a rather unambiguous solution. According to Polya (1973, p. 113), heuristics cannot be used as a tool for rigorous proof. Instead, heuristics belong to a problem-solving process as a part of it. Whereas algorithms are usually constructed of certain predetermined steps, heuristics involve a decision-making process. Students make assumptions on whether a certain approach would work or not and try out different ways to implement the method: for example, in this study, making first a diagram or table provides numerous chances to proceed in solving the problem.

Heuristics can be learned and practiced (Schoenfeld, 1985; Bruder & Collet, 2011) and are generally more applicable in different types of mathematical domains and problems than plain algorithms. Due to the nature of transferability, learning heuristics also supports the development of confidence in mathematical problem-solving (Goldenberg et al., 2003). The aim of teaching mathematics through problem-solving is to equip students with skills to apply previously learned techniques in non-routine and novel situations (Leppäaho, 2018, p. 379).

Polya's four-step model is still useful in today's mathematics classroom and was referred to as a framework to underline different phases of problem-solving; mathematics is more than just filling in the textbook, it could be understood as an activity. Devising a plan and choosing the most appropriate heuristic were supported by visual tools called Problem-solving Keys, which are introduced in Chapter 3.2.

2.2 Proportional reasoning as a problem-solving domain

Fifth graders' problem-solving skills were mapped by proportional reasoning tasks. It is an excellent domain to solve mathematical problems linked to everyday life. For example, adjusting the recipe, preparing juice from a concentrate, calculating the most beneficial buy or comparing discounts between two products, calculating the consumption of the petrol in a car trip, or using a map and its scale to calculate the distance between two targets require skills to reason proportionally. Traditional symbolic representations or algorithms linked to proportional reasoning are not familiar for Finnish fifth graders and was therefore chosen as a domain to assess students' intuitive problem-solving skills and strategies in non-routine problems.

Proportional reasoning is often described as a cornerstone to higher mathematical and scientific thinking and cognitive development (e.g. Lesh et al., 1988; Lamon, 2007; 2012). Understanding proportionality requires reasoning with ratios. In textbooks and mathematics dictionaries the word proportion is often defined as an equivalence of ratios or statement of equal ratios or fractions, written as follows:

$$\frac{a}{b} = \frac{c}{d} \text{ or } a : b = c : d.$$

Proportional reasoning requires skills to convey the same relationship for example in producing or comparing ratios or finding a missing value. Abilities to reason proportionally are a marker of a move towards more developed forms of reasoning and form a foundation for example for algebra. Previous research indicates that students are capable of solving proportional word problems already during their early years of primary school (e.g. Tourniaire, 1986; Van Dooren et al., 2005; Vanluydt et al., 2019).

Understanding ratio and proportion requires the ability to reason with multiplicative relationships and distinguish them from relationships, which are additive in nature (Van Dooren et al., 2010; Son, 2013). In an additive approach, the student operates with an invariant difference between two values, whereas a multiplicative approach requires an understanding of an invariant ratio between two values (Van Dooren et al., 2010). Even if some proportional reasoning tasks can be solved by additive approaches, also in those situations students need to understand the co-varying situation of given values. Building-up or scaling-down by skip-counting until the anticipated value is reached represents one of the strategies, which often bases in additive reasoning. These types of solution methods can be supported as steps towards multiplicative and proportional strategies.

Reasoning is an integral part of mathematical problem-solving and skills reach beyond solving routine problems. Reasoning requires logical and systematic thinking, being a process, which requires making conclusions on how to achieve certain goals; these conclusions guide problem-solving and decision-making behaviour (Leighton, 2004; Grønmo et al., 2013). Students make notions on patterns and regularities and use that information on making decisions on problem-solving approaches. Reasoning involves skills to make conjectures, logical deductions based on assumptions and rules, and abilities to justify results. (Grønmo et al., 2013, p. 27.) Teachers can help students to develop these skills by presenting mathematical problems linked to unfamiliar contexts and providing opportunities to solve open-ended or multi-step problems (e.g. Grønmo et al., 2013). This has not been typically encouraged in school culture (e.g. Pehkonen et al., 2013). Close-ended textbook examples do not necessarily support students' skills to apply the learned procedures or algorithms outside the school context, and the applications to real-world situations can seem rare to them.

3 Teaching experiment: Heuristics for problem-solving

Interest towards improving primary-aged students' mathematical problem-solving skills was based on data, which was collected in Finland and Indonesia in 2014-2015 for Kaitera's doctoral research. A preliminary analysis of the mentioned data indicated that Finnish students had severe difficulties in explaining their thinking in tasks. This led to wondering whether these skills could be developed by implementing a teaching approach, which provided tools for solving a wide variety of out-of-the-textbook problems. The teaching experiment was carried out during the following academic year in a class of fifth graders. The learning environment was designed to support the development of students' mathematical problem-solving skills. The quasi-experimental design was conducted in real-world learning settings, attempting to discover aspects that could be useful for example for teachers aiming to develop mathematics teaching practices.

Teaching heuristics for mathematical problem solving is often linked to working with students with challenges in learning mathematics (e.g. Gallagher Landi, 2001; Fuchs & Fuchs, 2003; Swanson et al., 2013). General heuristics are not associated directly to certain kinds of mathematical problems and therefore can facilitate integrating the given information with steps for action (Swanson et al., 2013, p. 170).

This report suggests that any student would benefit from getting familiar with a range of generalisable problem-solving approaches instead of just learning a variety of algorithms fit for certain types of mathematical problems.

3.1 Participants and background for the research

Research was carried out in a large urban school in Northern Finland with a class of 25 fifth graders (12 boys and 13 girls). In the beginning of academic year 2015-2016, students' skills and strategies were mapped by a pre-test with proportional reasoning problems. At that time, students' mean age was 11 years and 2 months (range from 10 years and 9 months to 11 years and 7 months). During the autumn semester, the class got familiar with a range of generalisable heuristics, which were used in solving a variety of mathematical problems. Participating class followed the guidelines of mathematics education outlined in the Finnish National Core Curriculum. Students had attended five years of elementary school, but not received any formal instructions in solving proportional reasoning tasks, which were the main domain for assessing the development of mathematical problem-solving skills in this research. The class-teacher had a degree as a Master of Education and had been teaching for 10 years in primary and secondary schools. She was working on her Doctoral research on mathematical problem-solving, and the study described in this report was carried out of an interest towards developing students' problem-solving skills.

At the beginning of the fifth school year, it appeared that many students seemed to struggle in mathematical tasks and especially in explaining their problem-solving processes in written form. Students were introduced to concrete tools called Problem-solving Keys, which were modified from Strategy Keys based on work by Herold-Blasius (2021). The aim was to provide students with a visual reminder of heuristics for mathematical problem-solving tasks. Similar heuristics were outlined also in the Singaporean Mathematics Syllabus 2013 and used as a reference when classifying and modifying the Keys for teaching purposes in Finland.

Fifth graders had three mathematics lessons every week. Mathematics textbooks were used, but in addition to those, during the autumn semester the class spent on a weekly basis on average one mathematics lesson on working with mathematical tasks in a practical context and learning a variety of general heuristics for problem-solving. Out-of-the-textbook problems were solved during the spring semester, too, but learning heuristics was not the focus anymore. Post-test data was collected at the end of the fifth grade in 2016 by using the same test than in the beginning of the school year. At that time, the mean age of students was 12 years (range from 11 years and 7 months to 12 years and 5 months).

3.2 Framework for practicing mathematical problem-solving

The central idea of study in a real-life context was to teach mathematics and general heuristics through solving a variety of out-of-the-textbook problems. Mathematical problems were sourced for example from everyday situations, children's literacy, and local, national and international news. In addition to that, students created word problems for their peers and learned to solve them in various ways. Problems were often integrated into other subjects, such as Environmental Studies and other Science themes, Finnish as a mother tongue and Arts and Crafts.

Exploration of mathematical problems followed a framework with different phases of problem-solving (Stein et al., 2008; OECD, 2014, p. 31): first, the task was presented by the teacher to the students (*a launch phase*), then students worked on problems either in small groups or individually (*an exploration phase, planning and executing*) and finally the outcomes were shared and discussed (*a summarising and reflecting phase*). In practice, the process was not a linear, step-by-step progressing path, but rather a flexible model for moving between different phases. Quite often discussing and sharing the ideas led to returning to the exploration phase and assessing the problem-solving approaches from new perspectives. Polya's (1945/1973) four step model was followed especially during the exploration and summarising phases. Problems were solved in collaborative settings always when it was possible: this enabled discussion and made the importance of justifying thinking more visible.

Heuristics or general techniques for solving mathematical problems were introduced to students by using a visual tool called Problem-solving Keys, which are based on for example Polya's (1945/1973) and Bruder and Collet's (2011) heuristics, and the same ideas were outlined in Singaporean Mathematics Curriculum 2013. These heuristics were modified into a concrete tool by Herold-Blasius and Rott (2016) and named as Strategy Keys. They describe these tools as "door openers" for a problem-solving process and reminders of general heuristics that students have learned (Herold-Blasius & Rott, 2016; Herold-Blasius, 2021).

Keys were modified for teaching experiment purposes, translated in Finnish, and renamed as Problem-solving Keys. Keys that were used in this study were chosen based on their generalisability, transferability and fit for the mathematics curriculum for this age group. The guidance progressed by introducing one or two keys (heuristics) at the time, linking them in a variety of out-of-the-textbook problems. As the new heuristic was introduced and practised, the key linked to that particular heuristic was added to a student's personal "Problem-solving key chain". Each key was linked to a mathematical problem, which was often open-ended, or at least had multiple different solution paths to choose from. The problem was chosen so that the heuristic in that Key worked well in solving a particular problem: for example, Gravett's The Rabbit Problem (2009) was based in Fibonacci's approach and used when practising the problem solving by using a table. Literature often offers an excellent context to bring abstract and complicated concepts closer to real-world situations. The following Figure 1 shortly illustrates the keys which were chosen as a focus area in this study, and some prompts, which were presented in guiding the learning processes.

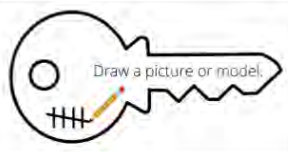
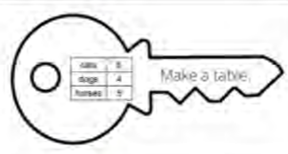
	**Draw a picture or model** Create a visual representation (drawing, icon, simplified picture etc.) of the situation. Use the picture or model as an aid in solving the problem.
	**Make a table** Arrange data by using a table. Fill in the data which is already known. Can you recognise conditions, which are affecting given data? How do these conditions help you in finding out the missing information?
	**Look for a pattern or rule** Arrange data by using e.g. tables or diagrams. Can you recognise rules or patterns? Can you make predictions and test them? This heuristic can often be linked to a previous Key, Make a table.
	**Draw a diagram** Create a suitable diagram based on the given information. Can you make assumptions based on that? How can you interpret the diagram? Are they any outliers and if yes, why?
	**Use different colours** Find and mark the important information by using different colours. Will this help you to progress in the process and understand the question better?

Figure 1. Examples of Problem-solving Keys and prompts presented to students.

In addition to the keys described in Figure 1, students had three additional keys, which were called "When I'm stuck" -keys:

- Read the task again,
- Guess and check,
- Solve part of the problem.

These ideas often enabled either using other heuristics or continuing with other steps in the process. Key called “Solve part of the problem” turned out to be well used. Breaking the problem down into more approachable steps and solving even a small part of the problem opened new insights on how to proceed in tasks. The notion is not new: Duncker (1945, p. 8) linked “reformulation of the original problem” as one of the important characteristics in the problem-solving process and referred to this reformulation as a step or phase on a path towards solution. As Kilpatrick (2016, p. 45) points out, it might be easier to solve the problem if it is broken into smaller pieces or modified into another form.

It is important that teachers value attempts for intuitive problem-solving methods and will be able to guide the student forward. Children often use everyday logic and apply that also to mathematical problems. They can be invited to justify their thinking and invent proofs for their ideas. Later students should learn about mathematical proof and formalities. They need to recognise that there is a difference between a guess, a conjecture, and a proven assertion. It is important to encourage students to wonder why things are as they are and guide them in providing a logical chain of reasons as the explanation. (Goldenberg et al., 2003, p. 24.) An educated guess differs from a random guess by its metacognitive aspects. True mathematical problem solving is challenging, but at the same time rewarding for both students and the teacher, as Schoenfeld (1992, p. 354) points out.

Students benefit from having opportunities to explain their thinking not only by using mathematical language, but also pictorial and natural language: possibilities to draw and write during the problem-solving process may strengthen the understanding of mathematical concepts and contribute to mathematical thinking skills (Joutsenlahti & Kulju, 2017). Open-ended problems or planted error tasks are excellent domains for developing students’ skills in negotiating and articulating their mathematical ideas to others. According to D’Ambrosio and Prevost (2008, p. 276) “all contributions should be valued and respected“. By assessing students’ solution methods, also the self-generated ones, teachers can correct the ones which are mathematically acceptable, or guide students forward in partially constructed explanations. Classroom discussions provide crucial information on students’ understanding on topic and problem-solving processes. Effective teaching includes listening to students' ideas and explanations and using that information as a guide in making decisions on instruction (e.g. Lester, 2013; Ivars et al. 2020; Shaughnessy et al., 2021). These views were at the centre point of the study, because a variety of out-of-the-textbook problems enabled interesting mathematical discussions in the classroom. Conversations were emphasized as important steps in learning problem-solving. Students were advised and expected to show their thinking in tasks by writing down the calculations or drawing the stages in solving the problem in a mathematically understandable way. That can be surprisingly difficult even for the 10-12-year-old students, who have already attended several mathematics lessons per week for multiple years.

4 Mapping the problem-solving skills

4.1 Data collection instruments

Students' performance was assessed by individually completed paper-and-pencil tests, which were taken in the beginning and in the end of fifth grade. Tasks included different types of proportional reasoning problems and are presented in more detail in Table 1 and Table 2. Students had a 45-minute lesson to complete the pre- and post-tests, but most of them used 20-30 minutes for tasks.

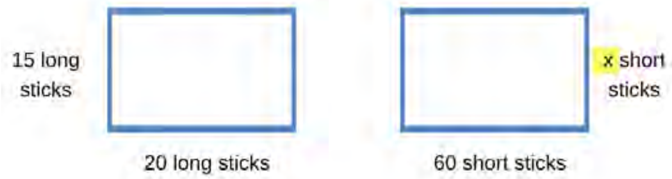
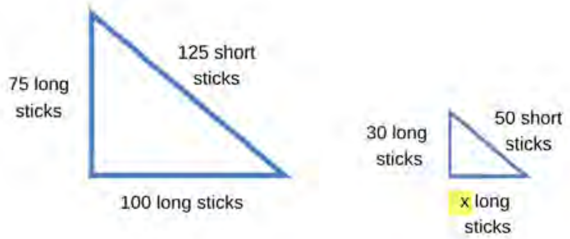
Multiple-choice questions 1-5 and 8 represented typical comparison problems, in which students needed to determine the relationship(s) of two or more ratios, for example by judging whether one ratio is greater or less than the other one(s) or are they equal. In task three with mixtures two of the given ratios were similar. The rationale for having two equivalent ratios in the task was to map whether students were favouring one of the choices over the other, in this case whether they took the first choice, 2:4 or rather chose 1:2, which is used in several everyday contexts.

Table 1. Proportional reasoning tasks with ratios (multiple-choice questions)

Item	Description	Context and source
1. Lemon tea	Mum is making lemon tea. She mixes tea and sugar in a jug. Which one tastes the most sweet? Choose. 1 glass of tea and 1 spoon of sugar 4 glasses of tea and 4 spoons of sugar 1 glass of tea and 3 spoons of sugar	Comparing ratios in mixtures (qualitative comparison), adapted from Noelting (1980) and Kaput and West (1994)
2. Lemon tea	Which one tastes the most sweet? Choose. 1 glass of tea and 2 spoons of sugar 2 glasses of tea and 2 spoons of sugar 2 glasses of tea and 1 spoon of sugar	
3. Lemon tea	Which one tastes the most sweet? Choose. 2 glasses of tea and 3 spoons of sugar 1 glass of tea and 2 spoons of sugar 2 glasses of tea and 3 spoons of sugar	
4. Lemon tea	Which one tastes the most sweet? Choose. 2 glasses of tea and 3 spoons of sugar 1 glass of tea and 2 spoons of sugar 1 glass of tea and 3 spoons of sugar	
5. Lemon tea	Which one tastes the most sweet? Choose. 6 glasses of tea and 3 spoons of sugar 5 glass of tea and 2 spoons of sugar 5 glasses of tea and 3 spoons of sugar	
8. Paint-mixture	Green paint is made by mixing two buckets of blue paint and three buckets of yellow paint. The painter needs to get more paint. How many buckets of blue and yellow paint does he need to get the exactly same shade of green? Choose one option. 3 buckets of blue and 4 buckets of yellow paint 4 buckets of blue and 6 buckets of yellow paint 5 buckets of blue and 6 buckets of yellow paint 6 buckets of blue and 8 buckets of yellow paint	Comparing ratios (quantitative comparison), similarity, adapted from Tourniaire (1986)

Tasks 6A, 6B, 7 and 9 required proportional reasoning with ratios, inverse proportionality or similarity of mixtures. Students were explicitly asked to record their thinking in these tasks and explain their problem-solving processes by mathematical, pictorial and/or natural language in written form.

Table 2. Tasks used in assessing strategies: Proportional reasoning tasks with ratios, inverse proportionality or similarity of mixtures

Item	Description	Context and source
6A. Rectangles	Students are building geometric shapes. They make two similar rectangles and triangles by using short and long sticks. How many sticks do they need in x?  	Determining a missing value with continuous ratio-preserving, geometric similarity, idea adapted from Mr. Tall and Mr. Small tasks by Karplus et al. (1974), Lamon (1993) and research by Son (2013)
6B. Triangles	How many sticks do they need in x?  	
7. Painters	Six painters paint a house in three days. If they all work at the same speed, how many painters would be needed to paint the same house in one day?	Inverse proportionality, item was created for this research
9. Paint-mixture	Orange paint is made by mixing four buckets of yellow paint and one bucket of red paint. To get exactly the same shade of orange, how many buckets of red would the painter need to mix to six buckets of yellow?	Comparing ratios (quantitative comparison), similarity, adapted from Tourniaire (1986)

Timing of pre-test was before students were introduced Problem-solving Keys, and therefore they were not used while solving the items. Post-test was in the end of the school year and students were allowed to use their “key chain”, if they wished, in a similar way they could during the ordinary mathematical tests as well. None of the students felt that they needed the Problem-solving Keys at the post-test in June. The aim of this study was not to map the role or usage of these tools for heuristics but would be another interesting viewpoint for the future research (see Herold-Blasius, 2021). Problem-solving Keys were on a very important role when teaching different ways to approach a wide variety of non-routine mathematical problems especially during the autumn semester.

4.2 Data analysis

First, students' overall test performance in tasks 1-9 was assessed by awarding points on correct and erroneous answers, but also on intermediate steps towards correct explanation. In multiple choice items 1-5 and 8, students received 1 point for a correct answer and 0 points for an erroneous one. In item 3, students were expected to choose both options A and B to gain 1 point, and 0,5 points were given, if they chose either A or B. Maximum points for multiple choice questions were 6. In items 6A, 6B, 7 and 9 students were expected to explain their thinking, and their answers were serving as a base for building a framework for correct and erroneous strategies from intuitive to more sophisticated ones. Maximum points for these items were 2 points for each. The in-between marks were the following:

- 0 points: erroneous explanation and/or answer, or no answer provided
- 0,5 points: some explanation towards correct answer provided, answer incorrect
- 1 point: no explanation provided, answer correct
- 1,5 points: some explanation provided, answer correct
- 2 points: correct explanation provided, answer correct.

With this grading, it was possible to gain a maximum of 8 points in items 6A, 6B, 7 and 9. This approach was close to rating used in school mathematics tests for this age group, and took also the partially correct answers into account. Numerical scores were used as indicators of overall performance and possible development between pre- and post-tests. Maximum points for the whole test were 14.

Exploring and mapping the strategies that students used in task began by dismantling the data (students' responses in items 6A, 6B, 7 and 9). This was done on a detailed level by creating codes based on how students justified their thinking and explained it by using numbers, drawings or written explanations. Coding was concluded with Grounded Theory methods, which provide systematic, yet flexible guidelines for collecting and analysing data (Charmaz, 2014; Birks & Mills, 2015; Chun Tie et al., 2019). Written explanations were worked through in three phases of analysis (Charmaz, 2014), and the framework for coding was created by classifying similar responses to sub-categories (focused coding phase) and core categories (theoretical coding phase). This scheme was used as the analytical tool to assess the strategies that students used in solving tasks and on the other hand, as an indicator on whether the teaching approach provoked a shift from intuitive to more sophisticated heuristics linked to proportional reasoning.

Table 3 illustrates students' correct answers in Task 6A, and how they were grouped as sub-categories during the focused coding phase.

Table 3. Dismantling the data during the initial coding phase and sub-categories in focused coding phase: example from correct approaches in Task 6A

Initial coding phase: observable behaviour	Focused coding phase: sub-category
Student understands that the long side on the second rectangle is "three times longer" than the corresponding side on the first rectangle. Demonstrates thinking by addition: 20+20+20=60 and 15+15+15=45, but cannot clearly explain where ”three times” comes from.	Demonstration of relative thinking between given quantities but failing to provide mathematically understandable explanations.
Student understands that 20 long sticks = 60 short sticks by comparing corresponding parts but cannot explain how he/she gets x=45.	
Student calculates the ratio between the sides of the first rectangle and applies the same logic to another picture.	Demonstration of relative thinking between quantities e.g. by using ratio as a unit in calculations, but not necessarily able to create generalisable formulas.
Student calculates that on the first rectangle the vertical side is ¾ of the horizontal side and applies the logic to the second rectangle to determine x (for example by deducting ¼ from 60).	
Student understands that long sticks are three times longer than short sticks, and is able to utilize the knowledge to solve missing value x.	
Student works with both rectangles simultaneously by using the ratio 3:4 to solve the missing value (ability to form generalisable calculations).	Use of formal operations based on ratio or use of a certain algorithm, such as cross-multiplication or "rule of three".
Student uses a formula, such as a cross-multiplication algorithm, "rule of three" or equivalent to solve the task.	

Consistency for the coding scheme was ensured by comparing the original data in several phases of the coding by student to another student, student's answer to anticipated strategy and strategy by strategy. This involved repeated visits to original answers to ensure that they were understood and interpreted correctly. Final scheme for coding can be found in Appendix 1.

5 Results

5.1 Performance in tasks

Tasks 1-5 and 8 were multiple choice mixture tasks, and even though the student's choice of option could give some indication on solution strategy as well, this report focuses on analysis and classification of solution approaches in tasks 6A, 6B, 7 and 9.

Students' performance in tests provided insights on whether the teaching approach, which focused on mathematical problem-solving, improved students' general skills in solving also proportional reasoning tasks. In the beginning of fifth grade, the mean for total score in the proportional reasoning test was 6,1 points (SD 2,5 p.). Boys (N=12) performed better than girls, their mean being 6,5 points (SD 2,1 p., minimum 3,5 p. and maximum 11 p.), whereas girls (N=12, one being absent) had a mean of 5,5 points (SD 2,9 p., minimum 0,5 p. and maximum 12 p.).

After getting familiar with a variety of different heuristics (but not explicitly algorithms) for solving mathematical problems, the post-test in June indicated positive results: the mean score of students had risen to 8,9 points (SD 3,6 p.). It was interesting to notice that this time girls performed better than boys. Female students' mean had risen from pre-tests' 5,5 points to 9,3 points (SD 3,3 p., min. 5 p. and max. 13,5 p.). Male students also improved their performance: in the pre-test they had a mean of 6,5 points and in the post-test 8,4 points (SD 4 p., min. 0,5 p., max. 13,5 p.). Development of total points is illustrated in Figure 2.

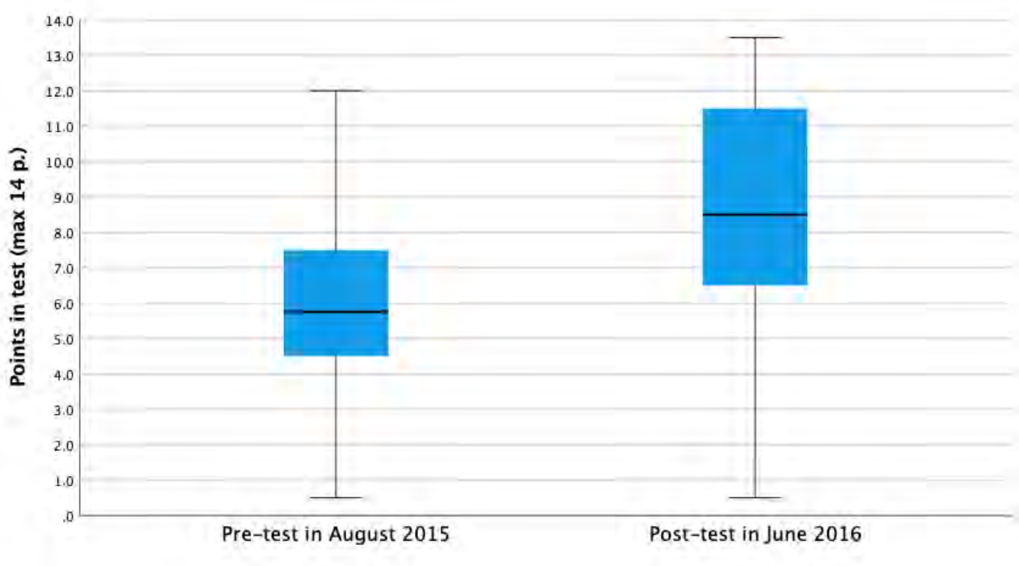


Figure 2. Total points in pre- and post-tests.

The following Figure 3 visualises individual students' performance. Blue marks indicate an individual's total points in the beginning of the fifth grade, whereas orange marks are for post-test points in the end of the school year. Development of skills was visible especially among those students, who in the pre-test scored below the average points, but it seems that the intervention had a positive influence on skills of almost all students[1].

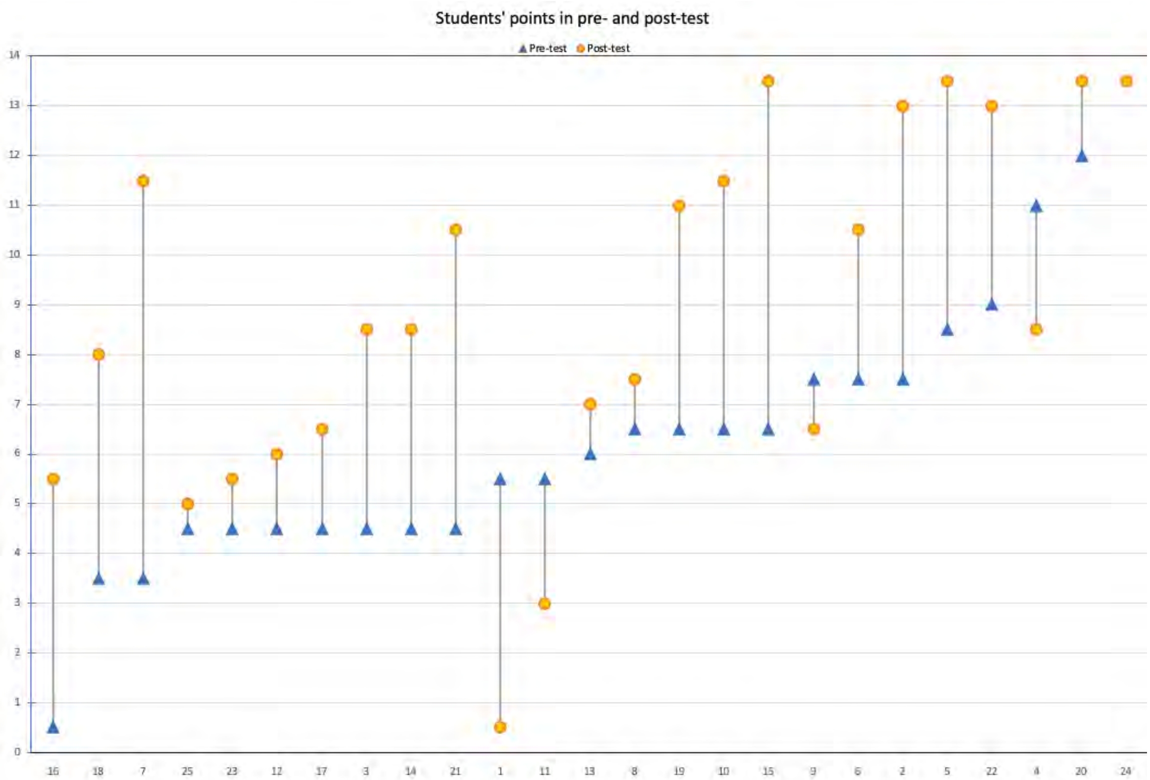


Figure 3. Development of total points by individual students in problem-solving pre-test in August and post-test in July.

Difficulty of tasks is often linked to the number structure and numerical complexity. For example, mathematical problems with small, integer ratios are easier than tasks with non-integer ratios (e.g. Tourniaire, 1986). For assessing the difficulty of items, students' answers were combined with a larger set of data from Finnish fifth graders', which completed the same test. Difficulty of items was done by assessing frequencies of correct and erroneous answers by 95 students. Items in the test sheets

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[1] Student number 24 was absent during pre-test tasks 7-9 and the total points are not calculated. In the post-test, student 1 did not answer any of the questions 6-9, which affected the final score. Student 11 left several tasks unanswered, or it was not possible to determine the answer.

were designed to get gradually more challenging, but it seems that the difficulty of items in this test did not match students' skills and therefore the interpretations on students' performance need to be addressed with reservations. Tasks 1-5 were too easy for fifth graders, and on the other hand the success rate in tasks 6A, 6B and 9 was 20-28%. With this setting, the difficulty of task 7 was fairly ideal (success rate 58%) and task 8 was almost too difficult. If the results of the post-test would be considered, too difficult items would appear to be more ideal also in tasks, which required skills to explain reasoning.

Results indicate that there was no significant improvement in how students performed in multiple-choice mixture tasks in pre- and post-tests. High success rates suggest that tasks 1-5 were easy for fifth graders at the first place. Development of skills is visible in more difficult tasks 6, 7, 8 and 9, which are discussed in more detail in Chapter 5.2. These results indicate that the teaching approach with a focus on problem-solving may have had a positive influence on students' abilities to solve tasks, which require proportional reasoning skills.

Mathematical problems were usually not presented in a written form in a similar way as typical word problems in mathematics books. Students did not get any extra training in solving word problems and therefore the development of skills cannot be explained by them getting more fluent in solving mathematical problems presented in written form. A table describing students' ability to solve tasks correctly can be accessed in Appendix 2 and will be discussed task by task in the next sub-chapter.

5.2 Strategies in tasks

One of the aims of the study was to find out if teaching approach, which offered tools for heuristics, improved students' skills in explaining their thinking in mathematical tasks. In addition to getting familiar with problem-solving techniques, the practical aim for the intervention was to build up students' mathematical self-confidence so that they would become active in describing their problem-solving processes. Questions 6A, 6B, 7 and 9 were assessed as indicators, if students were able to give an understandable explanation on how they processed the task. Informal techniques and strategies provided an insight on how students understood problem-solving concepts and were able to progress even in an unfamiliar type of a problem. Explanations and heuristics were also assessed to see if there were differences in students' use of correct and erroneous strategies between the pre-test and the post-test.

In the beginning of fifth grade, students had major difficulties in describing their problem-solving path and often left the explanation completely out. Teaching approach, which encouraged students to describe their thinking even with partially complete explanations and solving problems one step after another, seemed to have a positive impact on their performance during the later phases of the academic year.

5.2.1 Task 6A: Rectangles

Tasks 6A and 6B represented typical proportional reasoning problems with a missing value. According to Karplus et al. (1983, p. 21), these types of problems involve “reasoning in a system of two variables between which there exists a linear functional relationship”. To maintain proportional values, students carry out parallel transformations within or between variables (Son, 2013). The relation between quantities is invariant, whereas the quantities in the problem co-vary.

In task 6A, a correct approach required the ability to compare corresponding parts between two rectangles. 17 students (68%) provided an answer to the question 6A, and six students (24%) were able to solve the task correctly. 10 students out of 17 were able to explain their solution process, whether the answer was erroneous or correct. Almost a quarter of all students (N=6) were skilled enough to explain their thinking with the correct approach. Seven students provided an answer but did not explain how they ended up in that. Eight students (32%) did not answer the question at all.

It appears that the problem-based teaching approach had a positive impact on students’ skills: in the post-test 92% of students (N=23) answered the question and 68% (N=17) were able to provide a correct answer. 20 students out of 23, who answered the question, were able to explain their thinking in written form. Almost a half (N=12) of all students in the post-test approached the task with the correct strategy. Only three students answered the question but did not explain their thinking and two students (8%) did not answer the question at all.

In pre-test 16% (N=4) were able to implement a correct ratio or unit factor approach in task 6A, demonstrating relative thinking between quantities in solving the unknown quantity. In post-test the number of students using a correct strategy had more than doubled, being 40% (N=10). Even though in many cases an explanation for the solution process did not include all the mathematically correct steps, students were demonstrating the understanding of long sticks being three times longer than the short sticks.

During the pre-test, the most common erroneous strategy was additive reasoning (16%, N=4). It was typical to focus on dimensions within one rectangle, for example reasoning that because the difference between the sides of the first figure was five (20-15=5), the same difference applies for the second figure (60-5=55). In some cases, students calculated the perimeter of the first rectangle and tried to apply or modify the logic to find the missing value in the second rectangle. In both examples students failed to understand the relational nature of the task: if 20 long sticks equal the length of 60 short sticks, the same ratio should be maintained with 15 long sticks and *x* short sticks. According to the previous research, students often rely on additive strategies also in multiplicative situations (e.g. Tourniaire & Pulos, 1985; Nunes & Bryant, 1996; Van Dooren et al., 2010; Son, 2013). Still, it is not clear how students choose their preferations between additive and multiplicative relations (Vanluydt et al., 2019). Both approaches can be characterised as intuitive in nature, yet it is difficult to verbally describe reasoning; the given explanations are not necessarily in line with students' actual solution processes (Degrande et al., 2020).

Distinguishing multiplicative missing value problems from additive ones is challenging for students. Additive thinking is emphasized during the first years of school and the transition towards multiplicative ideas is not always straightforward. On the other hand, additive reasoning could support the development of multiplicative reasoning. Yet, the shift from additive to multiplicative thinking requires a qualitative change in thinking (e.g. Nunes & Bryant, 1996). In the beginning of the fifth grade, only one student approached the problem via multiplicative reasoning but ended up in an erroneous end-result. After the intervention, one fifth (N=5) of students turned into this approach. Even though these solution attempts were erroneous, they could be interpreted as a shift towards understanding the relative nature of the task. A more detailed description on the range of strategies that students used can be accessed in Appendix 3.

Development of solution approaches and possible shifts between the strategies was visualised as individual students' performance in tasks. In Figure 4, explanation categories are presented in an order, which suggests a hierarchy from erroneous and intuitive ones to more sophisticated and generalisable strategies. Light green area marks correct approaches. Opaque fill-ins in pre- and post-test markers indicate that the student was able to solve the task correctly.

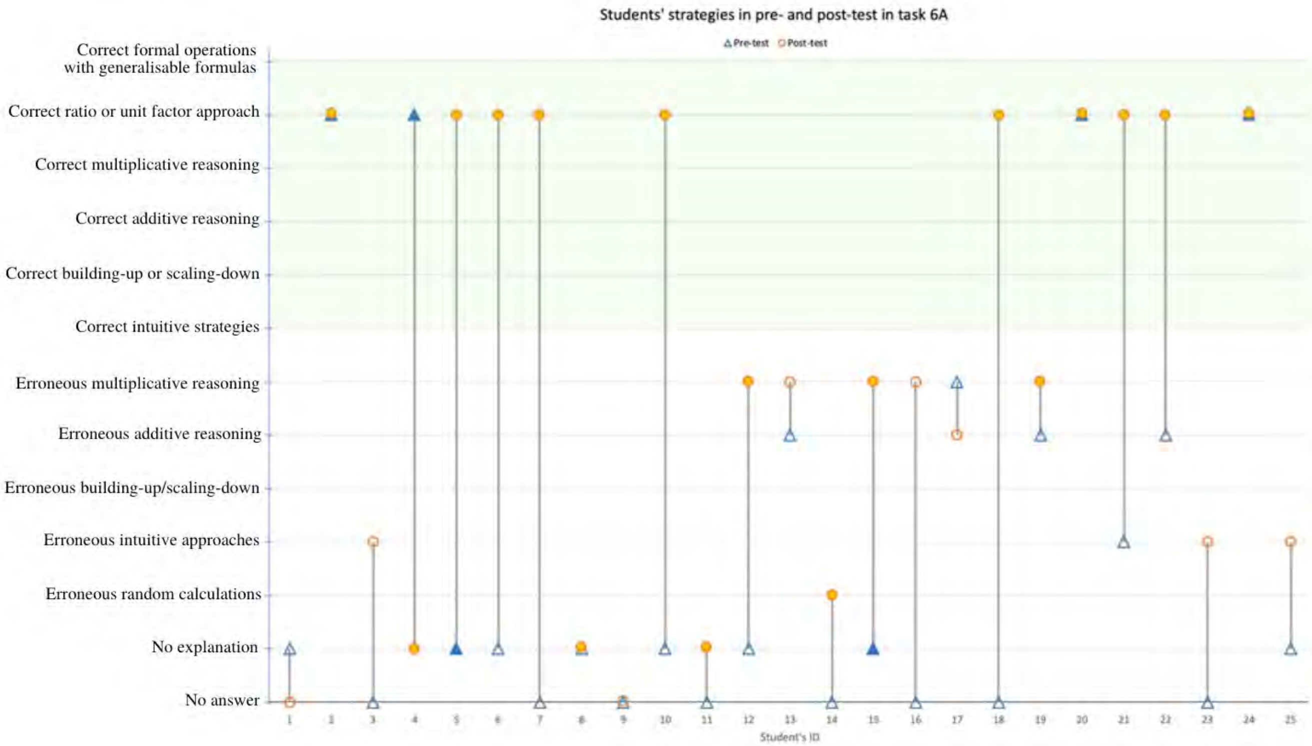

Figure 4. Individual students' strategies in pre- and post-tests in task 6A.

Correct solution approaches were rare in the pre-test, even though the task was relatively easy. In the end of the fifth school year the frequency for correct strategies had increased and students were able to approach the task by correct ratio or unit factor approach.

5.2.2 Task 6B: Triangles

Task 6B was more difficult than 6A. It would have been possible to solve the task only by focusing on dimensions on one triangle and using Pythagorean theorem, but that is a topic for Finnish secondary school curriculum and therefore not expected that any of the students would use that algorithm. In the pre-test 16 students (64%) answered task 6B and only two (8%) of them solved the task correctly. Seven students of 16 explained their thinking process in writing, but only one of them was able to choose a correct strategy. Nine students gave an answer, but no explanation. Nine students (36%) did not answer the question 6B in pre-test.

Before the intervention, students had difficulties in explaining their thinking, 72% of students (N=18) either leaving the explanation out (N=9) or not answering the question at all (N=9). By the end of the school year, the number of empty explanation spaces (36%, N=9) had decreased to half, even though the task was challenging. In the post-test 76% of students (N=19) answered the question and less than a quarter did not (N=6). 48% (N=12) of students were able to provide a correct answer. The majority, 16 students out of 19, tried to explain their thinking in a written form. Three students answered the question but did not provide any insights on the solution process. In the pre-test only one student was able to choose a correct strategy, but in the post-test the number increased to seven students (37% of 19 students answering this question). This was quite an interesting finding, because students had not encountered any similar mathematical problems during the academic year.

In the pre-test, only one student was able to explain thinking by demonstrating mathematically correct reasoning. During the intervention the variety of correct solution strategies increased. Students came to conclusions by additive reasoning or more sophisticated multiplicative reasoning, and there were also some examples of abilities to create correct, generalisable formulas to solve these types of problems. None of the students solved the task by using ratio or unit factor.

Students often relied on erroneous intuitive strategies, such as trying to solve the problem by random calculations on given numbers or basing the problem-solving process on visual observations on given pictures, and not mathematically valid concepts. The range of observable strategies in this task can be accessed in Appendix 3. Figure 5 illustrates the changes in used strategies that individual students had between from the pre-test and to the post-test.

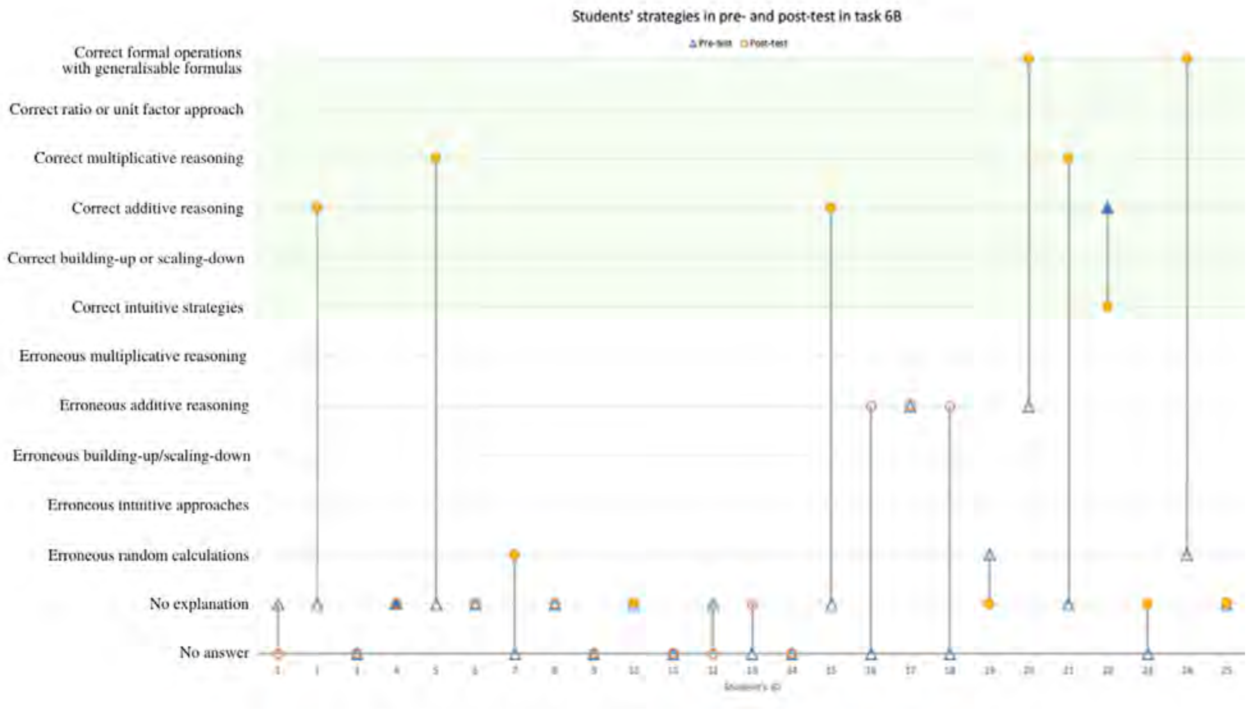

Figure 5. Individual students' strategies in pre- and post-tests in task 6B.

As one can see from Figure 5, task 6B was more difficult than 6A for this student group. After the teaching experiment, successful students chose usually correct additive or multiplicative reasoning, but many students left the explanation out still during the post-test.

5.2.3 Task 7: Painters

Seventh task was based on inverse proportionality. Painting a house involved a situation, in which the time spent on painting was reduced from three days to one day, and students were calculating the number of people needed in painting work. In this task, it was crucial to understand that it would take three times as many painters to complete the work in 1/3 of the time. The analysis of students' responses raised a question, whether many of them solved the task correctly without really understanding the concept. Due to the numerical structure in this task, it was possible to end up in a correct answer of 18 painters by simply multiplying the word problems' given numbers, six and three.

67% (N=16) of students in the pre-test solved the problem correctly, and in the post-test the frequency had increased to 80% (N=20). Only one student in both tests did not answer the question at all. In the pre-test 70% of students (N=16) who answered the questions also explained their thinking, but only one of them was able to choose a correct strategy. In the post-test 24 out of 25 students gave an explanation on their solution process, and at that point 83% (N=20) of them used the correct approach. In the post-test none of them left the explanation slot empty. High success rates in all student groups are possibly linked also to the possible bias caused by the number structure. Majority of students based their explanation on this particular task simply stating 3x6=18 but did not provide any additional information on how they were thinking, or where the numbers came from. Only a few of the participants with correct answers were able to express that they understood the concept instead of performing a random calculation. They, for example, reasoned the number of painters by building up or scaling down with the figures (e.g., 6 painters=3 days, 12 painters=2 days, 18 painters=1 day) or used the addition or multiplication, but were rarely able to justify, why they chose certain procedures. Even though multiplicative reasoning was the most common correct strategy, it is difficult to assess whether the concept of inverse proportionality was really understood. Range of strategies in task 7 can be accessed in Appendix 4 and development of strategies between pre- and post-tests in Figure 6.

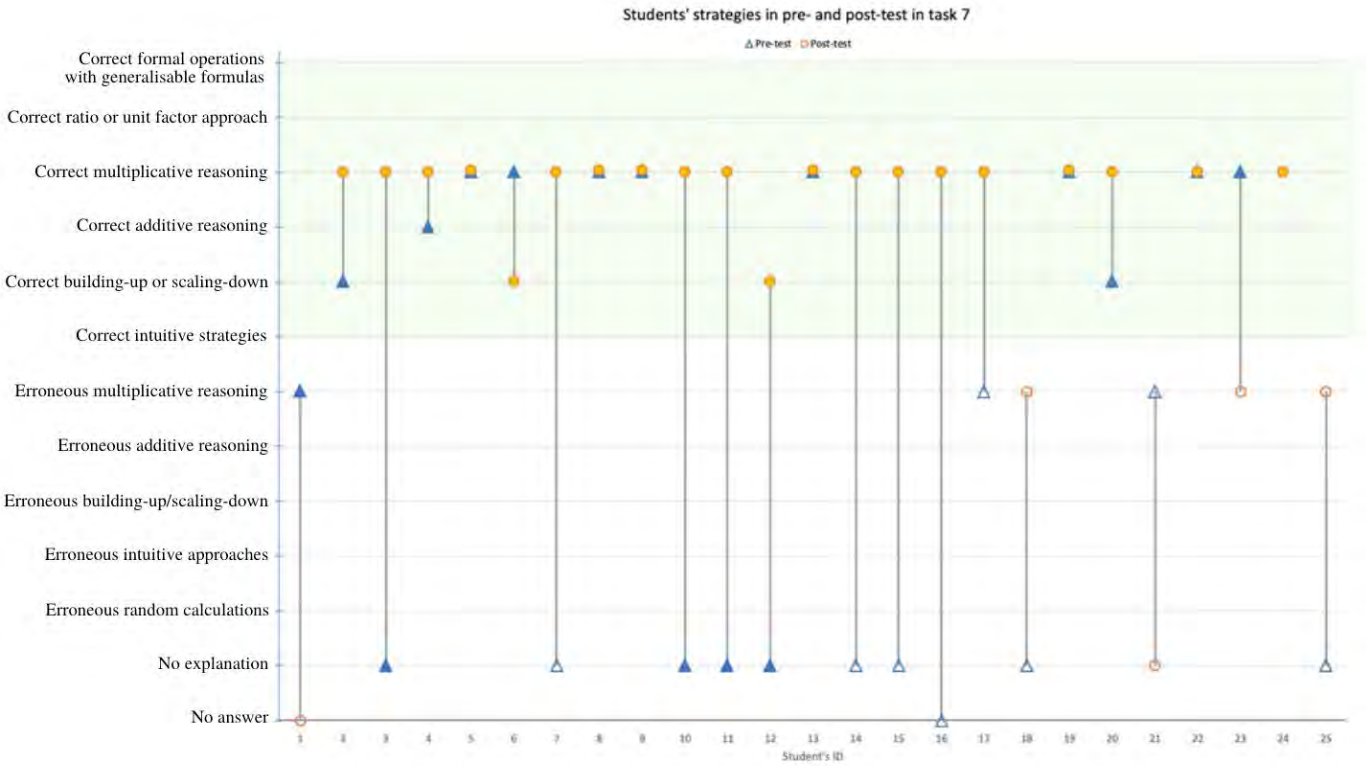

Figure 6. Individual students' strategies in pre- and post-tests in task 7.

It is likely that the wording and the number structure of this task also guided the choice of erroneous approaches: students often relied on erroneous multiplicative reasoning, which was the most common erroneous strategy. To map the real understanding of inverse proportionality, the task could be worded for example by "Two painters paint the house in three days. If they all work at the same speed, how many painters would be needed to paint the same house in two days?" In this case, multiplying two by three would not result in a correct answer.

5.2.4 Task 9: Paint-mixtures

Task 9 was a mixture task, in which students had to maintain the same ratio of paint buckets per mixture to determine the missing value (number of red paint buckets) for the similar mixture. This was a difficult test item for fifth graders, but on the other hand, provided interesting insights on students' development of problem-solving skills.

During the pre-test 29% of students (N=7) did not answer the question at all, whereas the percentage in the comparison group was lower, 14% (N=7). In the post-test, only two students did not answer the question and in both cases, they expressed their unwillingness to engage with the task at all.

17% of students (N=4) had a correct answer in pre-test, but clearly struggled in providing explanations on their reasoning processes: 12 out of 17 students, who answered the question, left the explanation out. Only one student was able to choose the correct strategy in this task, the other four relied on erroneous approaches. Post-test results indicated significant improvement. 52% of students (N=13) ended up with a correct answer, and 10 out of 23 students answering this question also described their thinking with a correct strategy.

When having a closer look on strategies that students used, in the pre-test only one student was able to provide an explanation while solving the problem correctly, turning into a building-up strategy. Development of skills was visible in the post-test: more students were able to not only explain their correct problem-solving process, but also use a more sophisticated strategy by working with the ratio. Even though students did not necessarily have skills to explain thinking with mathematically valid expressions, they became more confident in using different strategies. Figure 7 illustrates the ratio approach, in which student proceeds one step at the time. In this example, the student correctly reasons that because there is one red paint bucket in every four yellow paint buckets, you need to add 1,5 buckets of red to six buckets of yellow.

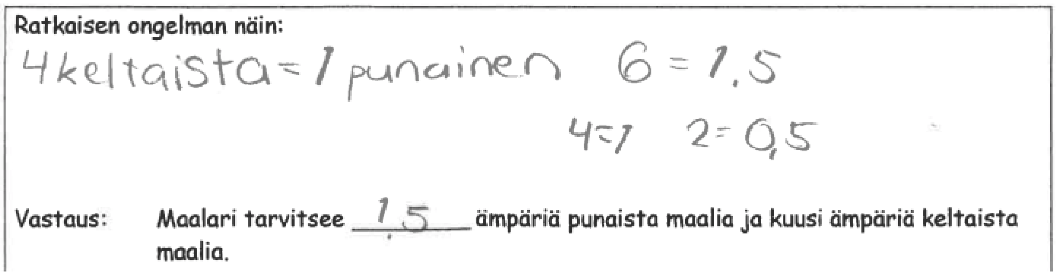
Ratkaisen ongelman näin:
4keltaista = 1 punainen   6 = 1,5
4 = 1   2 = 0,5
Vastaus: Maalari tarvitsee 1,5 ämpäriä punaista maalia ja kuusi ämpäriä keltaista maalia.

Figure 7. Correct example in item 9 (student 42159 in post-test).

During the pre-test, only one of the students was able to provide a correct explanation for the task, working with building-up strategy. In the end of the school year there were indications on improved skills of explaining thinking also visible: 32% (N=8) utilised either building-up or scaling-down strategy, ratio or unit factor approach (the most common) or even correct formal operations with generalisable formulas. In the post-test, five students (20%) were able to work through the task by expressing that for every two buckets of yellow you need 0,5 buckets of red paint.

54% (N=13) of students gave an erroneous answer in the pre-test. Three students relied on multiplicative reasoning but failed to understand the relative nature of the task. They multiplied the given amounts of yellow paint, 6x4=24 or only stated "Calculated by multiplication", without providing a more detailed explanation.

During the post-test the most common erroneous strategy was additive reasoning. 24% (N=6) of students chose that strategy. They often based their reasoning on the idea that "you need three more yellow than red", focusing on the difference between the given numbers in the original paint mixture and ignoring the need to maintain the same relationship for the second paint. More detailed frequencies for the strategies visible in task 9 can be accessed in Appendix 4.

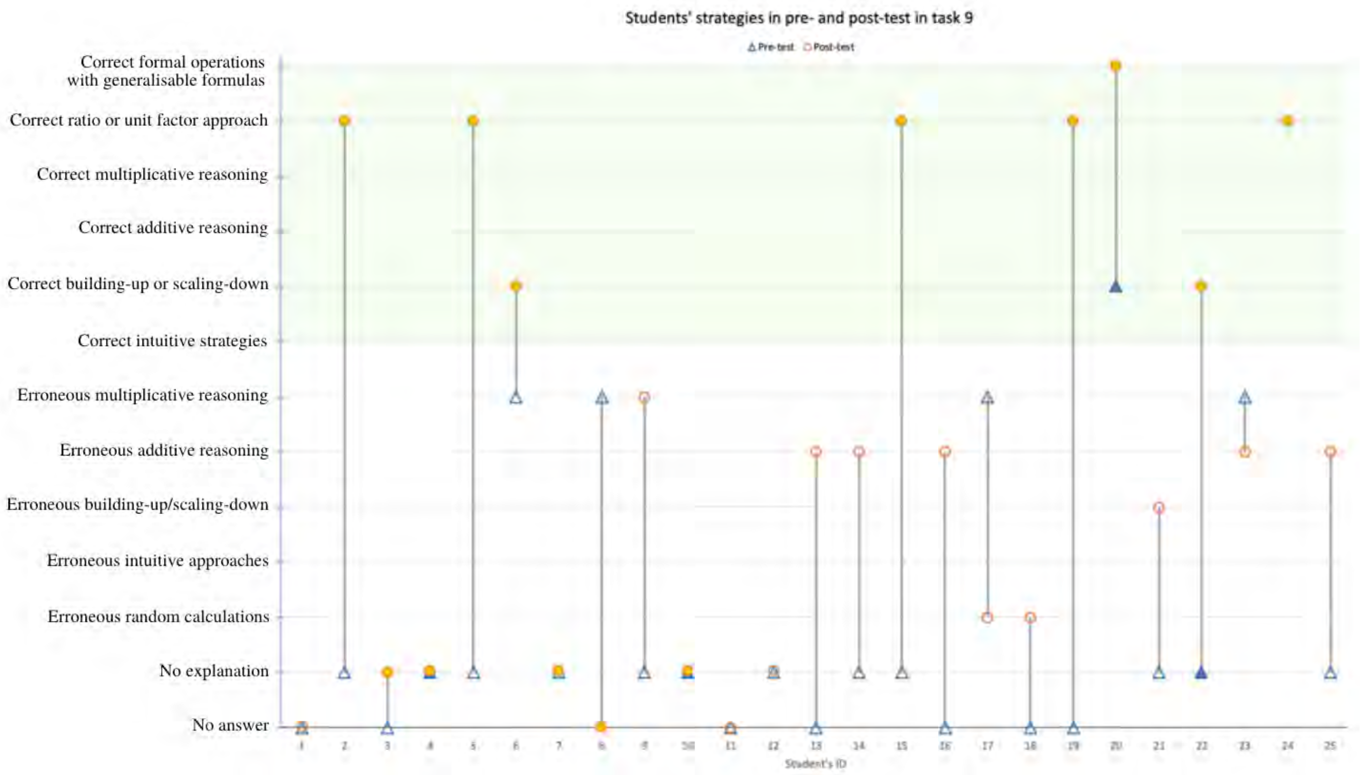

Figure 8. Individual students' strategies in pre- and post-tests in task 9.

Assessing and classifying students' strategies was not always straightforward. For example, student could state that multiplication was needed, but on the other hand, relied on additive reasoning when providing an answer: *"Because in the beginning you needed three more yellow buckets than red buckets, so you just need to multiply it"*, providing three as an answer.

5.2.5 Students with the lowest and highest points

To have a closer look on possible development of strategies of so-called low- and high-performing students, the performance of three students with lowest points and four students with the highest points in the pre-test were considered. Three low-performing students gained a maximum of 3,5 points in the pre-test and four high-performing students 8,5-12 points (for the overview of students' performance, see Figure 3).

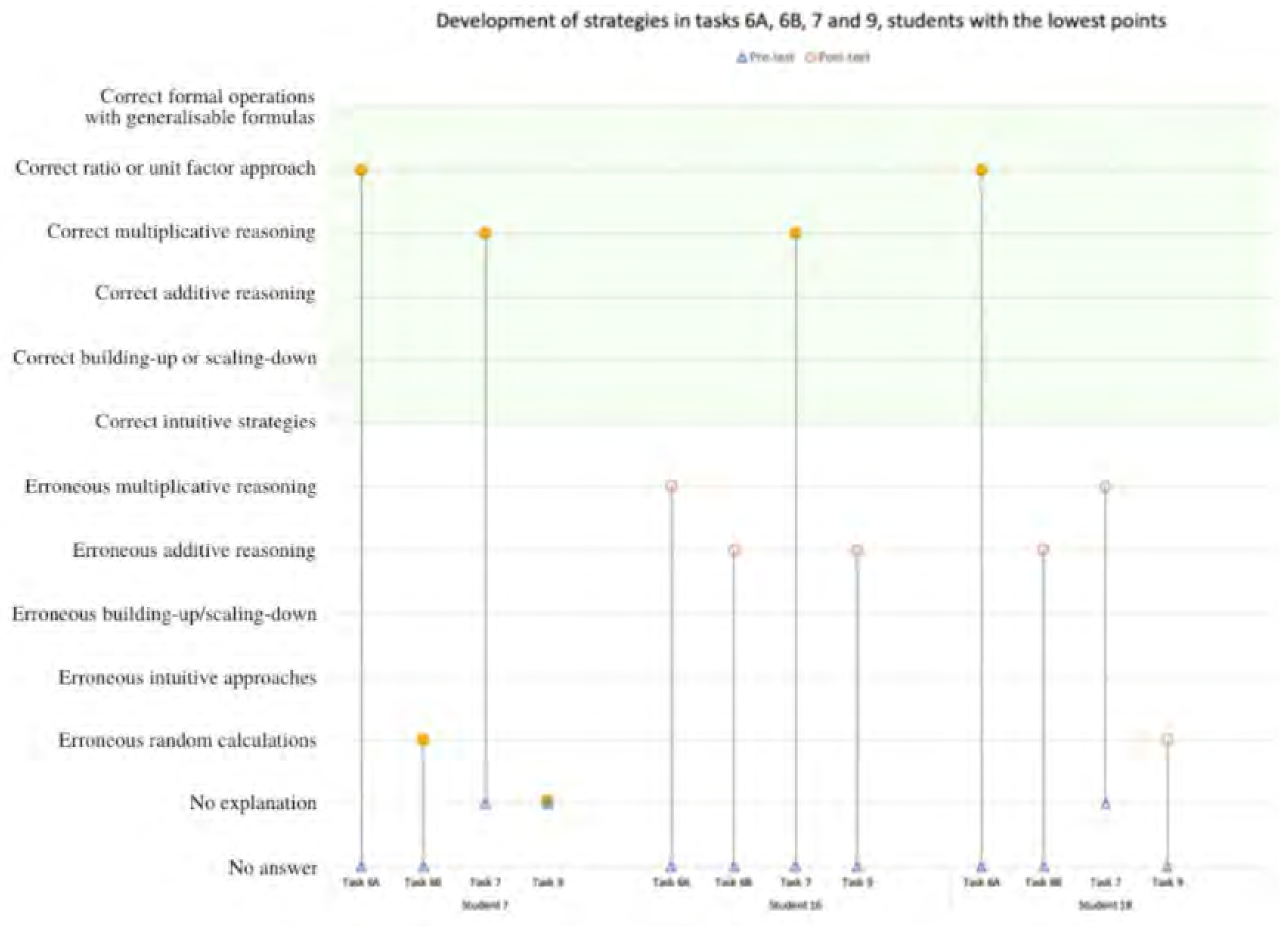

Figure 9. Development of strategies of three students scoring the lowest points in pre-test.

These three students tended to leave the answers completely out in the beginning of the fifth grade. By the end of the fifth grade, frequencies for solving the tasks correctly increased. With some individuals the difference was remarkable: for example, student 7 got correct answers in the post-test, but would still have needed a bit of support in explaining thinking (see Figure 9). After the problem-based teaching period, students were more willing to engage in attempts to solve mathematical problems, even though the strategies might not have been valid. With a correctly timed intervention the teacher has a change to support the shift from erroneous strategies to correct ones.

If lacking the skills to explain thinking with mathematically correct processes, students often started to explore the dimensions between the given values by implementing intuitive methods. Consider the explanation in Figure 10 that student 22 gave in Task 6B: the answer was correct 40 sticks, and in this case, the student seemed to calculate the solution by exploring the given values and their relationships within the first triangle. This student calculated the difference between the hypotenuse and opposite side is multiplied by two to get the adjacent side.

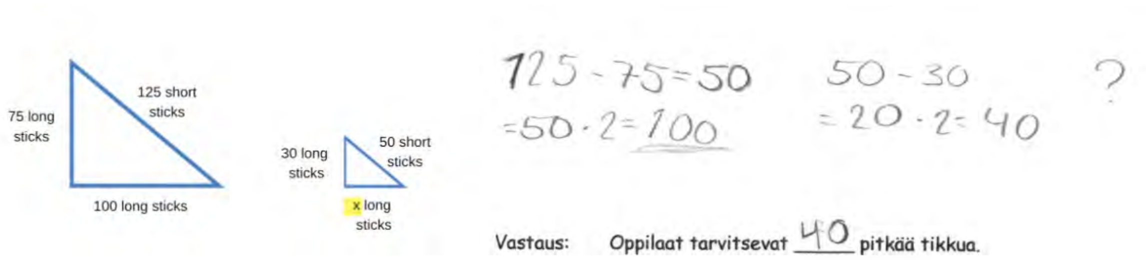

Figure 10. It was not uncommon that the answers were correct,
but not necessarily based on generalisable ideas.

In these kinds of examples, which are very common in primary school, students would benefit from opportunities to discuss their ideas with the teacher or with a peer; what is the purpose of short and long sticks in this task, and how should that information guide the solution process? If the strategy works with the given values, can that be generalised to all triangles with a 90-degree angle? How about all types of triangles?

Findings of the study indicated that especially students with lower points benefited from exploring different problems and heuristics to approach them. Students with high points in the pre-test were able to develop their skills in explaining their ideas and on the other hand, to move towards more advanced strategies (see Figure 11).

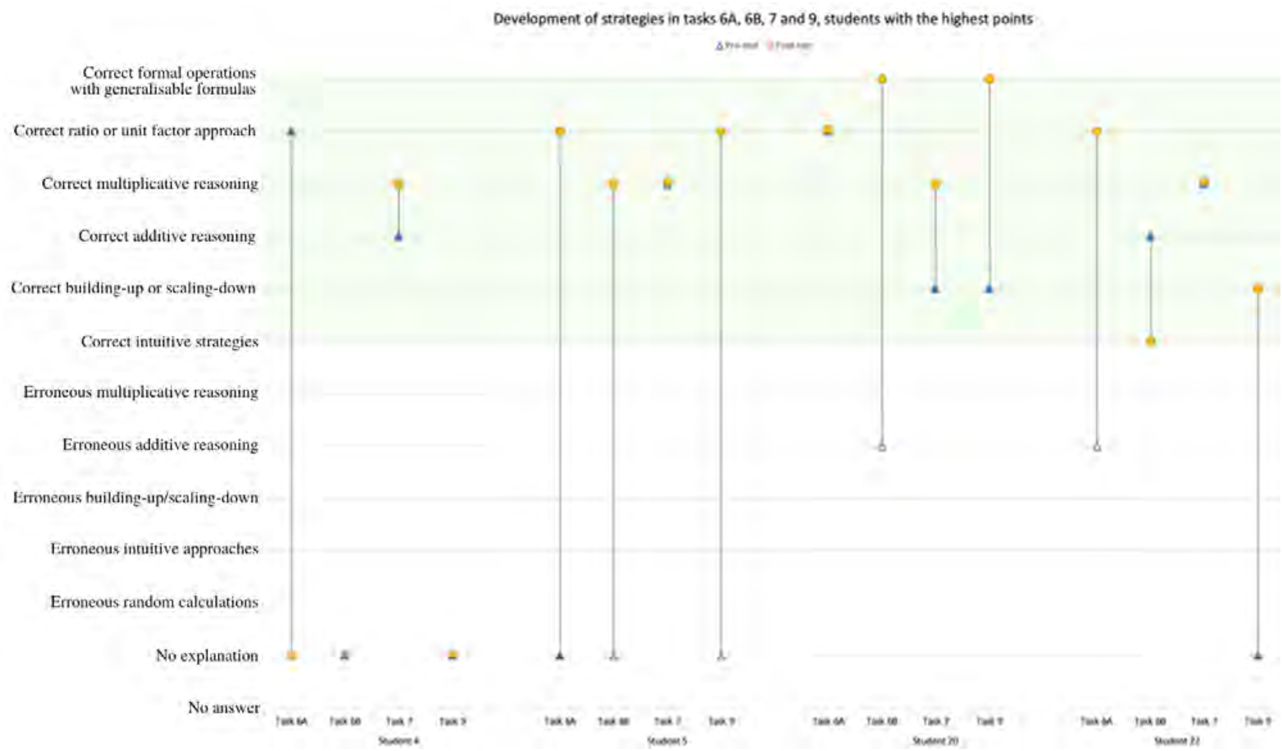

Figure 11. Development of strategies of four students scoring the highest points in pre-test.

Problem-oriented teaching approach, which emphasized the importance of discussing and explaining ideas, no matter even if they are just partially constructed or immature, seemed to have a positive influence on how students communicated their thinking in written tasks. By the end of the school year there was a significant improvement on students' reasoning skills, use of heuristics and abilities to explain their thinking. Some limitations on these observations needs to be addressed: with this research design, it is not possible to assess, whether the skills would have been improved by more traditional teaching approach as well. Another challenge is linked to the test items: several of them appeared to be too easy for Finnish students and tasks were completed in a shorter time than expected. A test with a wider variety of difficulty and more items would provide more reliable information on possible development of skills and strategies.

6 Discussion

The study focused on exploring whether students benefitted from a problem-solving focused teaching approach, which introduced them to a general set of heuristics as a concrete tool called Problem-solving Keys. This tool worked as visual reminders of a variety of generalisable approaches for mathematical problems. The study aimed to explore whether this kind of an active, heuristics-based teaching approach would improve fifth graders’ performance and use of strategies and develop students' skills in explaining their thinking in mathematical tasks.

The analysis indicated that skills to explain thinking improved. Before the intervention, students generally relied on intuitive strategies or opted to leave the justification completely out. After getting familiar with concrete tools for general heuristics, students became more confident in expressing their ideas and justifying their strategies and were also willing to help the others by explaining solution methods. Mathematical discourse helps students not only to develop their understanding of mathematical ideas, but also to build a personal relationship with mathematics on an emotional level (D’Ambrosio & Prevost, 2008). After the intervention students’ variety of heuristics increased and they were able to choose more sophisticated ones when solving different tasks. This can be interpreted as a positive development. This development was visible also in situations, in which the student still worked by implementing an erroneous strategy: in many cases, there was a shift from an intuitive approach towards a more sophisticated, yet erroneous approach. For example, an ability to base decisions on multiplication and demonstrate some understanding of the relative nature of the task can be interpreted as a step towards proportional reasoning, even though the student would not be able to expand the idea to cover the whole concept on a certain task. Findings of this study suggests that also the erroneous approaches can be viewed as hierarchical steps towards more sophisticated skills and correct reasoning. Teacher has a crucial role in recognising these small steps, for example student’s transition from erroneous intuitive approaches (for example drawing) towards additive and multiplicative reasoning and emerging skills in understanding relational nature of proportional reasoning tasks.

In Polya’s model (1945/1973), the last phase of “looking back” provides opportunities to assess and discuss ideas that emerged during the problem-solving process. This research underlines the importance of discussing different approaches and heuristics already during the earlier phases of problem-solving. This increases students’ confidence in presenting also the partially correct ideas, which can be seen as steps forwards. Teacher’s role is to make sure that students are not left with the impression that *any* answer is mathematically valid, and to guide them towards correct methods and mathematically correct language.

By teaching heuristics, students learn to solve complex word problems, reason mathematically in everyday situations and develop their thinking skills. Heuristics should be understood as general guidelines, methods, or possibilities to approach a diverse set of mathematical problems. Still, learning heuristics does not alone help students, and heuristics as such should not be reduced to learning certain techniques or sets of algorithms to choose from. Learning to describe and justify thought processes is equally important. It can be asked whether school mathematics in primary schools supports students' development in explaining their thinking, or is the focus still on finding the correct answer? This is problematic when considering the transition to secondary school, where students are expected to be able to explain their thinking by using mathematical language. Teaching approach, which guides students in justifying their ideas by using various methods, develops mathematical problem-solving skills and creates an excellent foundation for learning more complex mathematical concepts. Classroom discussions enable the teacher to make decisions on which state students benefit from teacher's guidance, and when it is more fruitful to let them find out the solution by themselves.

A few limitations of this study need to be addressed. The data for this research was collected from one sub-urban, monolingual primary school in Northern Finland. With a larger sample from Finnish schools, it would have been possible to gain more generalisable results on whether the students' performance and use of certain strategies would follow similar trends in schools in different areas. Another limitation is linked to the development of problem-solving skills and the possible effect that the teaching approach had on the results: it would have been beneficial to have the same pretest-posttest setting with a group of students without the intervention. At the point of implementing the teaching approach and collecting the data, this was not the main focus of the research, but the aim was to develop and assess the heuristics-based teaching approach, practicing teacher being also the researcher (e.g. Niemi & Nevgi, 2014). Further research is needed to understand the natural development of problem-solving skills and strategies, and whether and what kind of "out-of-the-textbook" approaches in mathematics classrooms could enhance these skills.

Students should be provided rich mathematical problems and taught a variety of problem-solving heuristics to tackle the demands of the 21st century. Mathematics should work as a tool, which would help in facing everyday situations. Even though not everyone becomes a mathematician, students' fluency as mathematical thinkers and problem solvers can be supported by paying attention in developing their skills already in primary school. Mathematics curriculum in Finland offers flexibility to shift from arithmetic "fill-in-the-book" exercises towards a meaningful problem-solving teaching approach. Teaching mathematics through problem-solving provides opportunities to develop a wide variety of problem-solving strategies and heuristics. Problem-solving Keys are one easily accessible tool to enhance these skills.

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