# The three worlds and two sides of mathematics and a visual construction for a continuous nowhere differentiable function

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A rigorous and axiomatic-deductive approach is emphasized in teaching mathematics at university-level. Therefore, the secondary-tertiary transition includes a major change in mathematical thinking. One viewpoint to examine such elements of mathematical thinking is David Tall's framework of the three worlds of mathematics. Tall's framework describes the aspects and the development of mathematical thinking from early childhood to university-level mathematics. In this theoretical article, we further elaborate Tall's framework. First, we present a division between the subjective-social and objective sides of mathematics. Then, we combine Tall's distinction to ours and present a framework of six dimensions of mathematics. We demonstrate this framework by discussion on the definition of continuity and by presenting a visual construction of a nowhere differentiable function and analyzing the way in which this construction is communicated visually. In this connection, we discuss the importance to distinguish the subjective-social from the objective side of mathematics. We argue that the framework presented in this paper can be useful in developing mathematics teaching at all levels and can be applied in educational research to analyze mathematical communication in authentic situations.

Keywords: mathematical thinking, advanced mathematics, the three worlds of mathematics, secondary-tertiary transition, nowhere differentiable functions

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## 1 Introduction

Research on mathematical thinking includes several approaches that focus on different aspects of the subject such as the pedagogical, cultural or cognitive (Sternberg, 1996). A recent review of research on mathematical thinking (Goos & Kaya, 2020) divides these different approaches into individual cognitive and constructivist perspective, cultural psychology perspective and discourse perspective. Mathematics education research has a long tradition in exploring the cognitive aspects of mathematical thinking (see e.g., Bingolbali & Monaghan, 2008; Fan & Bokhove, 2014; Tall, 1991). In this theoretical article, we draw upon that tradition and present a novel theoretical framework describing various aspects of mathematical thinking in mathematical discourse. We discuss the framework using examples from university-level mathematics, although we also consider the significance of our theoretical elaboration for primary and secondary school mathematics teaching and learning.

Several studies have described special features of the context of university mathematics. One of the main interests of such studies has been the secondary-tertiary transition. The transition includes a change in mathematical content, sociomathematical norms and educational culture (Education Committee of the EMS, 2013), and therefore causes both cognitive and pedagogical shocks to beginning undergraduates (Clark & Lovric, 2009). The secondary-tertiary transition has been found problematic for decades and consequently many beginning undergraduate become dropouts (Di Martino & Gregorio, 2019). Regarding cognitive aspects of the transition, a rigorous and axiomatic-deductive approach is emphasized at university, meaning that the transition includes a major change in mathematical thinking (Tall, 2008). For this reason, some universities have, for example, developed special bridging courses to ease the transition to advanced mathematical thinking.

Mathematics, however, is not only an art of axiomatic-deductive reasoning or manipulating symbols, neither in school mathematics nor in university-level mathematics. One influential framework to describe the variety of mathematical thinking is the framework of the three worlds of mathematics (Tall, 2013). Tall (2013) divides mathematical thinking into embodied world (pictures, gestures etc.), symbolic world (calculations, symbolic rules etc.) and formal world (axioms, proofs etc.). The interplay between these different worlds of mathematics has been found useful for developing undergraduate level mathematics teaching (Oikkonen, 2009), as well as teacher education (Hannula, 2018). That is to say, such interplay is important in terms of secondary-tertiary transition, as well as in terms of development of pre-service teachers' mathematical knowledge for teaching (see e.g., Dreher & Kuntze, 2015).

In this article, we elaborate the framework of three worlds of mathematics further. First, we shall discuss Tall's framework and combine it with a distinction of two sides of mathematics, that is, the subjective-social and objective sides of mathematics. Together these ways of looking at mathematics will lead to a division of 6 = 2 x 3 dimensions of mathematics. After discussing mathematical thinking in general, we present a discussion clarifying the definition of continuity and a construction for a continuous nowhere differentiable function. Such functions are related to advanced undergraduate level mathematics courses. The construction, and the way in which we present it, is a novel one and does not appear for instance in Thim's extensive review

(2003) of continuous nowhere differentiable functions. David Tall has been using a construction which he calls the Blancmange function in many of his writings (see e.g., Tall & Di Giacomo, 2000; Tall, 1982). We use our construction as an example to demonstrate six different dimensions of mathematics and their interplay.

The motivation of our theoretical considerations is connected to our experiences as a research mathematician and university lecturer (the first author) and as a mathematics teacher and teacher educator (the second author). In our work, we have found Tall's framework extremely fruitful in developing teaching and conducting educational research. However, we have come to the conclusion that Tall's distinction does not capture all aspects of mathematical discourse in authentic situations. Therefore, we find a theoretical elaboration of Tall's framework useful in developing teaching and in educational research. In this article, we aim to

1. introduce a novel theoretical framework of the six dimensions of mathematics
2. demonstrate the framework in the cases of the definition of continuity and a construction of a continuous nowhere differentiable function
3. discuss the possibilities of the framework for educational research and development of mathematics teaching and learning.

## 2 Theoretical framework

We examine the broad concept of mathematical thinking from a cognitive viewpoint. In the following subsections, we first present a summary of frameworks describing cognitive aspects of mathematical thinking. Second, we discuss in more detail the framework of the three worlds of mathematics (Tall, 2013). Third, we present our distinction between subjective-social and objective sides of mathematics. Finally, we elaborate Tall's framework further by combining the three worlds of mathematics with the distinction of two sides of mathematics.

### 2.1 Cognitive frameworks of mathematical thinking

Since the very beginning of the discipline, mathematics education researchers have presented several dichotomies and classifications of mathematical thinking and knowledge. Skemp (1976), for instance, divides mathematical understanding into *instrumental* understanding and *relational* understanding. Roughly speaking, instrumental understanding refers to *how* to carry out mathematical operations whereas relational understanding refers to *why* mathematical operations work.

Similarly, Hiebert (1986) divides between *conceptual* and *procedural* knowledge. Several similar dichotomies have been used in educational theories, and Haapasalo (2003), for instance, lists 20 such dichotomies presented in literature. Instead of discussing all these dichotomies, we summarize some of the most influential frameworks of mathematical thinking underlying present mathematics education research.

One of the most established frameworks in mathematics education research is the distinction between *concept image* and *concept definition* (Tall & Vinner, 1981). (It seems that the origin of David Tall's three worlds of mathematics lies there.) Concept image refers roughly to one's understanding of a mathematical concept and concept definition to the official definition of the concept. Tall and Vinner (1981) define concept image as the total cognitive structure that is associated with a mathematical concept. Thus, concept image may include mental pictures, symbolic processes and axioms etc.

Development of students' concept images have been widely studied in literature especially from the viewpoint of processes and concepts. The term *encapsulation*, originating from Piaget, means a change in thinking in which learner starts to think the concept itself instead of the process (Tall, 2013). For instance, Sfard (1991) considers the dualism between the operational and structural sides of mathematics. The encapsulation process, according to Sfard (1991), occurs in three steps: interiorization, condensation and reification. Similarly, Gray and Tall (1991) speak of *procepts* referring to an 'amalgam of process and concept in which process and product is represented by the same symbolism' (Gray & Tall, 1991, p. 73). Additionally, one influential framework explaining the encapsulation process is APOS-theory presented by Ed Dubinsky and colleagues (Asiala et al., 1996; Dubinsky & McDonald, 2002).

Frameworks presented above focus mostly on learning of algebra and calculus. In case of geometry, for instance, van Hiele levels (Burger & Shaughnessy, 1986) give a widely used framework to analyze students' learning. Some researchers have, however, presented more generic frameworks. Already in the 1960's Bruner (1967) divided mathematical representations into enactive, iconic and symbolic. Similarly, Fishbein (1994) classifies intuitive, algorithmic and formal approach to mathematical activity.

Furthermore, Viholainen (2008) separates mathematical reasoning into formal reasoning based on axioms, definitions and proven theorems, and informal reasoning based on visual or physical interpretations of mathematical concepts. Some researchers, such as Joutsenlahti (2005), have studied the overall picture of mathematical thinking including students' knowledge as well as their beliefs. In his doctoral dissertation, Joutsenlahti (2005) explored mathematical thinking from societal perspective, teacher's perspective and student's perspective.

Influenced by many frameworks presented above, Tall presented his framework of the three worlds of mathematics first in a conference paper (Tall, 2004). In that paper Tall divides mathematical thinking into *embodied*, *symbolic* and *formal* worlds of mathematics. Tall's framework aims to give an overall view to mathematical thinking and its development (Tall, 2013). Adapted from Chin (2013), some established frameworks of mathematical thinking are summarized in Table 1.

**Table 1.** Summary of frameworks adapted from Chin (2013)

| Researcher(s) | Key concepts of the framework | Focus |
|---|---|---|
| Sfard | operational – structural | encapsulation process |
| Gray & Tall | procept | procedural and conceptual knowledge |
| Dubinsky et al. | action - process - object - schema | cognitive development |
| Van Hiele | perceptions - operations - proofs | levels of knowledge in geometry |
| Bruner | iconic - enactive - symbolic | representations |
| Fischbein | intuitive - algorithmic - formal | approaches to mathematics |
| Viholainen | informal - formal | mathematical argumentation and reasoning |
| Joutsenlahti | knowledge - beliefs | aspects of mathematical thinking |
| Tall | embodiment - symbolism - formalism | modes of mathematical thinking |

The summary in Table 1 highlights the variety of frameworks describing cognitive aspects of mathematical thinking. In this article, we elaborate Tall's broad framework of the three worlds of mathematics.

## 2.2 The three worlds of mathematics

The idea of three worlds of mathematics is based on humans' capability to

i) *recognize* regularities, similarities and differences,
ii) *repeat* actions, and
iii) *use language* to name concepts (Tall, 2013).

Based on these humans' cognitive-physiological capabilities Tall divides mathematical thinking into conceptual-embodied, proceptual-symbolic, and axiomatic-formal worlds of mathematics. Tall has discussed his worlds in a great number of writings and the concepts have developed somewhat over the years.

In his book Tall (2013) gives an overall view of his work and describes the worlds as follows.

> A world of (conceptual) embodiment building on human perceptions and actions developing mental images verbalized in increasingly sophisticated ways to become perfect mental entities in our imagination;
> A world of (operational) symbolism developing from embodied human actions into symbolic procedures of calculation and manipulation that may be compressed into procepts to enable flexible operational thinking;
> A world of (axiomatic) formalism building formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof. (Tall, 2013, p. 133)

Later, we refer to these worlds simply as embodied, symbolic and formal.

The embodied world includes embodied thinking about mathematical concepts and processes such as pictures and physical objects, whereas the symbolic world includes symbolic thinking such as calculation rules. Formal world, on the other hand, includes rigorous mathematical theory including proofs and axioms. As an example, Tall (2013, p. 25) relates the system of the real numbers to these worlds. The real numbers have embodiment as a number line, symbolism as (infinite) decimals, and formalism as a complete ordered field (Figure 1).

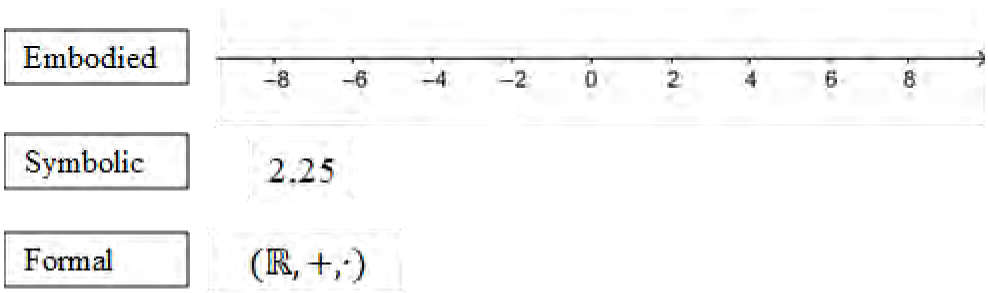



Figure 1. The concept of real number and the three worlds of mathematics

Although all these worlds are apparent in both school and university mathematics, the secondary-tertiary transition includes a change in emphasis from the embodied and symbolic world to the formal world (Tall, 2004; Tall, 2008). Therefore, the interplay between the different worlds is crucial in undergraduate mathematics and teacher education (Oikkonen, 2009; Hannula, 2018). Although the worlds are hierarchical in regard to cognitive development, all of them are more or less present in mathematical discourse at all levels of education.

Sfard's (1991) framework has some resemblance to the three worlds of Tall, but the ordering makes a big difference. The doctoral thesis of Hähkiöniemi (2006) is interesting in this respect. Tall (2008) remarks that Hähkiöniemi (2006) considered the routes of students towards learning the derivative. Tall say that Hähkiöniemi 'found that the embodied world offers powerful thinking tools for students' who 'consider the derivative as an object at an early stage'. According to Tall this questions Sfrad's suggestion that operational thinking precedes structural.

## 2.3 The subjective-social and objective sides of mathematics

As discussed above, several dichotomies and distinctions of mathematical thinking and activity have been presented in literature. These frameworks focus, for instance, on representations and cognitive development (e.g. Bruner, 1967; Fischbein, 1994). On the other hand, many of these frameworks somehow distinct between formal and informal aspects (e.g. Tall, 2013; Viholainen, 2008) or conceptual and procedural aspects (e.g. Gray & Tall, 1991) of mathematical thinking. Our distinction, presented in this section, is somewhat different to prior distinctions, and can actually be seen as 'orthogonal' to many of them.

Our distinction is based on the observation that there are aspects in mathematics that are objective and others that are subjective or social. To the first belong printed formulas and pictures that one can find in textbooks etc. To the latter belong my mental images that I as the author had in my mind while writing formulas or making pictures appearing in printed material, and your mental images that you as a reader had in your mind while reading the text. We refer to this distinction by speaking about the two sides of mathematics.

In our own work as university and schoolteachers, as well as mathematics and mathematics education researchers, such a division between two sides of mathematics has become important. But the emphasis seems to be somewhat different from those approaches referred to above. For us the division is related to what one does 'here and now' e.g., while working on a mathematical problem or teaching a mathematical concept: does one in the next moment speak about the ideas behind a mathematical concept or does one explicitly work with the formal definition of the concept. Our idea of the two sides was initially outlined several years ago

(Oikkonen, 2004) and it has been an important idea behind first author's development of university mathematics teaching (Oikkonen, 2008, 2009).

Let us consider as an example the continuity of a function $g$ at a point $a$. The idea is simple: $g(x)$ should be near $g(a)$ when $x$ is near $a$. This is often visualized by well-known pictures like Figure 2.

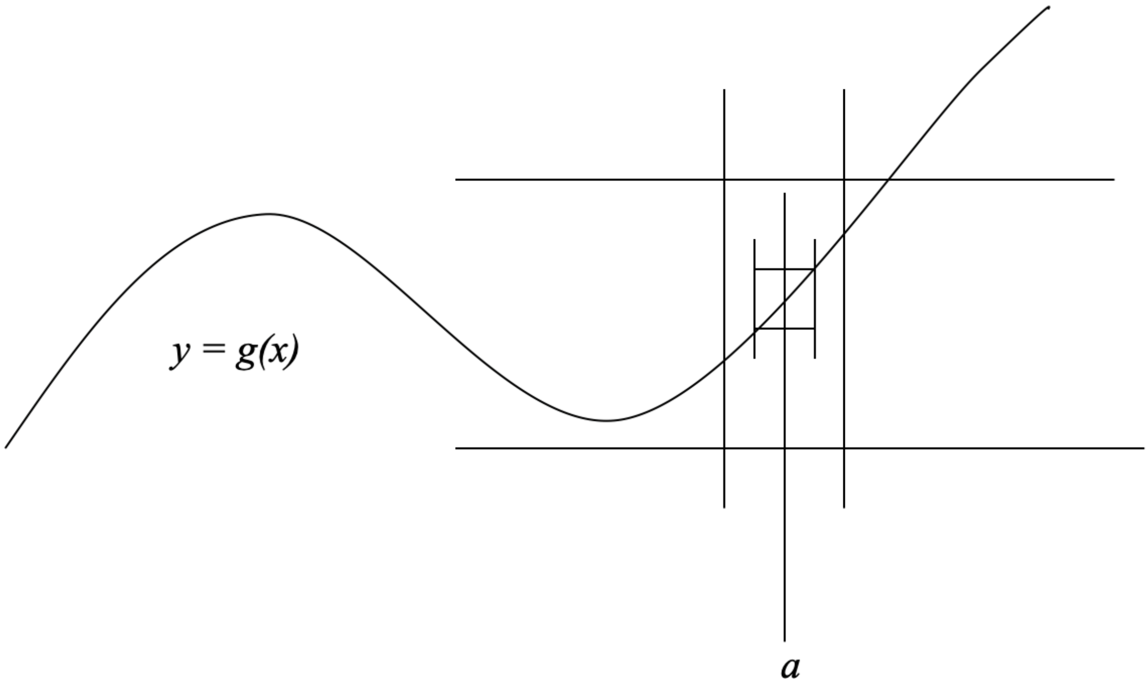



Figure 2. Continuity of *g* at *a*

Pictures like that in Figure 2 can be argued to represent the embodied world of mathematics while the exact epsilon-delta definition represents symbolic world of mathematics. But if we look closer at how these pictures are used in teaching, we see an example of the interplay between the subjective-social and the objective sides of mathematics.

We come now to our first main example. Consider an imaginary discussion between a teacher T and (a) student(s) S. The letters A to E refer to the pictures in Figure 3 (A-E).

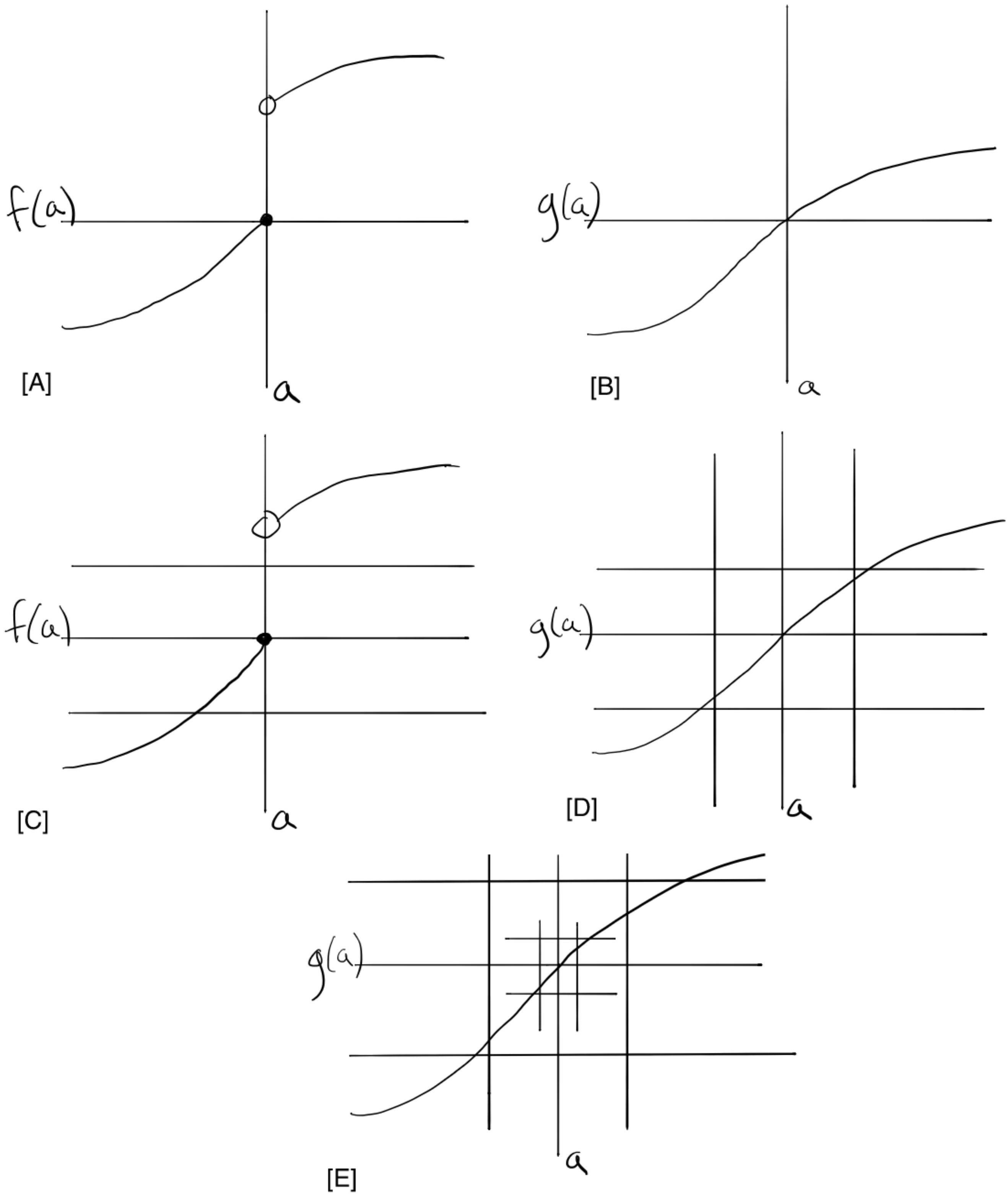



Figure 3. Discussion about continuity

The discussion goes in the following way.

T: Is f appearing in picture A continuous at a?

S: No!

T: Is g in B continuous at a?

S: Yes!

T: Why f is not continuous, but g is continuous?

S: g(x) goes near g(a) as x goes near a but f(x) does not go near f(a) as x goes near a.

T: Can you elaborate / be more exact?

S: ???

T: Let us draw horizontal lines shown in C and D. What can you see?

S: g(x) appears between these lines as x is close to a but a part of f stays outside the lines no matter how close to a we are.

T: Yes! For g we get in the bigger rectangle shown in E. (The graph of) g does not cut / go through the floor or roof of this rectangle. What happens if we draw new horizontal lines as in E closer to y = g(a)?

S: We can draw new vertical lines and get a smaller rectangle so that g does not cut the floor or the roof. This appears in the smaller rectangle of E.

T: Good! When the horizontal lines are near enough the line y = g(a) so that we can be sure of continuity?

S: ???

T: Never. The point in continuity is that no matter however close we draw the lines, there always are the vertical lines making a box such that g does not cut the floor or roof of the box. Can you say this in other words?

S: Could it work to say that for all horizontal lines...?

T: Yes. And it is enough to speak about vertical and horizontal distances. Actually, this is exactly what the epsilon-delta definition in your textbook says!

Pictures somewhat like Figure 2 appeared in the above discussion but here they had a role as a means of sharing thinking between the teacher and the student(s). Hence these pictures and the whole discussion are examples of the subjective-social side of mathematics. The discussion ends in a reference to the textbook of the students and the formal definition continuity. These are of course examples of the objective side of mathematics.

When combining the subjective-social vs. objective dichotomy to Tall's three worlds, we can say that the main part of the previous discussion lives on the subjective-social side and in Tall's embodied world.

Continuity has also an objective aspect, the well-known epsilon delta definition appearing in textbooks and mentioned at the end of the above discussion. In Tall's terminology, the definition belongs to the symbolic world, perhaps with a flavor of formal world. According to the definition, a function g is continuous at a, if (and only if) for every $\varepsilon > 0$ there is such a $\delta > 0$ that $|g(x) - g(a)| < \varepsilon$ for all x satisfying $|x - a| < \delta$.

One of the main purposes of an introductory course in analysis is to teach this kind of definitions and proofs of the main theorems of analysis based on such definitions. But it is not an easy task. This is not helped by the way how we too often begin solutions of examples or proofs of theorems: 'Assume that $\varepsilon > 0$. Let $\delta = \frac{3}{7}\varepsilon$...'

The first author's experience in teaching analysis supports the idea that it is helpful to change the viewpoint from which we look at mathematics. This takes place by combining the formal definitions with an active use of teacher's and students' mental images like the one described above. By doing this it is also possible to reveal in teaching the way in which an expert mathematician thinks.

In our experience this kind of an approach helps in making the content of a mathematics course meaningful and understandable to students. Thus, a course in mathematics is not only the polished formal content of the course but also – and to the authors essentially – the thinking and culture that lies behind the text. We believe that this approach explains partially the success shown in Oikkonen (2009). (There are also other pedagogical ideas involved in this paper.)

The first author's path to this kind of an approach results from the striking similarity between two seemingly quite different types of discussion on mathematics in which he has taken part: those taking place in math days in elementary schools and those taking place when experts discuss some problem in research mathematics. The 'here and now' choice between different kinds of action that was mentioned above seems to be characteristic to such discourses.

So, we have two sides of mathematics. But which of them is the correct one? Let us go back to continuity: which side is the correct one, the human (mental) images or the formal epsilon-delta definition? Our own answer is that neither of them is the correct one. The concept of continuity depends on both of its sides, and it is to us really a kind of interplay between these two sides.

## 2.4 There are six = 2 x 3 dimensions of mathematics

Above we discussed the ideas of the tree worlds of David Tall and the two sides of mathematics. In this section, we elaborate on how these ideas can be combined into a new way of looking at mathematics and mathematical thinking. We argue that this leads to new insight in mathematical thinking and communication. It may also help in better understanding of Tall's three worlds.

It seems to us that our distinction between subjective-social and objective and Tall's division between embodied, symbolic and formal look at similar features in mathematical thinking from two different standpoints. Moreover, the resulting 2 x 3 = 6 dimensions of mathematics help us to see some aspects more easily. Indeed, we shall consider some examples that show how each of Tall's three worlds seems to divide into two sides.

The case of the embodied world seems especially natural. Our own mental images of mathematical objects or situations are subjective embodiment. It becomes social when a group of people shares such images while working on a problem. Various objects like number sticks etc. made for teaching mathematics are examples of objective embodiment.

A number line was mentioned above as an embodied version of the system of the real numbers. It can belong to either side depending on what we actually mean. The idea of a line of numbers belongs to the subjective-social side whereas an actual line drawn on a blackboard belongs to the objective side.

But is the real line itself an objective 'mathematical object' belonging to the objective side of mathematics? What do we think about it and its existence? In a sense this is not an important question here. On the subjective-social side most mathematicians seem to behave as if the real line would actually 'be there'. But to us, it seems that we cannot distinguish those mathematicians who really believe that the real line "is there in a Platonic universe" from those who only behave as if it existed. The theorems concerning the reals are proved using the axioms of the reals in the objective side of Tall's formal world and they make no direct reference to the truth or meaning of the actual statement that the 'reals exist'. In this sense formalism and platonism are not very far from each other.

Moreover, it is not clear how to reply from a set-theoretic point of view to the question what the real line really is. Namely, there are different constructions (Dedekind-cuts of the rationals, certain equivalence classes of Cauchy sequences of the rationals etc.) leading to different sets.

The case of the symbolic world is more interesting. Rules for manipulating symbols and correct application of such rules belong to the objective side. These include long divisions in elementary school or solving equations or doing differentiation of expressions for functions in upper secondary school. Students' own minitheories and systematic errors seem to belong to the subjective-social side of the the symbolic world.

Perhaps also various routines applied in what is called street mathematics (see e.g. Resnick, 1995) in basic calculations can be seen also as examples of the subjective-social side of the symbolic world.

Written university level mathematics with its axioms, definitions and theorems is an example of objective side of the formal world of mathematics. Higher level strategic discussion on research mathematics belongs to the subjective-social side or the formal world. An example of this represents the comment 'she mixed ideas from physics to analysis to solve the problem'.

The step from the subjective-social side or the formal world to the subjective-social side of embodied mathematics with its mental images and gestures is very short. A nice example of this is in the Introduction of W. Hodges' book (1985) where he tells about a difficulty with his own doctoral thesis. His supervisor C. C. Chang made an up and down movement with his hand and said: 'This should help.' (see Figure 4) According to Hodges, it helped.

Figure 4. Supervisor's advice

Chang's gesture indicated a certain model theoretic back-and-forth construction and obviously Hodges understood Chang's suggestion. (Such constructions are the main theme of Hodges' book.)

Before leaving this section, we shall have closer look at the concept of continuity of a function discussed above in connection to our two sides of mathematics. There we considered the appearing in Figure 2.

The notion of continuity and the function studied is clearly embodied in such a drawing. (Of course, it is possible that there is no specific function that is considered and that the whole discussion concerns the concept of continuity.) This drawing is clearly objective in the sense that everybody can observe it. So, the drawing belongs to the objective side and to the embodied world of mathematics.

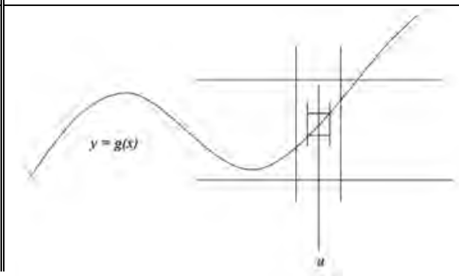
But these drawings are used either by oneself to think about continuity or by a group of people to discuss continuity. Such actions belong to the subjective-social side of the embodied world of mathematics.

When one works with examples of assertions concerning continuity, one usually has to manipulate mathematical formulas. As long as one thinks or discusses how to proceed, one acts in the subjective-social side of the symbolic world of mathematics. When these formulas are actually written they become observable and thus objective and so one acts in the objective side of the symbolic world of mathematics.

But usually, the real interest lies in understanding, teaching or using the 'epsilon-delta' -definition of continuity, and so the subjective-social or objective side of Tall's formal world is involved.

As a conclusion, while discussing the continuity of a function, all six dimensions of mathematics may be involved (Table 2).

**Table 2.** The six dimensions of mathematics in the case of continuity

| | **Embodied** | **Symbolic** | **Formal** |
|---|---|---|---|
| **Subjective-social** | What does one see in the picture and how is the picture used in a mathematical discussion? | How are the formulas manipulated and how are the symbols used? | How does one understand, teach and use the definition? |
| **Objective** |  | $\lvert g(x)-a\rvert$ $= \lvert x^2-4\rvert$ $= \lvert(x+2)(x-2)\rvert$ $= \lvert x+2\rvert\lvert x-2\rvert$ $\leq 5\lvert x-2\rvert$ | For every $\varepsilon > 0$ there is such a $\delta > 0$ that $\lvert g(x)-g(a)\rvert < \varepsilon$ for all x satisfying $\lvert x - a\rvert < \delta$. |

This framework gives a viewpoint in which mathematical activity is an interplay between six dimensions of mathematics.

# 3 A continuous nowhere differentiable function and the six dimensions of mathematics

In this section, we present a novel construction of a continuous nowhere differentiable function and discuss our construction from the viewpoint of six dimensions of mathematics.

## 3.1 Continuous nowhere differentiable functions

Continuous nowhere differentiable functions have an important role in the development of mathematics in the 19'th century. After the first discovery of a continuous nowhere differentiable function by Karl Weierstrass (1872), a great variety of constructions leading to such functions have been found (see e.g., Thim 2003). Being extremely counterintuitive such functions and their existence present also an interesting challenge for learning of the basic concepts of analysis and in mathematical thinking in general. For example, David Tall has been using a construction which he calls the Blancmange function in many of his writings (see e.g., Tall & Di Giacomo, 2000; Tall, 1982).

Such functions are related to first year analysis courses in university mathematics. Mostly their existence is only mentioned in analysis courses without going to details.

Our example of such a function is related to the use of pictures in communication mathematics. It seems that explicit reliance on Tall's embodied world is of special interest in connection to such technical mathematics. We shall present a new construction of a nowhere differentiable continuous function. We shall first discuss the construction of the function and the proofs of its special properties on the level of pictures. These pictures are not machine-made graphs of the function. Instead, they present the thinking behind the construction and therefore can be used as a basis of argumentation.

## 3.2 A visual construction of the function $f$

We shall give a construction of a continuous nowhere differentiable function by visual means. The construction of our continuous nowhere differentiable function $f$ and the discussion of its properties are written below so that the presentation suits for a group of students in a university course of analysis. Especially it is assumed that the students know in advance the basic properties of the real line including completeness and 'epsilon-delta'-definitions for continuity and differentiability.

Constructions of continuous nowhere differentiable functions usually rely on several theorems of analysis. The construction we present next is in this sense simpler. Besides the definitions of continuity and differentiability only a simple principle concerning nesting closed intervals will be used.

We shall consider the function $f$ defined during the following imaginary discussion between a Teacher (T) and a Student (S). Originally the function will be defined for $x$ satisfying $0 \leq x \leq 1$. Later a simple way of extending it to the whole real line is indicated.

In the discussion T and S look at the pictures appearing in Figure 5. While the pictures as such belong to the objective side of Tall's embodied world, they are used on the subjective-social side of mathematics in the discussion.

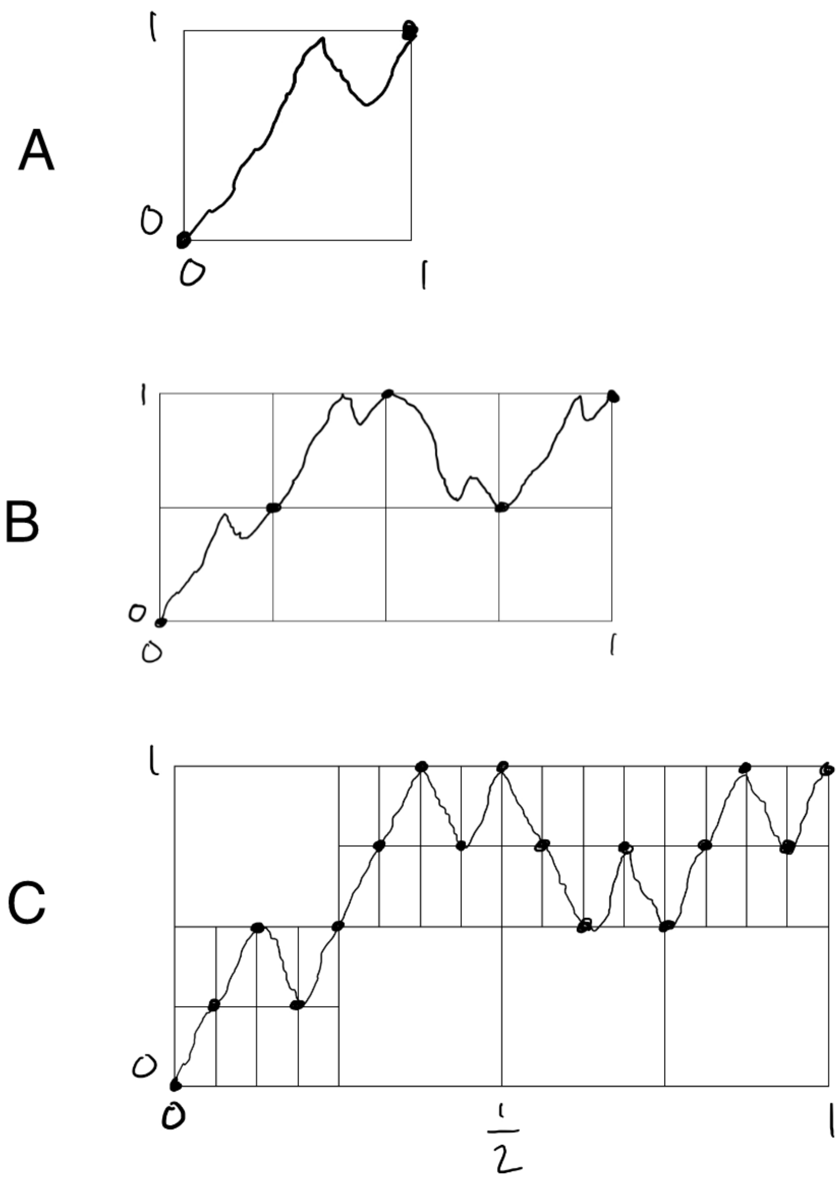



Figure 5. Explanation of the definition of $f$

T: Let me show you a very interesting function

S: Fine! What is the definition?

T: Actually, I am not going to give a simple definition. Rather I use pictures to describe a process of adding information so that all the values will eventually be determined.

S: Exciting!

T: Look at picture A (of Fig. 5). We start with the information that f goes from the bottom left corner to the top right corner of the unit square.

This means that at the beginning we know that $0 \leq f(x) \leq 1$ for $0 \leq x \leq 1$. Moreover, $f(0) = 0$ and $f(1) = 1$. In picture A, a sketch of a graph is drawn only to give a feeling of what kind of a function we have in mind.

S: OK. But this does not tell much.

T: Look at picture B. At the next step we cut the square horizontally into four and vertically into two. This gives the smaller rectangles shown in the picture. And the function goes through some of these small rectangles as the sketch of a graph indicates.

So $0 \leq f(x) \leq 1/2$ as $0 \leq x \leq 1/4$; $1/2 \leq f(x) \leq 1$ as $1/4 \leq x \leq 2/4$ $(= 1/2)$; $1/2 \leq f(x) \leq 1$ as $2/4 \leq x \leq 3/4$ and $1/2 \leq f(x) \leq 1$ as $3/4 \leq x \leq 1$. Moreover, $f(0) = 1$, $f(1/4) = 1/2$, $f(2/4) = 1$, $f(3/4) = 1/2$ and $f(1) = 1$.

S: The function seems to be in all these smaller rectangles somehow similar to the whole function in the original unit square with the exception of the third one.

T: Good! The third rectangle will be like the others, but everything is only upside down.

S: OK!

T: We know at this stage that we have rectangles in which the function goes from a left corner to the opposite right corner. To get more information we keep on cutting our rectangles to smaller. At each step we cut the rectangles horizontally in to four and vertically into two.

Look at picture C. There the next step / third step is drawn.

S: Yes, a similar idea seems really to repeat itself! But the small rectangles become all the time somehow different.

T: Can you say how they become different?

S: They become somehow more and more narrow!

T: indeed! Look at the ratio of the height to the width of these rectangles. Can you say what happens to it as our construction goes on?

S: It seems to increase all the time! The function becomes all the time somehow steeper and steeper.

To spare space, we end the dialogue and describe what happens. The above process is repeated infinitely many times. In cases where the function "goes from the upper left corner to the lower right corner" (which is above the case on the subinterval $\left[\frac{1}{2}, \frac{3}{4}\right]$ ), the picture is used "upside down" as in Figure 6.

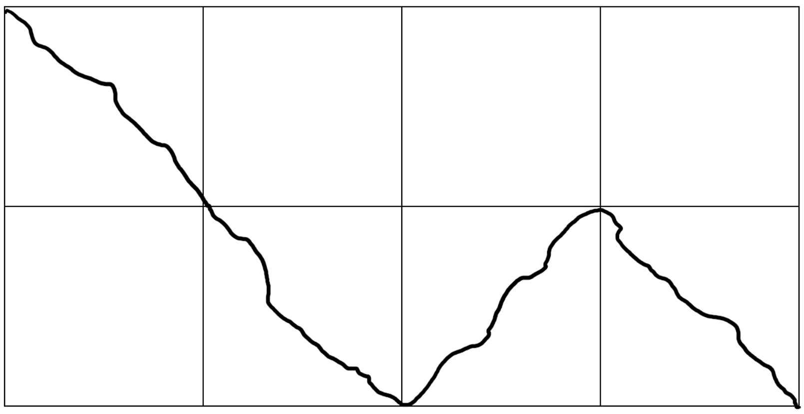


Figure 6. One more detail in the definition of f

Pictures can be used also for communicating proofs for the continuity and nowhere differentiability of $f$ – or at least for indicating the thinking behind the formal proofs.

To do this, some notation will help. Notice that at each step n of the construction we use rectangles with certain width $w_n$ and height $h_n$. Indeed,

$$w_1 = 1 \text{ and } w_{n+1} = \frac{1}{4} w_n;$$

$$h_1 = 1 \text{ and } h_{n+1} = \frac{1}{2} h_n.$$

Especially, the form of these rectangles is characterized by the ratio

$$\frac{h_{n+1}}{w_{n+1}} = 2^n.$$

The first immediate consequence of the construction is that whenever

$$|x - t| < w_n,$$

the points $(x, f(x))$ and $(t, f(t))$ of the graph of f must lie in the same or consecutive rectangles. (If we were discussing such pictures in front of us, it would be natural to show with one's finger the points discussed. So, gestures appear naturally on the subjective-social side of mathematics.)

Thus

$$|f(x) – f(t)| < 2h_n.$$

It follows from this observation that $f$ is uniformly continuous (see Figure 5, picture C).

To prove the nowhere differentiability of $f$, we take a new look at the pictures used before and make a small addition to them. This is done in Figure 7. To show that $f$ is not differentiable at a certain point $x_0$, we shall consider the difference quotients

$$\frac{f(x) - f(x_0)}{x - x_0}$$

for certain other values $x$.

In every stage n of the construction, we can locate $x_0$ in a picture like this. We can assume that f 'goes' from the bottom left corner to the top right corner. (The other case where $f$ 'goes' from the top left corner to the bottom right corner is quite similar.)

Assume first that $x_0$ is 'in' the rightmost quarter. Let the other value $x$ in the difference quotient correspond to the left bottom corner. For geometric reasons we see that the absolute value

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right|$$

is at least the slope for the rising line drawn in the picture. Thus

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| \geq \frac{1}{2} \cdot \frac{h_n}{w_n}.$$

But this ratio can be made as big as we like by choosing $n$ big enough! Notice that if $x_0$ is 'in' any other part of the picture, we have the same estimate. (If $x_0$ is 'in' the leftmost part, then we take $x$ to 'correspond to' the top right corner of the picture.)

This observation gives us the following result: For every $x_0$, every $\varepsilon > 0$ and every $M > 0$, there is x for which $|x - x_0| < \varepsilon$ and

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| > M.$$

Especially, $f$ is nowhere differentiable since the difference quotient corresponding to any $x_0$ cannot have a limit.

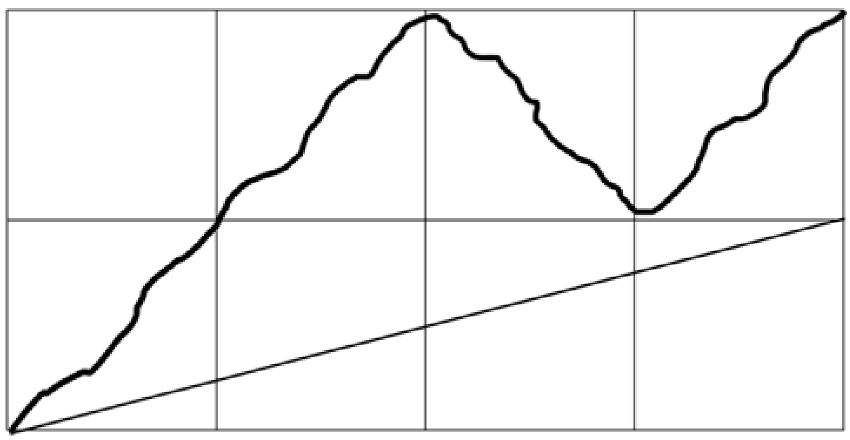


Figure 7. Why *f* is not differentiable

(The picture is of course too wide here, but it is meant to express the idea.)

One theoretical detail has been omitted so far. The reader may wonder how actually to prove that this construction leads to exact values $f(x)$. This follows from the simple principle that if we consider a nesting sequence of closed intervals $[a_1, b_1], [a_2, b_2], [a_3, b_3], \ldots$ where $a_1 \leq a_2 \leq a_3 \leq \ldots \leq b_3 \leq b_2 \leq b_1$ and where length $b_n - a_n$ tends to 0 as $n$ increases, then there is a unique number lying in all these intervals. Indeed, this number is the supremum of $a_1, a_2, a_3, \ldots$

This property is actually very interesting for several reasons. It is rather obvious when we 'look at' the real line. Hence it is very close to our 'visual image' of the real line. It is also rather easy to prove this property in an introductory course in analysis. Moreover, this property is a nice version of the compactness of closed intervals, and it can be used to give a uniform way of proving the main consequences of compactness in an analysis course by ‘cutting closed intervals into to halves’.

In this construction and in the arguments above the authors like especially the feature that all the thinking is completely visual (or embodied in the pictures, as will be said later in this paper). It would be wonderful to present this at a backboard!

More exactly, the visual proof consists of the above pictures and a discussion while observing the pictures. This will suffice to convince most novices and experts. In case we would like to write a formalized proof, we could use such a discussion as a recipe.

Since $f$ is continuous and nowhere differentiable, also that the function $g(x) = f(x) - x$ is continuous and nowhere differentiable. This function has the additional property that $g(0) = g(1)$. Therefore, extending $g$ on the whole real line is especially simple: just put $g(x) = g(x - n)$ when $n \leq x < n + 1$.

Finally, in Figure 8 there is a ‘realistic’ picture of the function $f$ produced by Maple using a code kindly written for us by Antti Rasila.

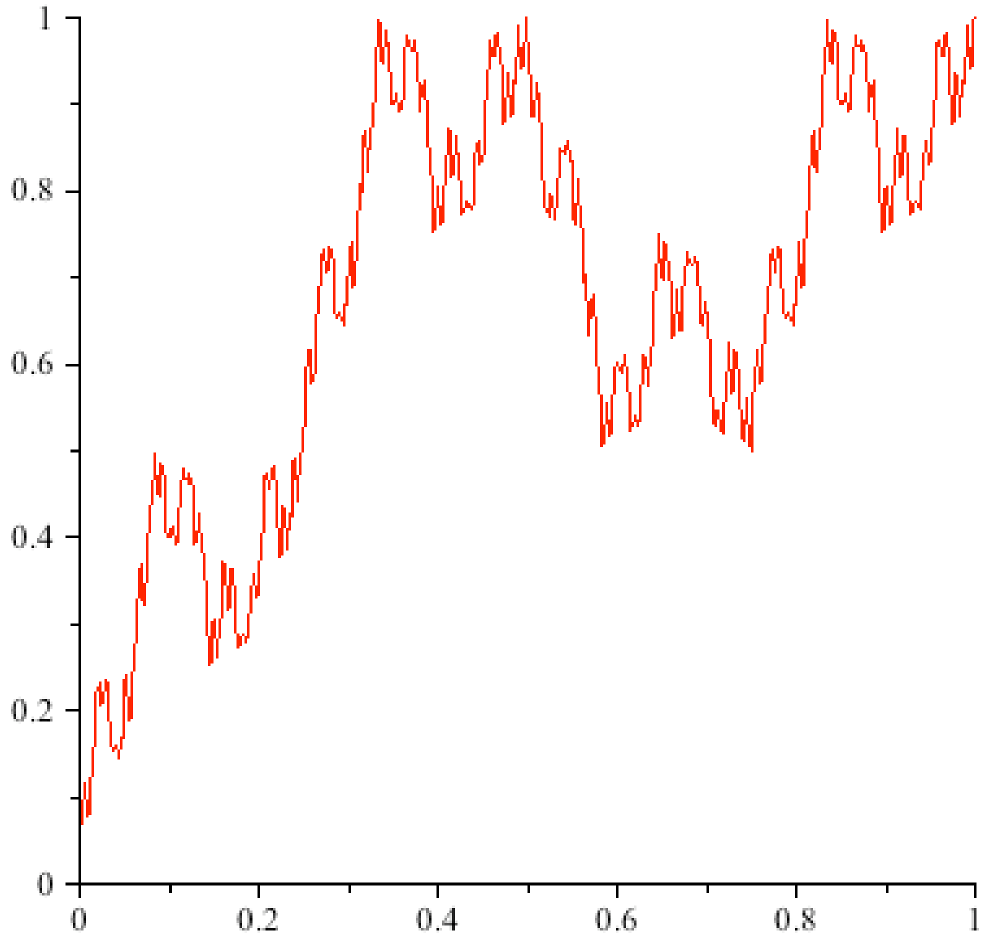



Figure 8. The portrait of the function $f$

## 3.3 The function $f$ from the point of view of the six dimensions of mathematics

We defined in the previous section a continuous nowhere differentiable function using pictures and dialogue. This was very much like the dialogue used earlier in connection to the notion of continuity. In this section we shall relate this to our idea of 2 x 3 (= 6) dimensions of mathematics.

A continuous nowhere differentiable function is a very theoretical object. As such it seems to belong strongly to Tall's formal world. There are many constructions of such functions in literature, and they are presented usually in a theoretical way.

The function whose construction and properties were discussed in the previous section is essentially quite similar. But the main interest in the previous section was how to think and communicate about our function. This was done by means of sketchy pictures and a dialogue.

The dialogue and thinking were strongly subjective-social. The pictures as part of the communication had also a subjective-social role. It is probable that the participants of the dialogue constructed several mental images of their own related to the pictures and sayings (and gestures) of the other participants.

From the point of view of Tall's worlds, this happened in the embodied world. Hence our construction was presented in the dimension of the subjective-social side of the embodied world.

The pictures and a description of how to interpret them would belong to the objective side of Tall's embodied world when printed. The meaning of the pictures as such would have been very hard to understand without the dialogue or a good written explanation.

There was also an explanation of how to prove the continuity and nowhere differentiability of $f$. This used simple calculations on the proportions of the rectangles appearing in the construction. There we added aspects of the subjective-social side of Tall's symbolic world. And when printed, this addition was in the objective side of the symbolic world.

Finally, if we would have continued the discussion to the meaning and interest in continuous nowhere differentiable functions, we would have entered the subjective-social side of Tall's formal world. And when printed, this would have happened in the objective side.

So, all the 2 x 3 dimensions of mathematics had a role in what was done.

## 4 Conclusion

We introduced a novel framework of six dimensions of mathematics by combining our view of two sides of mathematics with the framework of three worlds of mathematics. The idea of objective and subjective-social sides of mathematics does not as such appear in other distinctions presented in literature. However, the view of two sides of mathematics can be seen as an extension of many prior distinctions. Especially, the construction of the function $f$ presented in this paper supports the view that our two sides of mathematics and Tall's three worlds of mathematics fit nicely together in a sense that they look at the same mathematical scenery from two 'orthogonal' directions. Both of our two sides correspond to aspects of most of Tall's three worlds and each of Tall's three worlds has aspects of both of our two sides. This holds even for the formal world for example in the sense that reading and making proofs belong to our subjective-social world.

In terms of developing university-level mathematics teaching, we considered two main examples. First, we presented a discussion on the definition of continuity which shared the expert's thinking with the students. Later, we analyzed this discussion using our theory of the six dimensions of mathematics. To the second we gave a construction of the function $f$ presented in this paper and used it to give insight into the variety of mathematical thinking behind advanced mathematics. Tall's worlds can easily be seen as three steps of growth towards deeper and more abstract (expertise in) mathematics. But a more correct view seems to be that more than one of them are present in an expert's relation to mathematics. However, the secondary-tertiary transition includes a change in mathematical thinking as the formal world is emphasized at university (Tall, 2008). Therefore, explicit interplay between different worlds of mathematics is crucial in university-level teaching (cf. Oikkonen, 2009). One of the most interesting features of the construction and argumentation concerning the function $f$ in this paper is that it serves as an example of an unusual route through the six dimensions of mathematics to present a piece of higher mathematics.

Regarding school mathematics teaching and teacher education, we also suggest more explicit interplay between different worlds of mathematics. Both Tall's three worlds and our two sides of mathematics are closely related to attempts to understand how mathematics can be made meaningful to people. Several studies show that use of multiple representations is crucial in teacher's profession (e.g., Dreher & Kuntze, 2015) and teacher education would benefit from more explicit links between university mathematics and school mathematics (e.g., Hannula, 2018). Our framework is one viewpoint to develop such mathematical thinking both in pre-service and in-service teacher education.

Concerning further research, our framework can be utilized especially in analyzing mathematical discussion in authentic situations. For instance, the framework can be used in analyzing the elements of student groups' (un)successful problem-solving processes of in undergraduate mathematics courses. In addition, the framework gives a new lens to widely studied themes of representations and teacher knowledge (cf. Hannula, 2018). Therefore, the framework can be applied also in school mathematics and teacher education related research projects.

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